Article

On Hybrid Numbers with Gaussian Leonardo Coefficients

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Abstract: We consider the Gaussian Leonardo numbers and investigate some of their amazing characteristic properties, including their generating function, the associated Binet formula and Cassini identity, and their matrix representation. Then, we define the hybrid Gaussian Leonardo numbers and obtain some of their particular properties. Furthermore, we define nn Hessenberg matrices whose permanents yield the Leonardo and Gaussian Leonardo sequences.

Keywords: hybrid Gaussian Leonardo; generating function; Binet formula; Cassini identity; permanent; Hessenberg matrix

MSC: 11B39; 11C20; 05A15

1. Introduction

The Leonardo numbers [1] are defined by the following recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \,;\, n \geq 2$$

where Le_n denotes the nth Leonardo number and $Le_0 = 1$, $Le_1 = 1$. It is known that they can be represented via the Fibonacci numbers, as shown below:

$$Le_n = 2F_{n+1} - 1$$

where F_n is the nth Fibonacci number, defined by the following recurrence for $n \geq 2$:

$$F_n = F_{n-1} + F_{n-2}.$$

Although the Leonardo numbers have been recently defined, one can find a vast amount of papers about them in the literature. For example, in [2], the authors obtain new identities of the Leonardo numbers and give relationships among the Fibonacci, Lucas, and Leonardo numbers. Moreover, they give some matrix representations. In [3], the authors define the incomplete Leonardo numbers and obtain some properties. The authors give a generalization of the Leonardo numbers in [4]. In [5], the authors define the generalized Leonardo numbers and provide some properties of these numbers. Additionally, they give matrix representations and define the incomplete generalized Leonardo numbers. In [6], the authors present some properties of octonion numbers of the Leonardo sequence.

Recently, there has been a huge amount of interest in hybrid numbers, which can be considered a generalization of complex numbers and are composed of a combination of complex ($i^2 = -1$), hyperbolic ($h^2 = 1$), and dual numbers ($\varepsilon^2 = 0$). The set of hybrid numbers (for details, see [7]) are defined below:

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

Here, we want to draw your attention to the fact that the product of any two hybrid numbers is achieved by exploiting Table 1:

Table 1. Multiplication table.

.	**1**	*i*	ε	*h*
1	1	*i*	ε	*h*
i	*i*	-1	$1 - h$	$\varepsilon + i$
ε	ε	$h + 1$	0	$-\varepsilon$
h	*h*	$-\varepsilon - i$	ε	1

Recently, many researchers have been interested in the hybrid numbers with some well-known number sequence coefficients. For more details, please see [8–23] and the references therein.

The determinant and the permanent of an $n \times n$ matrix $A = (a_{ij})$ may be given by

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \,,$$

and

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

respectively. Here, S_n represents the symmetric group of the degree n. Note that if one omits the sign pattern in the definition of the determinant, we obtain the permanent of A.

In the literature, there are many researchers who are interested in determinant and permanent computations. For more details, please look at references [24–28]. The authors of [24] have defined an excellent method for computing matrix permanents, which is called the contraction method. In other words, it is defined as follows:

Let $A = (a_{ij})$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We say A is *contractible* on column k if that column contains exactly two nonzero elements, say $a_{ik} \neq 0$, $a_{jk} \neq 0$, and $i \neq j$. Then, the $(m-1) \times (n-1)$ matrix, $A_{ij:k}$, is obtained from A replacing the ith row with $a_{jk}r_i + a_{ik}r_j$ and deleting the jth row, and the kth column is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$, and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the *contraction* of A on row k relative to columns i and j. We know that if A is an integer matrix, and B is a contraction of A, then

$$\text{per}\, A = \text{per}\, B \,. \tag{1}$$

Inspired by these recent papers, we introduce a new hybrid number system with the Gaussian Leonardo coefficients. In the following section, we define the Gaussian Leonardo hybrid numbers and obtain some of their particular properties. In the next section, we define $n \times n$ Hessenberg matrices whose permanents are the Leonardo and Gaussian Leonardo numbers. Finally, we give a Maple 13 source code to verify the permanent computation (Appendix A).

2. The Gaussian Leonardo Sequence

Definition 1. *The Gaussian Leonardo numbers are defined as shown below:*

$$GL_n = Le_n + Le_{n-1}i \,;\, n \geq 1$$

where GL_n denotes the nth Gaussian Leonardo number, and Le_n denotes the nth Leonardo number. Clearly, it can be written as follows:

$$
\begin{aligned}
GL_n &= Le_n + Le_{n-1}i \\
&= (2Le_{n-1} - Le_{n-3}) \\
&\quad + (2Le_{n-2} - Le_{n-4})i,
\end{aligned}
$$

and

$$GL_{n-1} = 2Le_{n-1} + Le_{n-3}i,$$

$$GL_{n-2} = 2Le_{n-2} + Le_{n-4}i.$$

In other words, the Gaussian Leonardo sequence can be rewritten by the following recurrence, $n \geq 3$:

$$GL_n = 2GL_{n-1} - GL_{n-3}$$

where $GL_0 = 1 - i$, $GL_1 = 1 + i$, $GL_2 = 3 + i$, and $GL_3 = 5 + 3i$.

Some values of the Gaussian Leonardo numbers are given in Table 2.

Table 2. Gaussian Leonardo numbers.

n	0	1	2	3	4	5	6	7
GL_n	$1 - i$	$1 + i$	$3 + i$	$5 + 3i$	$9 + 5i$	$15 + 9i$	$25 + 15i$	$41 + 25i$

Theorem 1. *(Generating function) The generating function for the Gaussian Leonardo numbers is given by*

$$g(x) = \sum_{n=0}^{\infty} (GL_n)x^n = \frac{(1-i) + (-1+3i)x + (1-i)x^2}{1 - 2x + x^3}.$$

Proof. Taking into account the definition of the generating function and the Gaussian Leonardo numbers, we can write the following equalities:

$$g(x) = GL_0 + GL_1 x + GL_2 x^2 + \dots + GL_n x^n + \dots$$

$$2xg(x) = 2GL_0 x + 2GL_1 x^2 + 2GL_2 x^3 + \dots + 2GL_{n-1} x^n + \dots$$

$$-x^3 g(x) = -GL_0 x^3 - GL_1 x^4 - GL_2 x^5 + \dots - GL_{n-3} x^n + \dots$$

Then,

$$
\begin{aligned}
(1 - 2x + x^3)g(x) &= GL_0 + (GL_1 - 2GL_0)x \\
&\quad + (GL_2 - 2GL_1)x^2 \\
&\quad + (GL_3 - 2GL_2 + GL_0)x^3 \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\quad + (GL_n - 2GL_{n-1} + GL_{n-3})x^n + \dots
\end{aligned}
$$

and, therefore,

$$g(x) = \frac{GL_0 + (GL_1 - 2GL_0)x + (GL_2 - 2GL_1)x^2}{1 - 2x + x^3}$$
$$= \frac{(1 - i) + (-1 + 3i)x + (1 - i)x^2}{1 - 2x + x^3}.$$

So, the proof is completed. □

Theorem 2. *(Binet formula) For $n \geq 0$, the Binet formula for the Gaussian Leonardo numbers is*

$$GL_n = T(\frac{\alpha^n - \beta^n}{\alpha - \beta}) + 2\beta^n + K$$

where $T = 1 + 2i + \sqrt{5}, K = -1 - i, \alpha = \dfrac{1 + \sqrt{5}}{2}, \quad and \quad \beta = \dfrac{1 - \sqrt{5}}{2}.$

Proof. By exploiting the generating function and the definition of the Gaussian Leonardo numbers, and, as a result,

$$g(x) = \frac{(1 - i) + (-1 + 3i)x + (1 - i)x^2}{1 - 2x + x^3}$$
$$= \frac{Ax + B}{(1 - x - x^2)} + \frac{C}{(1 - x)}$$
$$= \frac{D}{(1 - \alpha x)} + \frac{E}{(1 - \beta x)} + \frac{C}{1 - x}$$

where $A = 2i, \quad B = 2, \quad C = (-1 - i), \quad D = \dfrac{1 + 2i + \sqrt{5}}{\sqrt{5}}, \quad E = \dfrac{\sqrt{5} - 1 - 2i}{\sqrt{5}},$
$\alpha = \dfrac{1 + \sqrt{5}}{2} \quad and \quad \beta = \dfrac{1 - \sqrt{5}}{2}.$
This can be rewritten as

$$\frac{(1 - i) + (-1 + 3i)x + (1 - i)x^2}{1 - 2x + x^3} = \frac{D}{(1 - \alpha x)} + \frac{E}{(1 - \beta x)} + \frac{C}{1 - x}$$
$$= \left(\sum_{n=0}^{\infty} D\alpha^n x^n\right) + \left(\sum_{n=0}^{\infty} E\beta^n x^n\right) + \left(\sum_{n=0}^{\infty} Cx^n\right)$$
$$= \sum_{n=0}^{\infty}(D\alpha^n + E\beta^n + C)x^n$$

where

$$g(x) = \sum_{n=0}^{\infty}(D\alpha^n + E\beta^n + C)x^n$$

i.e.,

$$GL_n = T(\frac{\alpha^n - \beta^n}{\alpha - \beta}) + 2\beta^n + K$$

where $T = 1 + 2i + \sqrt{5} \quad and \quad K = -1 - i.$ So, the proof is completed. □

Note that by using the Binet formula for the Fibonacci numbers and the relation $GL_n = (2F_{n+1} - 1) + (2F_n - 1)i$, we obtain another statement for the Binet formula.

Example 1. *For $n = 3$,*

$$GL_3 = T(\frac{\alpha^3 - \beta^3}{\alpha - \beta}) + 2\beta^3 + K = (1 + 2i + \sqrt{5})(\alpha^2 + \alpha\beta + \beta^2) + 2\beta^3 - 1 - i = 5 + 3i.$$

Theorem 3. *(Cassini identity) For $n > 0$, the following identity holds*

$$\begin{aligned}
GL_{n-1}GL_{n+1} - GL_n^2 &= \left(\frac{3+i}{5}\right)8^{-n}\big[(-2-i)(8\alpha)^n(2\alpha + 2i) \\
&\quad + (8\beta)^n\left((4-3i) + (2-i)\sqrt{5}\right) \\
&\quad + 2^{1+2n}\left((-5+5i)(-2)^n + \left((2+i) + i\sqrt{5}\right)(2\alpha)^n\right)\big].
\end{aligned}$$

Proof. According to the Binet formula and by applying a pile of operations, we have:

$$\begin{aligned}
GL_{n-1}GL_{n+1} - GL_n^2 &= \left[(1+2i+\sqrt{5})\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}\right) \right. \\
&\quad \left. + 2\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + (-1-i)\right]\left[(1+2i+\sqrt{5})\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}\right)\right. \\
&\quad \left. + 2\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + (-1-i)\right] - \left[(1+2i+\sqrt{5})\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}\right)\right. \\
&\quad \left. + 2\left(\frac{1-\sqrt{5}}{2}\right)^{n} + (-1-i)\right]^2 \\
&= \left(\frac{3+i}{5}\right)8^{-n}\big[(-2-i)\left(4\left(1+\sqrt{5}\right)\right)^n\left((1+2i) + \sqrt{5}\right) \\
&\quad + \left(4 - 4\sqrt{5}\right)^n\left((4-3i) + (2-i)\sqrt{5}\right) \\
&\quad + 2^{1+2n}\left((-5+5i)(-2)^n + \left((2+i) + i\sqrt{5}\right)\left(1+\sqrt{5}\right)^n\right)\big].
\end{aligned}$$

So, the proof is completed. □

Note that by using the relation $GL_n = (2F_{n+1} - 1) + (2F_n - 1)i$, the Binet formula for F_n, and exploiting the algebraic manipulations on the recurrence relation for the Fibonacci numbers, one can find the Cassini identity, as shown below:

$$GL_{n-1}GL_{n+1} - GL_n^2 = (8(-1)^{n-1} - 4F_n + 6F_{n-1}) + (4(-1)^n - 2F_{n-1})i.$$

3. Matrix Representation of the Gaussian Leonardo Sequence

In this section, we give the matrix representation of the Leonardo numbers. Note that, in [2], the authors give a matrix representation of the Leonardo numbers; this matrix can also be expressed by the sums of the Fibonacci numbers. Let us consider the matrices given below:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} GL_2 \\ GL_1 \\ GL_0 \end{bmatrix}.$$

Then, it is easy to see that the following equation holds:

$$
\begin{bmatrix} GL_3 \\ GL_2 \\ GL_1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} GL_2 \\ GL_1 \\ GL_0 \end{bmatrix}
$$

and, by induction, we say:

$$
\begin{bmatrix} GL_{n+2} \\ GL_{n+1} \\ GL_n \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} GL_2 \\ GL_1 \\ GL_0 \end{bmatrix}.
$$

Moreover, the following amazing property also holds:

$$
\begin{bmatrix} GL_{2n+2} \\ GL_{2n+1} \\ GL_{2n} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} GL_{n+2} \\ GL_{n+1} \\ GL_n \end{bmatrix}.
$$

Theorem 4. *For $n \geq 1$,*

$$
A^n = \begin{bmatrix} \mathbb{F}_{n+4} & -\mathbb{F}_{n+2} & -\mathbb{F}_{n+3} \\ \mathbb{F}_{n+3} & -\mathbb{F}_{n+1} & -\mathbb{F}_{n+2} \\ \mathbb{F}_{n+2} & -\mathbb{F}_n & -\mathbb{F}_{n+1} \end{bmatrix}
$$

where $\mathbb{F}_1 = -1$, $\mathbb{F}_2 = 0$, and $\mathbb{F}_k = \sum_{i=0}^{k-3} F_i = F_{k-1} - 1$ ($k \geq 3$), and F_k is the kth Fibonacci number.

Proof. By exploiting the well-known property for the Fibonacci sums

$$
\sum_{i=1}^{n} F_i = F_{n+2} - 1,
$$

we have, for $k \geq 3$,

$$
\mathbb{F}_k = \sum_{i=0}^{k-3} F_i = F_{k-1} - 1.
$$

Then, by using the Mathematical Induction Method, it is easy to see that this is verified for $n = 1$. Then, suppose that the following equation is true:

$$
A^n = \begin{bmatrix} \mathbb{F}_{n+4} & -\mathbb{F}_{n+2} & -\mathbb{F}_{n+3} \\ \mathbb{F}_{n+3} & -\mathbb{F}_{n+1} & -\mathbb{F}_{n+2} \\ \mathbb{F}_{n+2} & -\mathbb{F}_n & -\mathbb{F}_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+3} - 1 & -F_{n+1} + 1 & -F_{n+2} + 1 \\ F_{n+2} - 1 & -F_n + 1 & -F_{n+1} + 1 \\ F_{n+1} - 1 & -F_{n-1} + 1 & -F_n + 1 \end{bmatrix}.
$$

Then,

$$
A^{n+1} = A^n A = \begin{bmatrix} \mathbb{F}_{n+4} & -\mathbb{F}_{n+2} & -\mathbb{F}_{n+3} \\ \mathbb{F}_{n+3} & -\mathbb{F}_{n+1} & -\mathbb{F}_{n+2} \\ \mathbb{F}_{n+2} & -\mathbb{F}_n & -\mathbb{F}_{n+1} \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{F}_{n+5} & -\mathbb{F}_{n+3} & -\mathbb{F}_{n+4} \\ \mathbb{F}_{n+4} & -\mathbb{F}_{n+2} & -\mathbb{F}_{n+3} \\ \mathbb{F}_{n+3} & -\mathbb{F}_{n+1} & -\mathbb{F}_{n+2} \end{bmatrix}
$$

which shows that it holds for $n + 1$. $\square$

Proposition 1. $det(A^n) = (-1)^n$.

Proof. By Theorem 4 and its proof:

$$
A^n = \begin{bmatrix} \mathbb{F}_{n+4} & -\mathbb{F}_{n+2} & -\mathbb{F}_{n+3} \\ \mathbb{F}_{n+3} & -\mathbb{F}_{n+1} & -\mathbb{F}_{n+2} \\ \mathbb{F}_{n+2} & -\mathbb{F}_n & -\mathbb{F}_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+3} - 1 & -F_{n+1} + 1 & -F_{n+2} + 1 \\ F_{n+2} - 1 & -F_n + 1 & -F_{n+1} + 1 \\ F_{n+1} - 1 & -F_{n-1} + 1 & -F_n + 1 \end{bmatrix}.
$$

Applying the suitable elementary row operations, one obtains the determinant, as shown below:

$$det(A^n) = det \begin{bmatrix} 1 & -1 & -1 \\ F_n & -F_{n-2} & -F_{n-1} \\ F_{n+1}-1 & -F_{n-1}+1 & -F_n+1 \end{bmatrix} = F_{n-1}^2 - F_n F_{n-2} = (-1)^n$$

where the last equality follows from the Cassini identity for the Fibonacci numbers. □

4. On Hybrid Numbers with Gaussian Leonardo Coefficients

The main goal of this section is to define hybrid numbers with the Gaussian Leonardo coefficients and to present some amazing results involving them.

Definition 2. *Let us define hybrid numbers with the Gaussian Leonardo coefficients, as shown below:*

$$HGL_n = GL_n + GL_{n+1}i + GL_{n+2}\varepsilon + GL_{n+3}h \; ; \; n \geq 0$$

where HGL_n denotes the nth hybrid Gaussian Leonardo number.

By using the definition of the GL_n, we can write:

$$\begin{aligned} HGL_n &= GL_n + GL_{n+1}i + GL_{n+2}\varepsilon + GL_{n+3}h \\ &= (2GL_{n-1} - GL_{n-3}) \\ &\quad +(2GL_n - GL_{n-2})i \\ &\quad +(2GL_{n+1} - GL_{n-1})\varepsilon \\ &\quad +(2GL_{n+2} - GL_n)h \end{aligned}$$

and

$$HGL_{n-1} = GL_{n-1} + GL_n i + GL_{n+1}\varepsilon + GL_{n+2}h,$$

$$HGL_{n-2} = GL_{n-2} + GL_{n-1}i + GL_n\varepsilon + GL_{n+1}h.$$

In other words, the hybrid Gaussian Leonardo numbers can be rewritten by the following recurrence, $n \geq 3$:

$$HGL_n = 2HGL_{n-1} - HGL_{n-3}$$

with the initial conditions $HGL_0 = 1 + 3i + 6\varepsilon + 4h$, $HGL_1 = 3 + 9i + 10\varepsilon + 6h$, and $HGL_2 = 5 + 15i + 18\varepsilon + 10h$.

Some values of HGL_n are given in Table 3.

Table 3. Some hybrid Gaussian Leonardo numbers.

n	0	1	2	3	4	5
HGL_n	$1 + 3i + 6\varepsilon + 4h$	$3 + 9i + 10\varepsilon + 6h$	$5 + 15i + 18\varepsilon + 10h$	$9 + 27i + 30\varepsilon + 16h$	$15 + 45i + 50\varepsilon + 26h$	$25 + 75i + 82\varepsilon + 42h$

In order to find the generating function for the hybrid Gaussian Leonardo numbers, we have to write the sequence as a power series in which each term of the sequence corresponds to the coefficients of the series. For more details, please see [20].

Theorem 5. *(Generating function) The generating function for the hybrid Gaussian Leonardo numbers is given by*

$$g(x) = \sum_{n=0}^{\infty} (HGL_n)x^n = \frac{(1 + 3i + 6\varepsilon + 4h) + (1 + 3i - 2\varepsilon - 2h)x + (-1 - 3i - 2\varepsilon - 2h)x^2}{1 - 2x + x^3}.$$

Proof. The generating function of HGL_n with a formal power series is

$$g(x) = HGL_0 + HGL_1 x + HGL_2 x^2 + \dots + HGL_n x^n + \dots$$

and

$$2xg(x) = 2HGL_0 x + 2HGL_1 x^2 + 2HGL_2 x^3 + \dots + 2HGL_{n-1} x^n + \dots$$
$$-x^3 g(x) = -HGL_0 x^3 - HGL_1 x^4 - HGL_2 x^5 + \dots - HGL_{n-3} x^n + \dots$$

From here,

$$
\begin{aligned}
(1 - 2x + x^3)g(x) =\ & HGL_0 + (HGL_1 - 2HGL_0)x \\
& + (HGL_2 - 2HGL_1)x^2 \\
& + (HGL_3 - 2HGL_2 + HGL_0)x^3 \\
& \quad . \\
& \quad . \\
& \quad . \\
& + (HGL_n - 2HGL_{n-1} + HGL_{n-3})x^n + \dots
\end{aligned}
$$

and, as a result,

$$
\begin{aligned}
g(x) &= \frac{HGL_0 + (HGL_1 - 2HGL_0)x + (HGL_2 - 2HGL_1)x^2}{1 - 2x + x^3} \\
&= \frac{(1 + 3i + 6\varepsilon + 4h) + (1 + 3i - 2\varepsilon - 2h)x + (-1 - 3i - 2\varepsilon - 2h)x^2}{1 - 2x + x^3}.
\end{aligned}
$$

So, the proof is completed.　$\square$

Theorem 6. *(Binet formula) For $n \geq 0$,*

$$HGL_n = \frac{X(\alpha^n + \beta^n) + Y(\alpha^n - \beta^n)}{\sqrt{5}} + (-1 - 3i - 2\varepsilon)$$

where $X = \sqrt{5}(1 + 3i + 4\varepsilon + 2h), \quad Y = 3 + 9i + 8\varepsilon + 4h$.

Proof. By exploiting the generating function and the definition of the hybrid Gaussian Leonardo numbers, we have

$$
\begin{aligned}
g(x) &= \frac{HGL_0 + (HGL_1 - 2HGL_0)x + (HGL_2 - 2HGL_1)x^2}{1 - 2x + x^3} \\
&= \frac{Ax + B}{(1 - x - x^2)} + \frac{C}{(1 - x)}
\end{aligned}
$$

where $A = (2 + 6i + 4\varepsilon + 2h)$, $B = (2 + 6i + 8\varepsilon + 4h)$, and $C = (-1 - 3i - 2\varepsilon)$. Then, we write

$$\frac{Ax + B}{(1 - x - x^2)} = \frac{E}{(1 - x\alpha)} + \frac{F}{(1 - x\beta)}$$

where

$$E = \frac{(3 + 9i + 8\varepsilon + 4h) + \sqrt{5}(1 + 3i + 4\varepsilon + 2h)}{\sqrt{5}}$$

and

$$F = \frac{(1 + 3i + 4\varepsilon + 2h)\sqrt{5} - (3 + 9i + 8\varepsilon + 4h)}{\sqrt{5}}.$$

This can be rewritten as

$$
\begin{aligned}
g(x) &= \frac{(1+3i+6\varepsilon+4h)+(1+3i-2\varepsilon-2h)x+(-1-3i-2\varepsilon-2h)x^2}{1-2x+x^3} \\
&= \frac{(3+9i+8\varepsilon+4h)+\sqrt{5}(1+3i+4\varepsilon+2h)}{\sqrt{5}(1-x\alpha)} \\
&\quad + \frac{(1+3i+4\varepsilon+2h)\sqrt{5}-(3+9i+8\varepsilon+4h)}{\sqrt{5}(1-x\beta)} \\
&\quad + \frac{(-1-3i-2\varepsilon)}{(1-x)} \\
&= \left(\sum_{n=0}^{\infty} E\alpha^n x^n\right) + \left(\sum_{n=0}^{\infty} F\beta^n x^n\right) + \left(\sum_{n=0}^{\infty} Cx^n\right) \\
&= \sum_{n=0}^{\infty}(E\alpha^n + F\beta^n + C)x^n
\end{aligned}
$$

i.e.,

$$
HGL_n = \frac{X(\alpha^n+\beta^n)+Y(\alpha^n-\beta^n)}{\sqrt{5}} + (-1-3i-2\varepsilon).
$$

So, the proof is completed.   □

Example 2. *For $n=2$,*

$$
\begin{aligned}
HGL_2 &= \frac{X(\alpha^2+\beta^2)+Y(\alpha^2-\beta^2)}{\sqrt{5}} + (-1-3i-2\varepsilon) \\
&= \left(\frac{3X+Y\sqrt{5}}{\sqrt{5}}\right) + (-1-3i-2\varepsilon) \\
&= 5+15i+18\varepsilon+10h.
\end{aligned}
$$

5. On Permanents of Hessenberg Matrices Yielding the Leonardo and Gaussian Leonardo Sequences

In this section, we define one type of Hessenberg matrix family and compute its permanent by using the contraction method. Furthermore, we provide a Maple 13 source code to calculate the permanent using the contraction method (Appendix A).

Let $P_n = [p_{ij}]$ be an $n \times n$ Hessenberg matrix with $p_{11} = p_{12} = p_{13} = 1$, $p_{mm} = 2$, $p_{m,m-1} = -1$, and $p_{t,t+2} = -1$ for $m = 2,3,\ldots,n$ and $t = 2,3,\ldots,n-2$. Clearly:

$$
P_n = \begin{bmatrix}
1 & 1 & 1 & & & & 0 \\
-1 & 2 & 0 & -1 & & & \\
& -1 & 2 & 0 & \ddots & & \\
& & -1 & 2 & \ddots & -1 & \\
& & & \ddots & \ddots & & 0 \\
0 & & & & & -1 & 2
\end{bmatrix}. \tag{2}
$$

Then, we have the following theorem.

Theorem 7. *Let P_n be a matrix as in (2). Then,*

$$
per P_n = Le_{n-1}.
$$

Proof. From the definition of the matrix P_n, this can be contracted on the first column. Let us denote the kth contraction of P_n by $P_n^{(k)}$. Namely,

$$P_n^{(1)} = \begin{bmatrix} 1 & -1 & -1 & & & & 0 \\ -1 & 2 & 0 & -1 & & & \\ & & -1 & 2 & 0 & \ddots & \\ & & & -1 & 2 & \ddots & -1 \\ & & & & \ddots & \ddots & 0 \\ 0 & & & & & -1 & 2 \end{bmatrix} \tag{3}$$

and by continuing with this process with $(n-3)$ steps, we obtain for $2 \leq k \leq n-3$;

$$P_n^{(k)} = \begin{bmatrix} Le_k & Le_{k-2} & -Le_{k-1} & & & & 0 \\ -1 & 2 & 0 & -1 & & & \\ & & -1 & 2 & 0 & \ddots & \\ & & & -1 & 2 & \ddots & -1 \\ & & & & \ddots & \ddots & 0 \\ 0 & & & & & -1 & 2 \end{bmatrix} \tag{4}$$

and the $(n-2)$th step is given below:

$$P_n^{(n-2)} = \begin{bmatrix} Le_{n-2} & Le_{n-4} \\ -1 & 2 \end{bmatrix}.$$

Then, we have

$$per P_n = per P_n^{(n-2)} = 2Le_{n-2} - Le_{n-4} = Le_{n-1}.$$

□

Let $K_n = [k_{ij}]$ be an $n \times n$ Hessenberg matrix with $k_{11} = k_{12} = k_{13} = 1$, $k_{m,m} = 2$, $k_{nn} = 2 + i$, $k_{m,m-1} = -1$, and $k_{t,t+2} = -1$ for $m = 2, 3, 4, \ldots, n-1$ and $t = 2, 3, 4, \ldots, n-2$. Clearly:

$$K_n = \begin{bmatrix} 1 & 1 & 1 & & & & 0 \\ -1 & 2 & 0 & -1 & & & \\ & & -1 & 2 & 0 & \ddots & \\ & & & -1 & 2 & \ddots & -1 \\ & & & & \ddots & \ddots & 0 \\ 0 & & & & & -1 & 2+i \end{bmatrix}. \tag{5}$$

Then, we have the following theorem.

Theorem 8. *Let K_n be a matrix as in* (5). *Then*

$$per K_n = GL_{n-1}.$$

Proof. The matrix K_n can be contracted on column 1, and, by following the same steps as in the previous theorem, we obtain the following matrix for $1 \leq r \leq n-3$;

$$K_n^{(r)} = \begin{bmatrix} Le_k & Le_{k-2} & -Le_{k-1} & & & 0 \\ -1 & 2 & 0 & -1 & & \\ & -1 & 2 & 0 & \ddots & \\ & & -1 & 2 & \ddots & -1 \\ & & & \ddots & \ddots & 0 \\ 0 & & & & -1 & 2+i \end{bmatrix} \tag{6}$$

and the $(n-2)$th step is given below:

$$K_n^{(n-2)} = \begin{bmatrix} Le_{n-2} & Le_{n-4} \\ -1 & 2+i \end{bmatrix}.$$

Then, we have

$$per K_n = per K_n^{(n-2)} = (2+i)Le_{n-2} - Le_{n-4} = GL_{n-1}.$$

$\square$

6. Conclusions

In this study, we initially present the Gaussian Leonardo numbers. Then, we investigate some of their amazing characteristic properties, such as the Binet formula, generating function, Cassini identity, etc. Furthermore, we give some matrix representations. In the following section, we define the hybrid Gaussian Leonardo numbers and obtain some particular properties for them. Additionally, we define $n \times n$ Hessenberg matrices whose permanents are the Leonardo and Gaussian Leonardo numbers. Finally, we give a Maple 13 source code to verify the permanent computation (Appendix A).

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Appendix A

Using the following Maple 13 source code, it is possible to obtain the matrix and the steps of the contraction method.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,k,c,C;
c:=(i,j)->piecewise(i=1 and j=1, 1, i=1 and j=2, 1, i=1
and j=3, 1, i=j,2,i=j-2,-1,
i=j+1, -1, 0);
C:=Matrix(n,n,c):
for k from 0 to n-3 do
print(k,C):
for j from 2 to n-k do
```

```
C[1,j]:=C[2,1]C[1,j]+C[1,1]C[2,j]:
od:
C:=DeleteRow(DeleteColumn(Matrix(n-k,n-k,C),1),2):
od:
print(k, eval(C)):
end proc:with(LinearAlgebra):
permanent(n);
```

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