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Approximate Solutions of a Fixed-Point Problem with an Algorithm Based on Unions of Nonexpansive Mappings

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Abstract: In this paper, we study a fixed-point problem with a set-valued mapping by using an algorithm based on unions of nonexpansive mappings. We show that an approximate solution is reached after a finite number of iterations in the presence of computational errors. This result is an extension of the results known in the literature.

Keywords: convergence analysis; fixed point; nonexpansive mapping; set-valued mapping

MSC: 47H09; 47H10; 54E35

1. Introduction

The study of fixed-point problems is an important topic in nonlinear analysis [1–15]. These problems have various applications in mathematical analysis, optimization theory, engineering, medicine, and the natural sciences [14–20]. In particular, in [21], a novel framework for the investigation of iterative algorithms was introduced. This framework was given in terms of a certain nonlinear set-valued map T defined on a space X. For every $x \in X$, $T(x)$ is a finite union of values of single-valued paracontracting operators. Tam [21] established a convergence for this algorithm. Note that his result was a generalization of the result attained by Bauschke and Noll [22]. In our recent paper [23], we obtained an extension of a result of [21]. It should be mentioned that in [21], X is a finite-dimensional Euclidean space, while in [23] and in the present paper, X is an arbitrary metric space. The main result of [23] was obtained for inexact iterations of operators under the assumption that the common fixed-point problem has a solution. In the present paper, we prove an extension of this result in a case in which the common fixed-point problem has only an approximated solution.

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Citation: Zaslavski, A.J. Approximate Solutions of a Fixed-Point Problem with an Algorithm Based on Unions of Nonexpansive Mappings. *Mathematics* **2023**, *11*, 1534. https://doi.org/10.3390/math11061534

Academic Editor: Manuel Sanchis

Received: 14 February 2023
Revised: 3 March 2023
Accepted: 20 March 2023
Published: 22 March 2023

2. Preliminaries

Assume that (X, ρ) is a metric space endowed with a metric ρ and that $C \subset X$ is its nonempty closed set. For every $u \in X$ and every $\Delta \in (0, \infty)$, we set

$$B(u, \Delta) = \{v \in X : \ \rho(u, v) \leq \Delta\}.$$

For every map $A : C \to C$, we define

$$\mathrm{Fix}(A) = \{u \in C : \ A(u) = u\}.$$

Assume that $T_i : C \to C$, $i = 1, \ldots, m$, where $m \geq 1$ is an integer, $0 < \bar{c} \leq 1$, and that for every $j \in \{1, \ldots, m\}$, every $u \in \mathrm{Fix}(T_j)$, and every $v \in C$,

$$\rho(u, v)^2 - \rho(u, T_j(v))^2 \geq \bar{c}\rho(v, T_j(v))^2. \tag{1}$$

It should be mentioned that inequality (1) is true for many nonlinear operators [14,15].

Assume that

$$\phi : X \to 2^{\{1,\ldots,m\}} \setminus \{\emptyset\}. \tag{2}$$

We set

$$T(u) = \{T_j(u) : j \in \phi(u)\}. \tag{3}$$

for each $u \in C$ and

$$F(T) = \{u \in C : u \in T(u)\}. \tag{4}$$

In this paper, we study the fixed-point problem

$$\text{Find } x \in X \text{ such that } x \in T(x).$$

This problem was introduced and studied in [21]. It should be mentioned that in [21], X was a finite-dimensional Euclidean space, and the mappings T_i, $i = 1, \dots, m$ were paracontracting. Tam [21] considered a sequence of iterations $\{x_k\}_{k=0}^{\infty} \subset X$ satisfying $x_{k+1} \in T(x_k)$ for every integer $k \geq 0$ and established its convergence under the assumption that the mappings T_i, $i = 1, \dots, m$ had a common fixed point. In [21], this convergent result was applied to sparsity-constrained minimization. Note that the result in [21] was a generalization of the result attained by Bauschke and Noll [22]. In our recent paper [23], we considered mappings acting on a general metric space and obtained two extensions of the result from [21]. In the first result, we studied exact iterations of the set-valued mapping, while in the second one, we dealt with its inexact iterations while taking computational errors into account. More precisely, in [23], for a given computational error $\delta > 0$, we considered a sequence $\{x_k\}_{k=0}^{\infty} \subset X$ satisfying $B(x_{k+1}, \delta) \cap T(x_k) \neq \emptyset$ for every integer $k \geq 0$ and analyzed its behavior. This result was also obtained under the assumption that the mappings T_i, $i = 1, \dots, m$ had a common fixed point. In the present paper, we generalize this result. Instead of assuming the existence of a common fixed point, we suppose that there exists an approximate common fixed point z such that

$$B(z, \gamma) \cap \mathrm{Fix}(T_i) \neq \emptyset, \ i = 1, \dots, m,$$

where γ is a given small positive constant. In other words, a small neighborhood of z contains a fixed point of every mapping.

We fix

$$\theta \in C.$$

For any $u \in R^1$, we set

$$\lfloor u \rfloor = \max\{j : \ j \text{ as an integer and } j \leq u\}.$$

We prove the following theorem in the presence of computational errors. This theorem shows that after a certain number of iterations, we obtain an approximate solution to our fixed-point problem. The number of iterations depends on the computational error.

Theorem 1. *Let $M > 0$, $\epsilon \in (0, 1]$,*

$$\gamma \in (0, (18)^{-1}(4M+4)^{-1}\epsilon^2\bar{c}), \tag{5}$$

$$z \in B(\theta, M) \tag{6}$$

satisfy

$$B(z, \gamma) \cap Fix(T_i) \neq \emptyset, \ i = 1, \dots, m, \tag{7}$$

$$Q = \lfloor 8\epsilon^{-2}M^2\bar{c}^{-1} \rfloor + 1, \tag{8}$$

and $\delta \in (0, \gamma)$. Assume that $\{x_k\}_{k=0}^{\infty} \subset C$,

$$\rho(\theta, x_0) \leq M \tag{9}$$

and that

$$B(x_{k+1}, \delta) \cap T(x_k) \neq \emptyset, \ k = 0, 1, \dots. \tag{10}$$

Then, there is a nonnegative integer $p < Q$ for which

$$B(x_p, \epsilon) \cap T(x_p) \neq \emptyset. \tag{11}$$

In the theorem above, we assume the existence of a point z that satisfies (7), which means that z is an approximate fixed point for all of the mappings T_i, $i = 1, \dots, m$. This result has a prototype in [23], which was obtained under the assumption that z is a common fixed point for all T_k, $k = 1, \dots, m$.

3. Proof of Theorem 1

Proof. Assume that for every nonnegative integer $k < Q$, relation (11) is not true. Then, for every nonnegative integer $k < Q$,

$$B(x_k, \epsilon) \cap T(x_k) = \emptyset. \tag{12}$$

We set

$$M_0 = 2M + 1. \tag{13}$$

According to (7), for every $k \in \{1, \dots, m\}$, there is

$$z_k \in \mathrm{Fix}(T_k) \tag{14}$$

such that

$$\rho(z, z_k) \leq \gamma. \tag{15}$$

According to (6) and (9),

$$\rho(x_0, z) \leq 2M. \tag{16}$$

Let $i \in [0, Q-1]$ be an integer. According to (10), there is

$$\widehat{x}_{i+1} \in T(x_i) \tag{17}$$

for which

$$\rho(x_{i+1}, \widehat{x}_{i+1}) \leq \delta. \tag{18}$$

Equations (3) and (17) imply that there is an integer $j \in [1, m]$ for which

$$\widehat{x}_{i+1} = T_j(x_i). \tag{19}$$

It follows from (1) and (19) that

$$\rho(z_j, x_i)^2 \geq \rho(z_j, \widehat{x}_{i+1})^2 + \bar{c}\rho(x_i, \widehat{x}_{i+1})^2. \tag{20}$$

According to (12) and (19),

$$\rho(x_i, \widehat{x}_{i+1}) > \epsilon. \tag{21}$$

In view of (20) and (21),

$$\rho(z_j, x_i)^2 \geq \rho(z_j, \widehat{x}_{i+1})^2 + \bar{c}\epsilon^2. \tag{22}$$

Assume that

$$\rho(z, x_i) \leq M_0. \tag{23}$$

(In view of (13) and (16), equation (23) holds for $i = 0$.) Equations (15) and (23) imply that

$$\rho(z_j, x_i) \leq \rho(z_j, z) + \rho(z, x_i) \leq M_0 + \gamma. \tag{24}$$

It follows from (5), (13), (22), and (24) that

$$\rho(z_j, \widehat{x}_{i+1})^2 \leq \rho(z_j, x_i)^2 - \epsilon^2\bar{c} \leq (M_0 + \gamma)^2 - \epsilon^2\bar{c}$$

$$= M_0^2 + \gamma(\gamma + 2M_0) - \epsilon^2\bar{c} \le M_0^2 + \gamma(1 + 2M_0) - \epsilon^2\bar{c}$$
$$\le M_0^2 - 7\gamma(1 + 2M_0) \le (M_0 - 2\gamma)^2$$

and

$$\rho(z_j, \widehat{x}_{i+1}) \le M_0 - 2\gamma. \tag{25}$$

According to (15) and (25),

$$\rho(z, x_{i+1}) \le \rho(z, z_j) + \rho(z_j, \widehat{x}_{t+1}) + \rho(\widehat{x}_{i+1}, x_{i+1})$$
$$\le M_0 - 2\gamma + \gamma + \gamma \le M_0$$

and

$$\rho(z, x_{i+1}) \le M_0. \tag{26}$$

According to (22),

$$\rho(z_j, \widehat{x}_{i+1})^2 \le \rho(z_j, x_i)^2 - \epsilon^2\bar{c}. \tag{27}$$

Equations (15) and (23) imply that

$$|\rho(x_i, z)^2 - \rho(x_i, z_j)^2|$$
$$\le (\rho(x_i, z) + \rho(x_i, z_j))|\rho(x_i, z) - \rho(x_i, z_j)|$$
$$\le (\rho(x_i, z) + \rho(x_i, z) + \gamma)\rho(z_j, z) \le \gamma(2M_0 + 1). \tag{28}$$

It follows from (15), (18), and (26) that

$$|\rho(x_{i+1}, z)^2 - \rho(\widehat{x}_{i+1}, z_j)^2|$$
$$\le (\rho(x_{i+1}, z) + \rho(\widehat{x}_{i+1}, z_j))|\rho(x_{i+1}, z) - \rho(\widehat{x}_{i+1}, z_j)|$$
$$\le (2M_0 + \gamma + \delta)(\rho(z_j, z) + \rho(\widehat{x}_{i+1}, x_{i+1})) \le (2M_0 + 2)(\gamma + \delta). \tag{29}$$

By (5), (13), (22), and (29),

$$\rho(x_{i+1}, z)^2 \le \rho(z_j, \widehat{x}_{i+1})^2 + 2\gamma(2M_0 + 2)$$
$$\le \rho(z_j, x_i)^2 - \bar{c}\epsilon^2 + 2\gamma(2M_0 + 2)$$
$$\le \rho(x_i, z)^2 - \epsilon^2\bar{c} + \gamma(2M_0 + 1) + 2\gamma(2M_0 + 2)$$
$$\rho(x_i, z)^2 - \epsilon^2\bar{c} + 3\gamma(2M_0 + 1)$$
$$\le \rho(x_i, z)^2 - \epsilon^2\bar{c}/2. \tag{30}$$

Thus, we have shown by induction that (23) and (30) hold for $i = 0, \ldots, Q - 1$. By (16) and (30),

$$4M^2 \ge \rho(z, x_0))^2$$
$$\ge \rho(z, x_0)^2 - \rho(z, x_Q)^2$$
$$= \sum_{i=0}^{Q-1} (\rho(z, x_i)^2 - \rho(z, x_{i+1})^2) \ge Q\bar{c}\epsilon^2/2,$$

and

$$Q \le 8M^2\bar{c}^{-1}\epsilon^{-2}.$$

This contradicts (8). The contradiction that we have reached proves Theorem 1. □

4. Extensions

We use the notation and definitions introduced in Section 2.

Lemma 1. *Assume that* $M_0 > 0$,

$$z \in B(\theta, M_0), \tag{31}$$

$$B(z,1) \cap \, Fix(T_i) \neq \emptyset, \; i = 1, \dots, m, \tag{32}$$

$$x_0 \in B(\theta, M_0), \tag{33}$$

$x_1 \in C$, *and*

$$B(x_1, 1) \cap T(x_0) \neq \emptyset. \tag{34}$$

Then,

$$\rho(x_1, \theta) \leq 3M_0 + 3.$$

Proof. According to (3), there is an integer $j \in [1, m]$ for which

$$\rho(x_1, T_j(x_0)) \leq 1. \tag{35}$$

According to (32), there is

$$z_j \in \mathrm{Fix}(T_j) \tag{36}$$

for which

$$\rho(z, z_j) \leq 1. \tag{37}$$

Equations (1), (31), (33), and (35)–(37) imply that

$$\begin{aligned}
\rho(x_1, \theta) &\leq \rho(\theta, T_j(x_0)) + \rho(T_j(x_0), x_1) \\
&\leq 1 + \rho(\theta, z) + \rho(z, z_j) + \rho(z_j, T_j(x_0)) \\
&\leq 1 + M_0 + 1 + \rho(z_j, x_0) \\
&\leq 2 + M_0 + \rho(\theta, x_0) + \rho(\theta, z) + \rho(z, z_j) \\
&\leq 3 + 3M_0.
\end{aligned}$$

Lemma 1 is proved. □

Theorem 2. *Let* $M > 0$, $\epsilon \in (0, 1]$,

$$\gamma \in (0, (18)^{-1}(12M + 12)^{-1}\epsilon^2\bar{c}),$$

$$z \in B(\theta, M)$$

satisfy

$$B(z, \gamma) \cap Fix(T_i) \neq \emptyset, \; i = 1, \dots, m,$$

$$Q = \lfloor 8\epsilon^{-2}(3M + 3)^2\bar{c}^{-1} \rfloor + 1,$$

and $\delta \in (0, \gamma)$.

Assume that $\{x_k\}_{k=0}^{\infty} \subset C$,

$$\rho(\theta, x_0) \leq M,$$

and that

$$B(x_{k+1}, \delta) \cap T(x_k) \neq \emptyset, \; k = 0, 1, \dots.$$

Then, there is $j \in \{1, \dots, Q\}$ *for which*

$$B(x_j, \epsilon) \cap T(x_j) \neq \emptyset.$$

Proof. Lemma 1 implies that

$$\rho(x_1, \theta) \leq 3M + 3.$$

The application of Theorem 1 to the sequence $\{x_{i+1}\}_{i=0}^{\infty}$ implies our result. □

Theorem 3. *Let* $M > 0$, $\epsilon \in (0, 1]$,

$$\{\xi \in C : \ B(\xi, \epsilon) \cap T(\xi) \neq \emptyset\} \subset B(\theta, M), \tag{38}$$

$$\gamma \in (0, (18)^{-1}(12M + 12)^{-1}\epsilon^2\bar{c}),$$

$$z \in B(\theta, M)$$

satisfy

$$B(z, \gamma) \cap Fix(T_i) \neq \emptyset, \ i = 1, \dots, m,$$

$$Q = \lfloor 8\epsilon^{-2}(3M + 3)^2\bar{c}^{-1} \rfloor + 1,$$

and $\delta \in (0, \gamma)$.

Assume that $\{x_k\}_{k=0}^{\infty} \subset C$,

$$\rho(\theta, x_0) \leq M,$$

and that

$$B(x_{k+1}, \delta) \cap T(x_k) \neq \emptyset, \ k = 0, 1, \dots.$$

Then, there exists a strictly increasing sequence of natural numbers $\{q_j\}_{j=1}^{\infty}$ *such that*

$$1 \leq q_0 \leq Q \tag{39},$$

and for each integer $j \geq 0$,

$$q_{j+1} - q_j \leq Q \tag{40}$$

$$B(x_{q_j}, \epsilon) \cap T(x_{q_j}) \neq \emptyset. \tag{41}$$

Proof. Theorem 2 implies that there exists $q_0 \in \{1, \dots, Q\}$ for which

$$B(x_{q_0}, \epsilon) \cap T(x_{q_0}) \neq \emptyset.$$

Assume that $p \in \{0, 1, \dots\}$, q_j, $j = 0, \dots, p$ are natural numbers such that for any integer j satisfying $0 \leq j < p$, (40) holds, and assume that (41) is true for all $j = 0, \dots, p$. We set

$$y_i = x_{i+q_p}, \ i = 0, 1, \dots.$$

According to (38) and (41),

$$\rho(\theta, y_0) \leq M.$$

Clearly, all of the assumptions of Theorem 2 hold with $x_i = y_i$, $i = 0, 1, \dots$, and Theorem 2 implies that there is $j \in \{1, \dots, Q\}$ for which

$$B(y_j, \epsilon) \cap T(y_j) \neq \emptyset.$$

We set

$$q_{p+1} = q_p + j.$$

Clearly,

$$B(x_{q_{p+1}}, \epsilon) \cap T(x_{q_{p+1}}) \neq \emptyset.$$

Thus, by induction, we have constructed the sequence of natural numbers $\{q_j\}_{j=1}^{\infty}$ and proved Theorem 3. □

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bejenaru, A.; Postolache, M. An unifying approach for some nonexpansiveness conditions on modular vector spaces. *Nonlinear Anal. Model. Control* **2020**, *25*, 827–845. [CrossRef]
2. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, UK, 1990.
3. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA; Basel, Switzerland, 1984.
4. Kassab, W.; Turcanu, T. Numerical reckoning fixed points of (E)-type mappings in modular vector spaces. *Mathematics* **2019**, *7*, 390. [CrossRef]
5. Khamsi, M.A.; Kozlowski, W.M. *Fixed Point Theory in Modular Function Spaces*; Birkhäser/Springer: Cham, Switzerland, 2015.
6. Khamsi, M.A.; Kozlowski, W.M.; Reich, S. Fixed Point Theory in Modular Function Spaces. *Nonlinear Anal.* **1990**, *14*, 935–953. [CrossRef]
7. Kirk, W.A. Contraction mappings and extensions. In *Handbook of Metric Fixed Point Theory*; Kluwer: Dordrecht, The Netherlands, 2001; pp. 1–34.
8. Kozlowski, W.M. An introduction to fixed point theory in modular function spaces. In *Topics in Fixed Point Theory* Springer: Cham, Switzerland, 2014; pp. 15–222.
9. Kubota, R.; Takahashi, W.; Takeuchi, Y. Extensions of Browder's demiclosedness principle and Reich's lemma and their applications. *Pure Appl. Func. Anal.* **2016**, *1*, 63–84.
10. Okeke, G.A.; Abbas, M.; de la Sen, M. Approximation of the fixed point of multivalued quasi-nonexpansive mappings via a faster iterative process with applications. *Discret. Dyn. Nat. Soc.* **2020**, *2020*, 8634050. [CrossRef]
11. Okeke, G.A.; Ugwuogor, C.I. Iterative construction of the fixed point of Suzukis generalized nonexpansive mappings in Banach spaces. *Fixed Point Theory* **2022**, *23*, 633–652.
12. Rakotch, E. A note on contractive mappings. *Proc. Am. Math. Soc.* **1962**, *13*, 459–465. [CrossRef]
13. Reich, S.; Zaslavski, A.J. Genericity in nonlinear analysis. In *Developments in Mathematics*; Springer: New York, NY, USA, 2014; Volume 34.
14. Zaslavski, A.J. Approximate solutions of common fixed point problems. In *Springer Optimization and Its Applications*; Springer: Cham, Switzerland, 2016.
15. Zaslavski, A.J. Algorithms for solving common fixed point problems. In *Springer Optimization and Its Applications*; Springer: Cham, Switzerland, 2018.
16. Censor, Y.; Zaknoon, M. Algorithms and convergence results of projection methods for inconsistent feasibility problems: A review. *Pure Appl. Func. Anal.* **2018**, *3*, 565–586.
17. Gibali, A. A new split inverse problem and an application to least intensity feasible solutions. *Pure Appl. Funct. Anal.* **2017**, *2*, 243–258.
18. Gibali, A.; Reich, S.; Zalas, R. Outer approximation methods for solving variational inequalities in Hilbert space. *Optimization* **2017**, *66*, 417–437. [CrossRef]
19. Takahashi, W. The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces. *Pure Appl. Funct. Anal.* **2017**, *2*, 685–699.
20. Takahashi, W. A general iterative method for split common fixed point problems in Hilbert spaces and applications. *Pure Appl. Funct. Anal.* **2018**, *3*, 349–369.
21. Tam, M.K. Algorithms based on unions of nonexpansive maps. *Optim. Lett.* **2018**, *12*, 1019–1027. [CrossRef]
22. Bauschke, H.H.; Noll, D. On the local convergence of the Douglas-Rachford algorithm. *Arch. Math.* **2014**, *102*, 589–600. [CrossRef]
23. Zaslavski, A.J. An algorithm based on unions of nonexpansive mappings in metric spaces. *Symmetry* **2022**, *14*, 1852. [CrossRef]