mathematics

MDPI

Article

Characterization Results on Lifetime Distributions by Scaled Reliability Measures Using Completeness Property in Functional Analysis

Mohamed Kayid and **Mansour Shrahili** *

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: msharahili@ksu.edu.sa

Abstract: In this article, using the scaled (weighted) residual life variable, some scaled measures, the scaled mean residual life and the scaled hazard rate, are introduced. Several scales are considered as examples of the derivation of the scaled measures. The measures are developed for the case of a weighted residual life at a random time, and it is shown that the measures are scale-free in these cases. This property proves useful in situations where a relative comparison of the lifetime distribution is studied. Some characterization properties are derived in terms of scaled measures evaluated at some sequences of random time points that follow a typical distribution. Examples are used to illustrate, examine, and satisfy the obtained characterizations.

Keywords: hazard rate; mean residual lifetime; characterization; residual lifetime

MSC: 60E05; 46N30; 62N05

check for updates

Citation: Kayid, M.; Shrahili, M. Characterization Results on Lifetime Distributions by Scaled Reliability Measures Using Completeness Property in Functional Analysis. *Mathematics* **2023**, *11*, 1547. https://doi.org/10.3390/math11061547

Academic Editor: Paul Bracken

Received: 16 February 2023
Revised: 16 March 2023
Accepted: 19 March 2023
Published: 22 March 2023

1. Introduction and Preliminaries

A characterization is a particular distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model. The study of the characterization properties of probability distributions plays an essential role in many areas of statistics and applied probability. In the context of lifetime research, such properties have proven useful in testing the fit of particular lifetime distributions. There are many aspects in a lifetime distribution that may not be observable. However, a particular characterization property of a particular distribution reveals a particular aspect or feature of that distribution. This aspect may be useful in finding further applications of distributions as they are used in modeling a natural phenomenon (cf. Nassar and Mahmoud [1], Navarro et al. [2], Nagaraja [3], Sunoj et al. [4], and Iwińska and Szymkowiak [5]).

The concept of aging as a qualitative behavior that can be realized using lifetime distributions is often used in reliability and survival analysis. There are many classes of lifetime distributions that quantitatively account for various aspects of aging (see Lai and Xie [6]). The residual lifetime random variable as a concept related to the aging process of a lifespan plays a central role in reliability theory (see, e.g., Jeong [7]). The distribution of the residual lifetime random variable can induce an aging property that sometimes results from stochastic comparisons of the lifetimes of units with different ages. There are many characterizations of distributions based on the residual lifetime random variable, and also the corresponding quantities derived from the residual lifetime random variable, such as the hazard rate function (HR) and the mean residual lifetime (MRL) function (Ruiz and Navarro [8] and Gupta and Kirmani [9]). More recently, some authors have developed some characterizations of the exponential distribution and other specific distributions using the concept of residual lifetime at random time (RLRT) (see, e.g., Kayid and Izadkhah [10], Shrahili and Kayid [11], Shrahili and Kayid [12], and Alomani and Kayid [13]).

The aim of the present paper is, first, to propose, study, and argue two scaled reliability measures based on the weighted residual lifetime variable and, second, to present some characterizations that are developments to the characterization properties discussed in Shrahili and Kayid [11] and Alomani and Kayid [13] and are more applicable in many situations.

Let X be a non-negative random variable (RV) with probability density function (PDF) f_X, cumulative distribution function (CDF) F_X, and survival function (SF) $\bar{F}_X$. The RV $X_t := [X - t \mid X > t]$, which is valid for all t after which survival is possible, i.e., for all t, for which $\bar{F}(t) > 0$, is called the residual lifetime of a device with original random lifetime X after the age t. Note that $[X \mid A]$, where A is an event, denotes a conditional RV whose distribution is identical with the distribution of X conditional on A. The RV X_t has SF

$$\bar{F}_t(x) = \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}, \tag{1}$$

and the corresponding PDF is given by

$$f_t(x) = \frac{f_X(t+x)}{\bar{F}_X(t)}. \tag{2}$$

Based on the residual lifetime, the RV X_t, the MRL function of the RV X when it has a finite mean, is defined as follows:

$$m_X(t) = E(X_t) = \frac{\int_t^{+\infty} \bar{F}_X(x)dx}{\bar{F}_X(t)}, \quad \bar{F}_X(t) > 0. \tag{3}$$

The MRL function is a useful characteristic of distribution, which summarizes the entire residual lifetime in an average number. This characteristic has found many applications in the context of reliability theory, system sciences, and survival analysis. Some authors have discussed different models in the context of distribution theory in the context of MRL function (see, e.g., Finkelstein [14], Nanda et al. [15], and Alshehri, Kayid [16]). The MRL function determines the underlying distribution uniquely, as the following inversion formula affirms it:

$$\bar{F}_X(t) = \frac{\mu}{m_X(t)} \exp\left(-\int_0^t \frac{1}{m_X(x)}dx\right), \tag{4}$$

where $\mu = m_X(0)$ is the distribution mean. In terms of (4), there is a one-to-one correspondence between the SF and the MRL function of a distribution, so that if $m_X(t) = m_Y(t)$ for all $t \geq 0$, then $\bar{F}_X(t) = \bar{F}_Y(t)$ for all $t \geq 0$, i.e., X and Y are equal in distribution and vice versa (cf. Nanda et al. [17]).

Another reliability measure which is closely related to residual lifetime is the HR function. The HR function is an instantaneous risk measure defined as below:

$$h_X(t) = \lim_{\delta \downarrow 0} \frac{P(X_t \leq \delta)}{\delta} = \frac{f_X(t)}{\bar{F}_X(t)}, \quad \bar{F}_X(t) > 0. \tag{5}$$

The HR function, in parallel with the MRL function, has had a central role in studies accomplished in survival analysis and reliability modelling. For some preliminary discussion on the distribution theory using HR function, we refer the reader to Prentice and Kalbfleisch [18] and Aarset [19]. The HR function also characterizes the underlying distribution uniquely. The next inversion relation validates it:

$$\bar{F}_X(t) = \exp\left(-\int_0^t h_X(x)dx\right), \tag{6}$$

From (6), there is a direct correspondence between the SF and the HR function of a distribution, so that if $h_X(t) = h_Y(t)$ for all $t \geq 0$, then $\bar{F}_X(t) = \bar{F}_Y(t)$ for all $t \geq 0$, i.e., X and Y are equal in distribution and vice versa.

In the literature there are many results on characterization of lifetime distributions. To justify the new proposal of characterization properties, our work contributes to characterize lifetime distributions by mathematical expectations of some other reliability measures instead of hazard rate function and mean residual lifetime function evaluated at some random points as proposed by Alomani and Kayid [13], Shrahili and Kayid [11], and Shrahili and Kayid [12]. To answer the question of why the results enable someone to do what they cannot do by other means, it is worth mentioning here that our study leads in particular to the characterization of lifetime distributions using mathematical expectations of the relationship between the hazard rate and the mean remaining lifetime of a lifespan and also mathematical expectations of the coefficient of variation of the remaining lifetime when the expectations are evaluated at some random points according to a sequence of probability distributions with a commonly used construction. To highlight the novelty of our study, here we further develop the work of Alomani and Kayid [13], Shrahili and Kayid [11], and Shrahili and Kayid [12] to characterize lifetime distributions with a number of additional reliability measures.

The contents of the paper are organized as follows. In Section 2, based on a scaled residual lifetime, two scaled reliability measures are presented. Some preliminary properties of these measures are given with some typical examples. The proposed measures are developed for the case where the weighted residual lifetime is evaluated at random time points. The property of being scale-free is established for the scaled measures at random time points. In Section 3 using the concept of scaled measures evaluated at random time points, some characterization properties are presented, with the property of completeness playing an essential role in functional analysis. Examples of characterization of certain parametric distributions are given. In Section 4, we conclude the paper with further remarks and mention some possible future work.

2. Scaled Reliability Measures

In this section, we propose two scaled reliability measures using the residual lifetime variable. The age of a fresh component with random lifetime X is scaled with respect to a weight function, w. Then, using the weighted residual lifetime variable we define two scaled reliability measures, namely the scaled hazard rate and the scaled mean residual lifetime. Properties of these measures and the connection they have are investigated.

Let us suppose that w is a non-negative function such that $w(t) > 0$ for all $t \geq 0$. Then, the RV $X_{w,t} := w(t)X_t$ is called the weighted residual lifetime, which contributes to a change in time scale of the item under consideration at the age t. The RV $X_{w,t}$ has CDF

$$F_{w,t}(x) = 1 - \frac{\bar{F}_X(t + \frac{x}{w(t)})}{\bar{F}_X(t)}, \quad x \geq 0, \tag{7}$$

and, consequently, it has PDF

$$f_{w,t}(x) = \frac{f_X(t + \frac{x}{w(t)})}{w(t)\bar{F}_X(t)}, \quad x \geq 0. \tag{8}$$

The SF of $X_{w,t}$ is also given by

$$\bar{F}_{w,t}(x) = \frac{\bar{F}_X(t + \frac{x}{w(t)})}{\bar{F}_X(t)}, \quad x \geq 0. \tag{9}$$

The residual lifetime of a lifetime unit can be scaled with respect to the current age of the unit. In this case, $w(t) = \frac{1}{t}$ is an appropriate choice. In this context, consider a situation where the residual lifetime of a human is considered according to his current age after a medical operation. The performance of the person's remaining life after surgery depends not only on the benefit of the surgery itself, but also on his current age. Therefore, the RV $X_{w,t} = \frac{X_t}{t}$, which measures the residual life by separating the age effect, is a useful tool to evaluate the residual life process. The RV $X_{w,t} = \frac{X_t}{t}$ has an essential role in the study conducted by Righter et al. [20] where several ageing notions based on the scaled conditional lifetime $X_{SC} = 1 + \frac{X_t}{t}$ have been introduced.

If the residual life is scaled with respect to the mean residual life at the age t, then $w(t) = \frac{1}{m_X(t)}$. In parallel, if the residual life is scaled with respect to the standard deviation of residual life after the age t, then $w(t) = \frac{1}{\sqrt{V(X_t)}}$, where $V(X_t)$ represents variance residual lifetime function defined by $V(X_t) = E(X_t)^2 - E^2(X_t)$. For all $x \geq 0$ and for all $t \geq 0$, from (9) one can get

$$\bar{F}_{w,t}(w(t)x) = \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)} = \bar{F}_t(x),$$

therefore, it becomes apparent that the weighted residual lifetime at age t, i.e., the RV $X_{w,t}$, follows an accelerated life model in terms of the residual lifetime distribution at age t. The accelerated life distributions have had an important role in reliability theory and life testing strategies (see, e.g., Oakes and Dasu [21], Bagdonavicius and Nikulin [22], Zhao and Elsayed [23], Gebraeel et al. [24], and Ling and Hu [25]).

We apply the weighted residual life distribution to obtain two reliability measures. Let us fix $t \geq 0$. The first quantity is the scaled hazard rate at time t given by

$$\begin{aligned} Sh_X(t) &:= \lim_{\delta \downarrow 0} \frac{F_{w,t}(\delta)}{\delta} \\ &= \lim_{\delta \downarrow 0} \frac{P(t < X \leq t + \frac{\delta}{w(t)})}{\delta P(X > t)} \\ &= \lim_{\delta_t \downarrow 0} \frac{F_X(t+\delta_t) - F_X(t)}{\delta_t w(t) \bar{F}_X(t)} \\ &= \frac{h_X(t)}{w(t)}, \quad t \geq 0, \end{aligned} \tag{10}$$

where $\delta_t := \frac{\delta}{w(t)}$. If $w(t) = \frac{1}{t}$, then $Sh_X(t) = th_X(t)$, which is known as proportional hazard rate function. The monotonicity of $Sh_X(t)$ based on t in this case induces some aging properties for a lifetime distribution (see, e.g., Oliveira and Torrado [26]). If $w(t) = \frac{1}{m_X(t)}$, then $Sh_X(t) = h_X(t)m_X(t) = 1 + m'_X(t)$. In this case, the monotonicity of $Sh_X(t)$ in terms of t indicates convexity (concavity) of the MRL function. These properties for MRL function have found some potential interest in the literature (see, e.g., Belzunce and Shaked [27] and the references therein). If $w(t) = m_X(t)$, then the interest centers on the risk of instantaneous death in terms of the mean residual life, the measure $Sh_X(t) = \frac{h_X(t)}{m_X(t)}$ is of relevance (see, Gupta and Kirmani [9]).

The second quantity on the basis of weighted residual lifetime distribution is the scaled mean residual lifetime, defined by

$$Sm_X(t) = E(X_{w,t}) = w(t)m_X(t). \tag{11}$$

If $w(t) = \frac{1}{t}$, then $Sm_X(t) = tm_X(t)$ is known as the proportional mean residual lifetime function. In this case, the monotonicity of $Sm_X(t)$ proposes some new aging perspectives (see, e.g., Kayid et al. [28]). In this case, a new stochastic order has also been proposed based on $Sm_X(t)$ (see, Kayid et al. [29]). If $w(t) = \frac{1}{m_X(t)}$, then $Sm_X(t) = 1$ and thus it does not provide any information on the distribution of X. Let us choose $w(t) = \frac{1}{\sqrt{V(X_t)}}$. Then, $Sm_X(t) = \frac{1}{CV_X(t)}$, where $CV_X(t)$ is the coefficient of variation of residual lifetime X_t. The monotonicity of $Sm_X(t)$ may be of interest in reliability and life testing (see, for instance, Gupta and Kirmani [30]). The scaled mean residual life can also be useful in insurance and risk analysis. In this context, let us choose $w(t) = \bar{F}_X(t)$, and observe that $Sm_X(t) = E(X-t)_+$, where $a_+ := \max\{a, 0\}$. Note that for any non-negative RV Z with finite mean, $E(Z-t)_+ = \int_t^{+\infty} \bar{F}_Z(z)dz$ where $\bar{F}_Z$ is the SF of Z. The scaled mean residual life could be a relative measure. For example, for two non-negative RVs X and Y if $w(t) = \frac{\bar{F}_X(t)}{E(Y-t)_+}$, then

$$Sm_X(t) = \frac{E(X-t)_+}{E(Y-t)_+} = \frac{\int_t^{+\infty} \bar{F}_X(x)dx}{\int_t^{+\infty} \bar{F}_Y(y)dy}. \tag{12}$$

Note that the scaled MRL function given in (12) is decreasing in $t \geq 0$ if and only if X is less that Y in the MRL order, denoted by $X \leq_{MRL} Y$. For the definition of MRL order we refer the reader to Shaked and Shanthikumar [31].

It is known in the literature that $h_X(t) = \frac{1+m'_X(t)}{m_X(t)}$, where $m'_X(t) = \frac{d}{dt}m_X(t)$. To see how the scaled HR in (10) is connected to the scaled MRL in (11), when $w(\cdot)$ is a positive differentiable function, one has

$$Sh_X(t) = \frac{w(t) + Sm'_X(t)}{Sm_X(t)} - \frac{d}{dt}\ln\{w(t)\}, \tag{13}$$

where $Sm'_X(t) = \frac{d}{dt}Sm_X(t)$. Various properties of scaled HR measures following the scaled MRL measure can be revealed via (13). However, that whether the scaled HR or the scaled MRL characterizes the underlying distribution uniquely depends on the weight $w(t)$. For instance, if $w(t)$ does not depend on F_X (the underlying distribution) then $Sh_X(t)$ and $Sm_X(t)$ each characterizes F_X uniquely. That is, if $Sm_X(t) = Sm_Y(t)$, for all $t \geq 0$, then X and Y have identical distributions, i.e., for all $t \geq 0$, $F_X(t) = F_Y(t)$. Further, if $Sh_X(t) = Sh_Y(t)$, for all $t \geq 0$, then X and Y are identical in distribution. This is because in this case using the inversion formulas, we have

$$\bar{F}_X(t) = \exp\left(-\int_0^t w(x)Sh_X(x)\,dx\right)$$

and,

$$\bar{F}_X(t) = \frac{w(t)Sm_X(0)}{w(0)Sm_X(t)}\exp\left(-\int_0^t \frac{w(x)}{Sm_X(x)}\,dx\right).$$

However, there may be cases where $w(\cdot)$ depends on F_X and $Sh_X(t)$ or $Sm_X(t)$ characterizes the distribution in a unique way. For example, if $w(t) = \frac{1}{\bar{F}_X(t)}$ then $Sh_X(t) = f_X(t)$, which is obviously a unique characteristic of F_X. In parallel, if $w(t) = \bar{F}_X(t)$ then $Sm_X(t) = E(X-t)_+$, which uniquely determines F_X. Gupta and Kirmani [9] showed that $Sh_X(t) = \frac{h_X(t)}{m_X(t)}$, which is the scaled HR when $w(t) = m_X(t)$, determines the distribution uniquely. It has also been proved by Gupta and Kirmani [30] that the coefficient of variation of residual lifetime and hence $Sm_X(t) = \frac{m_X(t)}{\sqrt{V(X_t)}} = \frac{1}{CV_X(t)}$, which is the scaled MRL function when $w(t) = \frac{1}{\sqrt{V(X_t)}}$, characterizes the distribution uniquely.

To develop the weighted residual lifetime to the case where the ages are random we consider the residual lifetime at random time. Let X and T be two non-negative RVs with CDFs F_X and F_T, respectively. The conditional random variable $X_T := [X - T | X > T]$ when $P(X > T) > 0$ is called residual lifetime at random time (RLRT). For example, if T is interpreted as a moment of a stroke or a traumatic accident the law of the residual lifetime after T is an important measure. The RV X_T is also useful to represent the additional lifetime of a device with lifetime X compared to the lifetime of another device with lifetime T, assuming the first device survives when the second device fails. The concept of RLRT has found many applications in reliability and systems science (see, e.g., Patra and Kundu [32], Amini-Seresht et al. [33], and Patra and Kundu [34]). The idle time in a classical GI /G/1 queuing system can be interpreted as RLRT. The results obtained involving this variable find application in the study of this type of queuing system (see, Dequan and Jinhua [35]).

Suppose that X and T are two independent RVs such that $P(X > T) > 0$, then the RV $X_{w,T} := w(T)X_T$, which we call weighted inactivity time at random time has SF

$$\begin{aligned}\bar{F}_{w,T}(x) &= P(w(T)(X-T) > x \mid X > T)\\ &= \frac{P(X > T + \frac{x}{w(T)})}{P(X > T)}\\ &= \frac{\int_0^{+\infty} \bar{F}_X(t + \frac{x}{w(t)}) f_T(t)dt}{\int_0^{+\infty} \bar{F}_X(t) f_T(t)dt},\end{aligned} \tag{14}$$

where f_T denotes the PDF of T. It is clear from (14) that $\bar{F}_{w,T}(x) = \frac{E[\bar{F}_X(T+\frac{x}{w(T)})]}{E[\bar{F}_X(T)]}$. If $w(t) = 1$ then $X_{w,t}$ is equal in distribution with X_t, and also in this case $X_{w,T}$ is identical in distribution with X_T. The PDF of $X_{w,T}$ is derived as

$$f_{w,T}(x) = \frac{\int_0^{+\infty} (f_X(t + \frac{x}{w(t)})/w(t)) f_T(t)dt}{\int_0^{+\infty} \bar{F}_X(t) f_T(t)dt}, \tag{15}$$

in which f_X represents the PDF of X, i.e., $f_{w,T}(x) = \frac{E[(f_X(T+\frac{x}{w(T)})/w(T))]}{E[\bar{F}_X(T)]}$ and $a/b = \frac{a}{b}$. Let us now derive the scaled HR and the scaled MRL in the context of RLRT. Let $F_{w,T}$ is the CDF of $X_{w,T}$. From (15), we define and obtain random scaled HR measure as follows:

$$\begin{aligned}RSh_X(T) &:= \lim_{\delta \downarrow 0} \frac{F_{w,T}(\delta)}{\delta}\\ &= f_{w,T}(0)\\ &= \frac{\int_0^{+\infty} (f_X(t)/w(t)) f_T(t)dt}{\int_0^{+\infty} \bar{F}_X(t) f_T(t)dt}\\ &= \int_0^{+\infty} \frac{f_X(t)}{w(t)\bar{F}_X(t)} \frac{\bar{F}_X(t) f_T(t)}{\int_0^{+\infty} \bar{F}_X(t) f_T(t)dt}\\ &= \int_0^{+\infty} \frac{h_X(t)}{w(t)} f_v(t)dt\\ &= E[Sh_X(T_v)],\end{aligned} \tag{16}$$

where T_v is an RV with pdf $f_v(t) = \frac{v(t) f_T(t)}{\int_0^{+\infty} \bar{F}_X(t) f_T(t)dt}$ which is the PDF of a weighted distribution of T (cf. Nanda and Jain [36]) in which $v(t) = \bar{F}_X(t)$ is the underlying weight function. In the same line, from (14), random scaled MRL measure is defined as follows:

$$
\begin{aligned}
RSm_X(T) &:= E[X_{w,T}] \\
&= \int_0^{+\infty} \bar{F}_{w,T}(x)dx \\
&= \frac{\int_0^{+\infty}\int_0^{+\infty} \bar{F}_X(t+\frac{x}{w(t)})f_T(t)dtdx}{\int_0^{+\infty} \bar{F}_X(t)f_T(t)dt} \\
&= \int_0^{+\infty}\left(\int_0^{+\infty} \bar{F}_X(t+\frac{x}{w(t)})dx\right)\frac{f_T(t)}{\int_0^{+\infty} \bar{F}_X(t)f_T(t)dt}dt \\
&= \int_0^{+\infty} w(t)\left(\int_t^{+\infty} \bar{F}_X(x')dx'\right)\frac{f_T(t)}{\int_0^{+\infty} \bar{F}_X(t)f_T(t)dt}dt \\
&= \int_0^{+\infty} w(t)\frac{\int_t^{+\infty} \bar{F}_X(x')dx'}{\bar{F}_X(t)}\frac{\bar{F}_X(t)f_T(t)}{\int_0^{+\infty} \bar{F}_X(t)f_T(t)dt}dt \\
&= \int_0^{+\infty} Sm_X(t)f_v(t)\,dt \\
&= E[Sm_X(T_v)].
\end{aligned} \tag{17}
$$

Remark 1. *In the case when $w(t) = 1$, the scaled HR given in (10) corresponds to the ordinary HR function of X. In this case the random scaled HR in (16) reduces to the random hazard measure defined in Shrahili and Kayid ([12]). In this case where $w(t) = 1$, the scaled MRL given in (11) becomes the MRL function X has. In the random case, we can see that random scaled MRL in (17) corresponds to $E[X_T] = E[m_X(T_v)]$ (cf. Kayid and Izadkhah [10]). Let us denote by h_Y the HR of a lifetime variable Y and take $w(t) = h_Y(t)$. Then, (10) becomes the ratio of the HR function of X divided by the HR function of Y, and also (16) corresponds to the random relative HR proposed by Alomani and Kayid [13]. If $w(t) = \frac{1}{m_Y(t)}$ where m_Y is the MRL of the RV Y, then (11) becomes the ratio of MRL of X and the MRL of Y and, further, (17) corresponds to the random relative MRL introduced by Shrahili and Kayid [11].*

One of the advantages of using (16) and (17) is that both of these measures are scale free. Suppose X and T are random lifetimes of two devices and assume that $w(\cdot)$ does not depend on F_X. Then, for any $k > 0$, if $w(kt) = \frac{w(t)}{k}$ then $RSh_{kX}(kT) = RSh_X(T)$ and also for every $k > 0$, $RSm_{kX}(kT) = RSm_X(t)$ as will be proved in the sequel. Therefore, the introduced random scaled measures could be a scale-free relative measure for comparison of two devices and their performance. Let us observe from (16) that, for any $k > 0$ whenever $w(\cdot)$ is such that $w(kt) = \frac{w(t)}{k}$ for all $t \geq 0$,

$$
\begin{aligned}
RSh_{kX}(kT) &= \int_0^{+\infty} \frac{f_{kX}(t)f_{kT}(t)}{w(t)P(kX > kT)}dt \\
&= \int_0^{+\infty} \frac{f_X(\frac{t}{k})f_T(\frac{t}{k})}{k^2 w(t)P(X > T)}dt \\
&= \int_0^{+\infty} \frac{f_X(t')f_T(t')}{kw(kt')P(X > T)}dt' \\
&= \int_0^{+\infty} \frac{f_X(t')f_T(t')}{w(t')P(X > T)}dt' = RSh_X(T).
\end{aligned}
$$

In parallel from (17), for any $k > 0$ if $w(kt) = \frac{w(t)}{k}$ for all $t \geq 0$, then

$$\begin{aligned} RSm_{kX}(kT) &= \int_0^{+\infty} w(t) \left(\int_t^{+\infty} \bar{F}_{kX}(x')dx' \right) \frac{f_{kT}(t)}{P(kX > kT)} dt \\ &= \int_0^{+\infty} w(t) \int_{\frac{t}{k}}^{+\infty} \bar{F}_X(x'')dx'' \frac{f_T(\frac{t}{k})}{P(X > T)} dt \\ &= \int_0^{+\infty} kw(kt') \int_{t'}^{+\infty} \bar{F}_X(x'')dx'' \frac{f_T(t')}{P(X > T)} dt' \\ &= \int_0^{+\infty} w(t') \int_{t'}^{+\infty} \bar{F}_X(x'')dx'' \frac{f_T(t')}{P(X > T)} dt' = RSm_X(T). \end{aligned}$$

For example, if $w(t) = \frac{1}{t}$ then for every $k > 0$, the property $w(kt) = \frac{w(t)}{k}$ holds true for all $t > 0$. This concludes that the random scaled HR $RSh_X(T) = E(T_v h_X(T_v))$ and the random scaled MRL $RSm_X(T) = E\left[\frac{m_X(T_v)}{T_v}\right]$ are both scale-free measures to evaluate the performance of a system or lifespan with lifetime X relative to another system or lifespan having lifetime T. Eryilmaz and Tutuncu [37] used the measure $E[T_{S_1} - T_{S_2} \mid T_{S_1} > T_{S_2}]$ as a quantity for measuring relative behaviour of a coherent system with lifetime T_{S_1} with respect to another coherent system with lifetime T_{S_2}. This measure is not scale-free under the transformation $(T_{S_1}, T_{S_2}) \mapsto (kT_{S_1}, kT_{S_2})$ when k is an arbitrary positive value. However, the measure $RSm_{T_{S_1}}(T_{S_2}) = E\left[\frac{T_{S_1}}{T_{S_2}} - 1 \mid T_{S_1} > T_{S_2}\right]$ is a scale-free measure under the foregoing transformation and, therefore, it is a more appropriate relative measure.

Remark 2. *In the definition of $X_{w,t}$ the weighted residual life, if $w(t)$ depends on F_X, is written as $w_X(t)$. It can be proved quite similarly as above that if $kw_{kX}(kt) = w_X(t)$ for all $t \geq 0$, then for any $k > 0$, $RSh_{kX}(kT) = RSh_X(T)$ for all $t \geq 0$ and also $RSm_{kX}(kT) = RSm_X(t)$ for all $t \geq 0$. For example, if w_X is either the HR $h_X(t) = \frac{f_X(t)}{\bar{F}_X(r)}$ or the reversed hazard rate (RHR) of X $\widetilde{h}_X(t) = \frac{f_X(t)}{F_X(t)}$, then the required condition on w_X is fulfilled. Thus, the random scaled HR and the random scaled MRL are both scale-free quantities.*

3. Characterization Results

In this section, the scaled random quantities proposed in Section 2 are developed to a random sequence on T. It is proved that if such sequence follows a typical family of distribution then some characterization properties are produced. We utilize the concept of completeness in a particular Hilbert space as a well-known methodology in functional analysis.

Definition 1. *The sequence $\psi_1, \psi_2, \cdots$ in a Hilbert space H is complete if the only member in H which is orthogonal to each and every ψ_n is the null member, so that*

$$\prec \phi, \psi_n \succ = 0, \;\; \forall\, n \in \mathbb{N} \Rightarrow \phi = 0,$$

in which 0 stands for the zero element of H.

The symbol $\prec \cdot, \cdot \succ$ indicates an inner product of H. The Hilbert space $L^2[a, b]$, along this paper, is supposed to be accompanied with the following inner product

$$\prec \phi_1, \phi_2 \succ = \int_a^b \phi_1(x)\phi_2(x)dx,$$

in which $\phi_i, i = 1, 2$ is a real-valued function which is square integrable in $[a, b]$. Note that if $\psi_1, \psi_2, \cdots$ is a complete sequence in the Hilbert space H, then $\sum_{n=1}^{+\infty} b_n \psi_n$ in which $b_n = \prec f, \psi_n \succ$ converges in H provided that $\sum_{n=1}^{+\infty} |b_n|^2 < +\infty$, and the limit corresponds to f.

Let us set a sequential family of probability distributions to play the role of the law of T in the weighted residual life at random time. Based on the random scaled HR and the random scaled MRL, a couple of characterization properties is presented.

Assume that the random time T_n for $n \in \mathbb{N}$, follows the PDF

$$f_{T_n}(t) = \frac{r(t)s^{\nu_n}(t)}{\int_0^{+\infty} r(t)s^{\nu_n}(t)dt}, \tag{18}$$

in which $r(\cdot)$ and $s(\cdot)$ are two non-negative functions for which $0 < \int_0^{+\infty} r(t)s^{\nu_n}(t)dt < +\infty$. We will assume that in the family (18), we have $1 \leq \nu_1 < \nu_2 < \ldots$ such that $\sum_{n=1}^{+\infty} \frac{1}{\nu_n} = +\infty$ and also will assume that $s(t)$ is a monotone function throughout. The family of distributions recognized by the PDF (18) is fulfilled in many situations in statistics and reliability modelling. To see some typical families of distributions we refer the reader to Shrahili and Kayid [11], Shrahili and Kayid [12] and also Alomani and Kayid [13].

We will denote the CDF of T_n by F_{T_n}. Let us consider $w_i(\cdot), i = 1, 2$ for which the corresponding weighted residual life at random time, i.e.,

$$X_{w_i,T_n} = [w_i(T_n)(X - T_n) \mid X > T_n]$$

is taken to consideration. As given in (10) and also in (11), we consider

$$Sh_X^{[i]}(t) = \frac{h_X(t)}{w_i(t)} \text{ and } Sm_X^{[i]}(t) = w_i(t)m_X(t).$$

Let us set, accordingly, $RSh_X^{[i]}(T_n) = E[Sh_X^{[i]}(T_{n,v})]$ and $RSm_X^{[i]}(T_n) = E[Sm_X^{[i]}(T_{n,v})]$ where $T_{n,v}$ follows a weighted distribution with baseline (18) which has pdf

$$f_{T_{n,v}}(t) = \frac{v(t)r(t)s^{\nu_n}(t)}{\int_0^{+\infty} v(t)r(t)s^{\nu_n}(t)dt}, \tag{19}$$

where $v(t) = \bar{F}_X(t)$. We will show below that if the random scaled HR (or the random scaled MRL) at the random time T_n with PDF (18) does not vary with $n = l, l+1, \ldots$ (for some $l \in \mathbb{N}$) then the function $w_i(\cdot)$ also does not vary with $i = 1, 2$ and vice versa.

Theorem 1. *Let $T_n, n = 1, 2, \ldots$ denote a sequence of RVs which is independent of X. Then,*

(i) If there exists an $l \in \mathbb{N}$ such that $RSh_X^{[1]}(T_n) = RSh_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$, then $w_1(t) = w_2(t)$ for all $t \geq 0$ and vice versa.

(ii) If there exists an $l \in \mathbb{N}$ such that $RSm_X^{[1]}(T_n) = RSm_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$, then $w_1(t) = w_2(t)$ for all $t \geq 0$ and vice versa.

Proof. The assertion (i) is firstly proved. It is trivial that if $w_1(t) = w_2(t)$ for all $t \geq 0$, then $Sh_X^{[1]}(t) = Sh_X^{[2]}(t)$, for all $t \geq 0$. Hence, $RSh_X^{[1]}(T_n) = RSh_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$. To prove the converse, assume that $T_{n,v}$ follows the PDF (18) with $v(t) = \bar{F}_X(t)$. We then have

$$\begin{aligned}
RSh_X^{[2]}(T_n) - RSh_X^{[1]}(T_n) &= E[Sh_X^{[2]}(T_{n,v})] - E[Sh_X^{[2]}(T_{n,v})] \\
&= \int_0^{+\infty} (Sh_X^{[2]}(t) - Sh_X^{[2]}(t)) \frac{v(t)r(t)s^{\nu_n}(t)}{\int_0^{+\infty} v(t)r(t)s^{\nu_n}(t)dt}\, dt \\
&= C_n \int_0^{+\infty} \left(\frac{1}{w_2(t)} - \frac{1}{w_1(t)}\right) h_X(t)s^{v_l - 1}(t)v(t)r(t)s^{v'_n}(t)dt \\
&= C_n \prec \phi, s^{v'_n} \succ = 0, \quad \text{for all } n = 1, 2, \ldots.
\end{aligned}$$

where $v_n' = v_{l+n-1} - v_l + 1$ for which $1 = v_1' < v_2' < \ldots,$

$$\phi(t) = \left(\frac{1}{w_2(t)} - \frac{1}{w_1(t)}\right) h_X(t) s^{v_l - 1}(t) v(t) r(t)$$

and $C_n = \frac{1}{\int_0^{+\infty} v(t)r(t)s^{v_n}(t)dt} > 0$. Therefore, we deduce that if $RSh_X^{[1]}(T_n) = RSh_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$, then $\prec \phi, s^{v_n'} \succ = 0$, for all $n = 1, 2, \ldots$. From Lemma 2 in Alomani and Kayid [13], since $s(t)$ is a monotone function, thus, $s^{v_n'}, n = 1, 2, \ldots$ constitutes a complete sequence and, therefore, $\phi(t) = 0$ which is equivalent to $w_1(t) = w_2(t)$, for all $t \geq 0$. The proof of (i) is complete. We now prove the assertion (ii). It is plain to see that if $w_1(t) = w_2(t)$ for all $t \geq 0$, then $Sm_X^{[1]}(t) = Sm_X^{[2]}(t)$, for all $t \geq 0$. Consequently, $RSm_X^{[1]}(T_n) = RSm_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$. We prove now the converse part. We get

$$\begin{aligned} RSm_X^{[2]}(T_n) - RSm_X^{[1]}(T_n) &= E[Sm_X^{[2]}(T_{n,v})] - E[Sm_X^{[2]}(T_{n,v})] \\ &= \int_0^{+\infty} (Sm_X^{[2]}(t) - Sm_X^{[2]}(t)) \frac{v(t)r(t)s^{v_n}(t)}{\int_0^{+\infty} v(t)r(t)s^{v_n}(t)dt}\, dt \\ &= C_n \int_0^{+\infty} (w_2(t) - w_1(t)) m_X(t) s^{v_l - 1}(t) v(t) r(t) s^{v_n'}(t) dt \\ &= C_n \prec \phi^*, s^{v_n'} \succ = 0, \quad \text{for all } n = 1, 2, \ldots. \end{aligned}$$

where

$$\phi^*(t) = (w_2(t) - w_1(t)) m_X(t) s^{v_l - 1}(t) v(t) r(t).$$

Hence, if $RSm_X^{[1]}(T_n) = RSm_X^{[2]}(T_n)$, for all $n = l, l+1, \ldots$, then $\prec \phi^*, s^{v_n'} \succ = 0$, for all $n = 1, 2, \ldots$. From Lemma 2 in Alomani and Kayid [13], as in the proof of assertion (i), it follows that $\phi^*(t) = 0$ which means $w_1(t) = w_2(t)$, for all $t \geq 0$. The proof is now complete. □

Remark 3. *In Theorem 1(i), let us take $w_1(t) = h_Y(t)$ and $w_2(t) = h_X(t)$. Then, for all $n = l, l+1, \ldots$, we get $RSh_X^{[1]}(T_n) = E\left[\frac{h_X(T_{n,v})}{h_Y(T_{n,v})}\right]$ and $RSh_X^{[2]}(T_n) = 1$. It is, therefore, deduced from Theorem 1 that $E\left[\frac{h_X(T_{n,v})}{h_Y(T_{n,v})}\right] = 1$, for all $n = l, l+1, \ldots$, if and only if $h_X(t) = h_Y(t)$, for all $t \geq 0$, which means that X and Y are equal in distribution. This is the result of Theorem 3(i) in Alomani and Kayid [13]. In Theorem 1(ii), if one take $w_1(t) = \frac{1}{m_Y(t)}$ and $w_2(t) = \frac{1}{m_X(t)}$. Then, for all $n = 1, 2, \ldots$, we get $RSm_X^{[1]}(T_n) = E\left[\frac{m_X(T_{n,v})}{m_Y(T_{n,v})}\right]$ and $RSm_X^{[2]}(T_n) = 1$. From Theorem 1 it follows that $E\left[\frac{m_X(T_{n,v})}{m_Y(T_{n,v})}\right] = 1$, for all $n = 1, 2, \ldots$, if and only if $m_X(t) = m_Y(t)$, for all $t \geq 0$, which means that X and Y are identically distributed. This is the result of Theorem 4(i) in Shrahili and Kayid [11].*

The result of Theorem 1 can be useful in many ways. In the sequel, before we close the paper, some examples of characterization properties are brought.

Example 1. *Let Z be a lifetime RV with CDF F_Z. Suppose that $w_Z(\cdot)$, a quantity of distribution, characterizes F_Z. Let us denote $H_{X,Z}(t) = \lim_{\delta \downarrow 0} \frac{P(X_{w_Z,t} \leq \delta)}{\delta}$ and $M_{X,Z}(t) := E[w_Z(t)X_t]$. Then, it is obvious that when X has a particular (known) distribution, then $H_{X,Z}(t)$ and also $M_{X,Z}(t)$ each characterizes F_Z, uniquely. Let us now suppose that Z_i follows CDF F_{Z_i}, $i = 1, 2$. From Theorem 1(i), we conclude that $H_{X,Z_1}(T_n) = H_{X,Z_2}(T_n)$, for all $n = l, l+1, \ldots$, if, and only if, $w_{Z_1}(t) = w_{Z_2}(t)$, for all $t \geq 0$, and equivalently, Z_1 is identical in distribution with Z_2. From Theorem 1(ii), it is realized that if $M_{X,Z_1}(T_n) = M_{X,Z_2}(T_n)$, for all $n = l, l+1, \ldots$, if, and only if, $w_{Z_1}(t) = w_{Z_2}(t)$, for all $t \geq 0$, and this means that Z_1 and Z_2 have a common distribution.*

In the following example, using Theorem 1 we characterize the exponential distribution. Gupta and Kirmani [30] proves that the coefficient of variation of residual lifetime determines the underlying distribution uniquely.

Example 2. *Suppose that* $w_1(t) = \frac{1}{\sqrt{V(X_t)}}$ *and* $w_2(t) = \frac{1}{m_X(t)}$*. Then,* $RSm_X^{[1]}(T_n) = E[(CV_X(T_{n,v}))^{-1}]$ *where* $(CV_X(t))^{-1} = \frac{m_X(t)}{\sqrt{V_X(t)}}$ *is the reciprocal coefficient of variation of* X_t*. Further,* $RSm_X^{[2]}(T_n) = 1$*. Let* $l \in \mathbb{N}$*. From Theorem 1(ii),* $E[(CV_X(T_{n,v}))^{-1}] = 1$*, for all* $n = l, l+1, \ldots$ *if, and only if,* $\frac{1}{\sqrt{V(X_t)}} = \frac{1}{m_X(t)}$*, for all* $t \geq 0$*, which holds if, and only if,* $CV_X(t) = 1$ *, for all* $t \geq 0$*, i.e., when* X *has an exponential distribution with unspecified mean.*

In the next example, another characterization of the exponential distribution using Theorem 1 is stated. Gupta and Kirmani [9] in their Theorem 3.1 proved that the ratio of the HR function of a distribution divided by the MRL function of that distribution determines the distribution uniquely. We can plainly see that this ratio is constant if, and only if, X has an exponential distribution.

Example 3. *Suppose that* $w_1(t) = m_X(t)$ *and* $w_2(t) = \theta h_X(t)$ *where* $\theta > 0$ *is a constant. Then,* $RSh_X^{[1]}(T_n) = E\left[\frac{h_X(T_{n,v})}{m_X(T_{n,v})}\right]$ *and* $RSh_X^{[2]}(T_n) = 1$*. Fix* $l \in \mathbb{N}$*. From Theorem 1(i),* $E\left[\frac{h_X(T_{n,v})}{m_X(T_{n,v})}\right] = 1$*, for all* $n = l, l+1, \ldots$ *if, and only if,* $m_X(t) = \theta h_X(t)$*, for all* $t \geq 0$*, which is satisfied if, and only if,* $Sh_X^{[1]}(t) = \frac{h_X(t)}{m_X(t)} = \frac{1}{\theta}$*, for all* $t \geq 0$*, i.e.,* X *has an exponential distribution with mean* $\sqrt{\theta}$*.*

Nanda et al. [38] introduced a reliability measure, called aging intensity (AI) function, defined as the ratio of the HR function to a baseline HR function. Let us take $H_X(t) = \frac{1}{t}\int_0^t h_X(x)dx$ for $t > 0$. The AI function is then defined as

$$AI_X(t) = \frac{h_X(t)}{H_X(t)} = \frac{th_X(t)}{\int_0^t h_X(x)dx}.$$

In the following example, the waxbill family of distributions is characterized using Theorem 1(i).

Example 4. *Suppose that* $w_1(t) = H_X(t)$ *and* $w_2(t) = \frac{h_X(t)}{\alpha}$ *where* $\alpha > 0$ *is free of the time* t*. Then,* $RSh_X^{[1]}(T_n) = E[AI_X(T_{n,v})]$ *and* $RSh_X^{[2]}(T_n) = \alpha$*. Let* $l \in \mathbb{N}$ *be fixed. By Theorem 1(i),* $E[AI_X(T_{n,v})] = \alpha$*, for all* $n = l, l+1, \ldots$ *if, and only if,* $\alpha H_X(t) = h_X(t)$*, for all* $t \geq 0$*, which is satisfied if, and only if,*

$$Sh_X^{[1]}(t) = AI_X(t) = \frac{h_X(t)}{H_X(t)} = \alpha, \quad \textit{for all} \quad t \geq 0,$$

which holds from Theorem 2.1 in Nanda et al. [38], equivalently if, X *has Weibull distribution with the shape parameter* α*.*

We say X follows a Hall–Wellner family of distributions, written as $X \sim HW(A, B)$ whenever it has the following SF:

$$\bar{F}_X(t) = \left(\frac{B}{At+B}\right)_+^{\frac{1}{A}+1}, \quad A > -1, B > 0;$$

where a_+ stands for $\max\{0, a\}$ when $a \in \mathbb{R}$. It is noticeable that if $A > 0, A = 0$ and $-1 < A < 0$ then the HW family corresponds to, respectively, a Pareto, an exponential and a resealed beta distribution. Note that in this case X has a linear MRL function obtained as

$m_X(t) = At + B$ and, thus, $m'_X(t) = A$. The example below presents a characterizations of the Hall-Wellner family of distributions.

Example 5. *Let us assume that* $w_1(t) = \frac{1}{m_X(t)}$ *and* $w_2(t) = \theta h_X(t)$ *where* $\theta > 0$. *Then,* $RSh_X^{[1]}(T_n) = E[1 + m'_X(T_{n,v})]$ *and* $RSh_X^{[2]}(T_n) = \frac{1}{\theta}$. *Take* $l \in \mathbb{N}$ *as a fixed number. From Theorem 1(i),* $E[1 + m'_X(T_{n,v})] = \frac{1}{\theta}$, *for all* $n = l, l+1, \ldots$ *if, and only if,* $\frac{1}{m_X(t)} = \theta h_X(t)$, *for all* $t \geq 0$, *which holds if, and only if,*

$$Sh_X^{[1]}(t) = h_X(t)m_X(t) = 1 + m'_X(t) = \frac{1}{\theta}, \quad \text{for all} \quad t \geq 0,$$

which holds, equivalently if, $X \sim HW\left(\frac{1}{\theta} - 1, \mu\right)$ *where* $\mu = E(X)$.

Remark 4. *Characterization relationships have been shown to be useful for constructing potential indices for testing statistical hypotheses and then performing the test on real data sets. For example, a characterization property of the exponential distribution can be used to construct goodness-of-fit tests for exponentiality (see, e.g., Angus [39], Ascher [40], Ahmad and Alwasel [41], Baringhaus and Henze [42], and Alwasel [43] among others). Characterization properties of other distributions can also be found in Metiri et al. [44] and the references there. In general, from a mathematical point of view, the characterization properties lead to typical functional equations and non-standard methods used to solve them. The practical aspect behind these properties can be seen in the model construction based on some properties of the observed statistics. Simulation studies are usually used to validate the characterization relationships. For example, in the context of Example 5, since X has distribution F that belongs to the Hall-Wellner family of distributions, simulated observations of X and then estimated the variance* $\delta_n(F) = RSh_X^{[1]}(T_n) - RSh_X^{[2]}(T_n)$ *for* $n = 1, 2, \ldots$ *using the simulated data show that for an* $l \in N$, *and* $n = l, l+1, \ldots$, $\delta_n(F)$ *is close to zero. However, the deviations* $\delta_n(F), n = 1, 2, \ldots$ *can be a potential index for constructing a goodness-of-fit test for the linear mean residual life model. In this context, using real data, if* $\delta_n(F)$, *for* $n = l, l+1, \ldots$ *or* $\sup_{n \geq l} \delta_n(F)$ *is sufficiently close to zero for some* $l \in N$, *then strong circumstantial evidence can be gathered that the model can be accepted as the underlying one.*

Remark 5. *To answer the question of how exactly a practitioner might use the results in a specific problem, some comments are made here. One of the advantages of the main characterization results (given in Theorem 1) is that the specification or deterministic determination of a problem is based on the criterion of sequences with matching tails. In the context of the characterization of exponential distributions, it is well known that* $E(X_t) = E(X)$, *for all* $t \geq 0$ *if and only if X is exponentially distributed. Moreover,* $CV_X(t) = 1$, *for all* $t \geq 0$ *if and only if X is exponentially distributed (see, e.g., Example 2). The statements* $E(X_t) = E(X)$, *for all* $t \geq 0$ *and* $CV_X(t) = 1$, *for all* $t \geq 0$ *each contain an unmanageable number of conditions because time is a continuous feature, so a practitioner cannot use these statements to characterize the exponential distribution. In the characterization results we have presented in this paper, countable conditions have been considered to characterize a particular situation. This can give the practitioner some real clues to work with. For example, consider a situation where the practitioner wants to test whether* $X_1, X_2, \ldots$, *which denote the identically distributed random lifetimes of a particular type of electronic device, follows an exponential distribution using a real data set. Let us perform a test and examine these devices in parallel with a series system containing n components with independent and identically distributed lifetimes* $Y_1, Y_2, \ldots, Y_n$, *which have a general (arbitrary) continuous lifetime distribution. The lifetime of the series system is* $T_n = \min\{Y_1, Y_2, \ldots, Y_n\}$, *whose density function satisfies the Equation (18). This is an empirical experiment to obtain a lot of data about the lifetime of the device and also the series system with n number of components. Note that the determination of n is also available to the practitioner and series systems with different numbers of components can be considered. The random variables* $[X - T_n \mid X > T_n]$ *for* $n = 1, 2, \ldots$ *are then observable, so the mean and coefficient of variation can be estimated. For example, if it turns out that the estimated mean and coefficient of variation for some* $l \in N$ *and the number of components of the series system*

$n = l, l+1, \ldots$ do not vary greatly, then one obtains a large practical proof of the claim that the devices have exponential lifetimes.

4. Conclusions

This work had two objectives. The first was to propose the concept of weighted remaining life, i.e., RV $X_{w,t} = w(t)X_t$, whereupon two scaled reliability measures, called scaled HR and scaled MRL, were introduced. The remaining life of a lifetime unit is not always affected by the current age of the unit, but perhaps in terms of functions of the current age. This function is called $w(t)$, which scales the remaining lifetime RV X_t at age t. Thus, the weighted residual life distribution has been shown to satisfy the properties of an accelerated life model. The residual lifetime RV is itself not scale-free and, therefore, it is not useful to perform, for example, a comparison of two lifetime units relatively. The introduced scaled measures, which refer to the residual lifetime at a random time point, have been shown to have the scale-free property. This property is necessary in comparative studies where the influence of the dimension of the variable is not so strong. The second goal of the work was to derive two characterization properties using the introduced scaled measures. The property of completeness of sequences in functional analysis is satisfied by a typical family of density functions of random times. The change of scale $w(t)$ in the scaled HR and the scaled MRL is shown to be invalid for an infinite number of random times according to the above typical density. The exponential distribution, the Weibull distribution, and the family of Hall–Wellner distributions have been characterized as examples where the results are applicable. The strength of our study is that our characterization results have a dynamic property, so they can incorporate many quantities that uniquely characterize the underlying lifetime distribution and are related to the residual lifetime variable, such as the residual lifetime coefficient of variation and also the ratio of the hazard rate and the mean residual lifetime functions. The conditions are straightforward and can be aligned with statistical lifetime test experiments, as illustrated in Remark 5. The weakness of the characterization properties derived in our work, and also of those obtained by Alomani and Kayid [13], Shrahili and Kayid [11], and Shrahili and Kayid [12], is that all characterization results require an infinite number of conditions. However, in the future, a reduction in the number of conditions could be considered.

For the future of this work, it is also hoped that the authors will apply the introduced scaled measures and weighted residual lifetime to the stochastic comparison of coherent systems and, in parallel, to the study of the relative aging behavior of the lifetime of systems. Other reliability models can also be built using the scaled HR and the scaled MRL, as well as the weighted residual lifetime. Scaled reliability measures related to the inactivity time of a lifetime can also be proposed and similar characterization results using these measures can be sought.

Author Contributions: Conceptualization, M.K.; Investigation, M.K.; Methodology, M.S.; Project administration, M.S.; Supervision, M.S.; Validation, M.K.; Writing—original draft, M.K.; Writing—review and editing, M.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by Researchers Supporting Project (number RSP2023R464), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank four anonymous reviewers for their constructive comments and suggestions. The authors acknowledge financial support from the Researchers Supporting Project number (RSP2023R464), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Nassar, M.M.; Mahmoud, M.R. On characterizations of a mixture of exponential distributions. *IEEE Trans. Reliab.* **1985**, *34*, 484–488.
2. Navarro, J.; Aguila, Y.D.; Ruiz, J.M. Characterizations through reliability measures from weighted distributions. *Stat. Pap.* **2001**, *42*, 395–402.
3. Nagaraja, H. Characterizations of probability distributions. In *Springer Handbook of Engineering Statistics*; Springer: London, UK, 2006; pp. 79–95.
4. Sunoj, S.M.; Sankaran, P.G.; Maya, S.S. Characterizations of distributions using log odds rate. *Statistics* **2007**, *41*, 443–451.
5. Iwińska, M.; Szymkowiak, M. Characterizations of distributions through selected functions of reliability theory. *Commun. Stat.-Theory Methods* **2017**, *46*, 69–74.
6. Lai, C.D.; Xie, M. *Stochastic Ageing and Dependence for Reliability*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2006.
7. Jeong, J.H. *Statistical Inference on Residual Life*; Springer: New York, NY, USA, 2014.
8. Ruiz, J.M.; Navarro, J. Characterization of distributions by relationships between failure rate and mean residual life. *IEEE Trans. Reliab.* **1994**, *43*, 640–644.
9. Gupta, R.C.; Kirmani, S.N.U.A. Some characterization of distributions by functions of failure rate and mean residual life. *Commun. Stat.-Theory Methods* **2004**, *33*, 3115–3131.
10. Kayid, M.; Izadkhah, S. Characterizations of the exponential distribution by the concept of residual life at random time. *Stat. Probab. Lett.* **2015**, *107*, 164–169.
11. Shrahili, M.; Kayid, M. Characterizations of Probability Distribution by Some Sequential Relative Reliability Measures: An Application of Completeness in a Hilbert Space. *J. Funct. Spaces* **2022**, *2022*, 8151159. [CrossRef]
12. Shrahili, M.; Kayid, M. Characterizations of the Exponential Distribution by Some Random Hazard Rate Sequences. *Mathematics* **2022**, *10*, 3052.
13. Alomani, G.; Kayid, M. Characterizations of Lifetime Distributions Using Two Relative Reliability Measures. *J. Funct. Spaces* **2022**, *2022*, 6476030. [CrossRef]
14. Finkelstein, M.S. On the shape of the mean residual lifetime function. *Appl. Stoch. Model. Bus. Ind.* **2002**, *18*, 135–146. [CrossRef]
15. Nanda, A.K.; Bhattacharjee, S.; Alam, S.S. Properties of proportional mean residual life model. *Stat. Probab. Lett.* **2006**, *76*, 880–890. [CrossRef]
16. Alshehri, M.A.; Kayid, M. Mean residual lifetime frailty models: A weighted perspective. *Math. Probl. Eng.* **2021**, *2021*, 3974858. [CrossRef]
17. Nanda, A.K.; Bhattacharjee, S.; Balakrishnan, N. Mean residual life function, associated orderings and properties. *IEEE Trans. Reliab.* **2009**, *59*, 55–65. [CrossRef]
18. Prentice, R.L.; Kalbfleisch, J.D. Hazard rate models with covariates. *Biometrics* **1979**, *35*, 25–39. [CrossRef] [PubMed]
19. Aarset, M.V. How to identify a bathtub hazard rate. *IEEE Trans. Reliab.* **1987**, *36*, 106–108. [CrossRef]
20. Righter, R.; Shaked, M.; Shanthikumar, J.G. Intrinsic aging and classes of nonparametric distributions. *Probab. Eng. Inf. Sci.* **2009**, *23*, 563–582. [CrossRef]
21. Oakes, D.; Dasu, T. A note on residual life. *Biometrika* **1990**, *77*, 409–410. [CrossRef]
22. Bagdonavicius, V.; Nikulin, M. *Accelerated Life Models: Modeling and Statistical Analysis*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2001.
23. Zhao, W.; Elsayed, E.A. Modelling accelerated life testing based on mean residual life. *Int. J. Syst. Sci.* **2005**, *36*, 689–696. [CrossRef]
24. Gebraeel, N.; Elwany, A.; Pan, J. Residual life predictions in the absence of prior degradation knowledge. *IEEE Trans. Reliab.* **2009**, *58*, 106–117. [CrossRef]
25. Ling, M.H.; Hu, X.W. Optimal design of simple step-stress accelerated life tests for one-shot devices under Weibull distributions. *Reliab. Eng. Syst. Saf.* **2020**, *193*, 106630. [CrossRef]
26. Oliveira, P.E.; Torrado, N. On proportional reversed failure rate class. *Stat. Pap.* **2015**, *56*, 999–1013. [CrossRef]
27. Belzunce, F.; Shaked, M. Stochastic comparisons of mixtures of convexly ordered distributions with applications in reliability theory. *Stat. Probab. Lett.* **2001**, *53*, 363–372. [CrossRef]
28. Kayid, M.; Izadkhah, S.; Alhalees, H. Combination of mean residual life order with reliability applications. *Stat. Methodol.* **2016**, *29*, 51–69. [CrossRef]
29. Kayid, M.; Izadkhah, S.; Alhalees, H. Reliability analysis of the proportional mean residual life order. *Math. Probl. Eng.* **2014**, *2014*, 142169. [CrossRef]
30. Gupta, R.C.; Kirmani, S.N.U.A. Residual coefficient of variation and some characterization results. *J. Stat. Plan. Inference* **2000**, *91*, 23–31. [CrossRef]
31. Shaked, M.; Shanthikumar, J.G. (Eds.) *Stochastic Orders*; Springer: New York, NY, USA, 2007.
32. Patra, A.; Kundu, C. Further results on residual life and inactivity time at random time. *Commun. Stat.-Theory Methods* **2020**, *49*, 1261–1271. [CrossRef]
33. Amini-Seresht, E.; Kelkinnama, M.; Zhang, Y. On the residual and past lifetimes of coherent systems under random monitoring. *Probab. Eng. Inf. Sci.* **2021**, *35*, 465–480. [CrossRef]

34. Patra, A.; Kundu, C. Stochastic comparisons and ageing properties of RLRT (ITRT) based on variance residual life. *Commun. Stat.-Theory Methods* **2022**, *51*, 4276–4295. [CrossRef]
35. Dequan, Y.; Jinhua, C. Some results on the residual life at random time. *Acta Math. Appl. Sin.* **2000**, *16*, 435–443. [CrossRef]
36. Nanda, A.K.; Jain, K. Some weighted distribution results on univariate and bivariate cases. *J. Stat. Plan. Inference* **1999**, *77*, 169–180. [CrossRef]
37. Eryilmaz, S.; Tutuncu, G.Y. Relative behavior of a coherent system with respect to another coherent system. *Stat. Pap.* **2015**, *56*, 519–529. [CrossRef]
38. Nanda, A.K.; Bhattacharjee, S.; Alam, S.S. Properties of aging intensity function. *Stat. Probab. Lett.* **2007**, *77*, 365–373. [CrossRef]
39. Angus, J.E. Goodness-of-fit tests for exponentiality based on a loss-of-memory type functional equation. *J. Stat. Plan. Inference* **1982**, *6*, 241–251. [CrossRef]
40. Ascher, S. A survey of tests for exponentiality. *Commun. Stat.-Theory Methods* **1990**, *19*, 1811–1825. [CrossRef]
41. Ahmad, I.A.; Alwasel, I.A. A goodness-of-fit test for exponentiality based on the memoryless property. *J. R. Stat. Soc. Ser. B (Stat. Methodol.)* **1999**, *61*, 681–689. [CrossRef]
42. Baringhaus, L.; Henze, N. Tests of fit for exponentiality based on a characterization via the mean residual life function. *Stat. Pap.* **2000**, *41*, 225–236. [CrossRef]
43. Alwasel, I. On goodness of fit testing of exponenttality using the memoryless property. *J. Nonparametr. Stat.* **2001**, *13*, 569–581. [CrossRef]
44. Metiri, F., Zeghdoudi, H. and Ezzebsa, A. On the characterisation of X-Lindley distribution by truncated moments: Properties and application. *Oper. Res. Decis.* **2022**, *32*, 97–109.