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Class of Crosscap Two Graphs Arising from Lattices–I

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Abstract: Let $\mathcal{L}$ be a lattice. The annihilating-ideal graph of $\mathcal{L}$ is a simple graph whose vertex set is the set of all nontrivial ideals of $\mathcal{L}$ and whose two distinct vertices I and J are adjacent if and only if $I \wedge J = 0$. In this paper, crosscap two annihilating-ideal graphs of lattices with at most four atoms are characterized. These characterizations provide the classes of multipartite graphs, which are embedded in the Klein bottle.

Keywords: crosscap; Klein bottle; lattice; annihilating-ideal graph

MSC: 05C75; 05C25; 05C10; 06A07; 06B99

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1. Introduction

According to the well-known theorem of Kuratowski and Wagner, a graph is planar if and only if it does not contain either of the two forbidden graphs K_5 and $K_{3,3}$. The Graph Minor Theorem of Robertson and Seymour [1] can be considered a powerful generalization of Kuratowski's Theorem. In particular, their theorem, which is the "deepest" and "most important" result in the arena of graph theory [2], implies that each graph property, *no matter what*, is characterized by a corresponding finite list of graphs. Thus, for surfaces (both orientable and non-orientable) in general, it is known that the set of forbidden minors is finite [3]. An analogous characterization for the embedding of graphs on surfaces is known for the crosscap one surface (Möbius strip) where 103 forbidden subgraphs (equivalently 35 forbidden minors) are characterized [4,5]. So, an open problem is to determine the several forbidden subgraphs for crosscap two surfaces (the Klein bottle). In this sequel, finding a family of graphs that has a crosscap two is an interesting one. Note that most of the 103 graphs contain a subgraph that is homeomorphic to $K_{3,3}$, and multipartite graphs play a vital role in finding these 103 forbidden subgraphs for the projective plane. It is worth mentioning that the crosscap value of bipartite and tripartite graphs are well known (refer to Proposition 1). The main goal of this paper is to identify a large class of crosscap two r-partite graphs where $r \geq 4$.

Let us introduce the concept of the *annihilating-ideal graph of a lattice*, a type of multipartite graph. Note that the annihilating-ideal graph is an extension of the concept of the zero-divisor graph. The idea of the zero-divisor graph of a ring structure is due to Beck [6]. In 2009, Halaš et al. [7] introduced the zero-divisor graph for a partially ordered set, and, in 2012, Estaji et al. [8] extended the concept of the zero-divisor graph to an arbitrary finite bounded lattice. For a clear exposition of the work completed in the area of zero-divisor graphs and their related areas, the reader is referred to the book by Anderson et al. [9]. In 2011, Behboodi et al. [10] defined and investigated the ideal theoretic version of the zero-divisor graph, called the *annihilating-ideal graph of a ring*, and, thereafter, many facts about zero-divisors were expressed in the language of ideals. The concept of an annihilating-ideal graph of a ring was extended to an arbitrary lattice by Afkhami et al. [11] in 2015. The *annihilating-ideal graph of a lattice* $\mathcal{L}$, denoted by $\mathbb{AG}(\mathcal{L})$, is defined to be a simple graph whose vertex set is the set of all non-trivial ideals of $\mathcal{L}$, and whose two distinct vertices I and J are adjacent if and only if $I \wedge J = 0$. The hope when studying the annihilating-ideal graph of a lattice is that the graph theoretic properties of the graph from the lattice will help us to better understand the lattice theoretic properties of the lattice.

One of the most important topological properties of a graph is its genus, which can be orientable or non-orientable (crosscap). The genus of graphs associated with algebraic structures has been studied by many authors (see [12–17]). The planar zero-divisor graph was first explicitly characterized by Smith [18], and the characterization of commutative rings with projective zero-divisor graphs was obtained by Chiang-Hsieh [15]. In 2019, Asir et al. [12] enumerated all commutative rings whose zero-divisor graph has a crosscap two. The planar and crosscap one annihilating-ideal graph of lattices were characterized by Shahsavar [19] and Parsapour et al. [20], respectively. Additionally, whether the line graph associated with the annihilating-ideal graph of a lattice is planar or projective was characterized by Parsapour et al. [21]. Moreover, the authors of [22] characterized all lattices $\mathcal{L}$ whose line graph of $\mathbb{AG}(\mathcal{L})$ is toroidal.

Now, this paper aims to classify lattices with a number of atoms less than or equal to four whose annihilating-ideal graph can be embedded in the non-orientable surfaces of crosscap two. The main results of this paper are Theorems 2, 3, and 5, in which we have obtained our classifications. As a result, this classification provides a large class of r-partite graphs that can be embedded in the Klein bottle. Further, in the proof of the main theorems, we have shown several minimal r-partite graphs that cannot be embedded in the Klein bottle. Possibly, these graphs may be realized as forbidden subgraphs for crosscap two surfaces (refer to Example 1). Further, in order to cover the missing cases in the proof of Theorem 2.6 [20], which affects the statement of the corresponding theorem, the modified version is included as Theorem 4.

2. Preliminaries

In this section, we present the definitions and results needed to prove the main results in the subsequent sections. First, we recall some definitions and notations on lattices. A *lattice* is an algebra $\mathcal{L} = (\mathcal{L}, \wedge, \vee)$, where $\wedge$ and $\vee$ are the binary operations, satisfying the following conditions: for all $a, b, c \in \mathcal{L}$

1. $a \wedge a = a, a \vee a = a$;
2. $a \wedge b = b \wedge a, a \vee b = b \vee a$;
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c); a \vee (b \vee c) = (a \vee b) \vee c$;
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

According to [23] (Theorem 2.1), we can define an order $\leq$ on $\mathcal{L}$ as follows: for any $a, b \in \mathcal{L}$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(\mathcal{L}, \leq)$ is an ordered set in which every pair of elements has the greatest lower bound (glb) and the least upper bound (lub). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, $glb(a, b)$ and $lub(a, b)$ belong to P. For each a and b in P, we define $a \wedge b = glb(a, b)$ and $a \vee b = lub(a, b)$. Then $(P, \wedge, \vee)$ is a lattice. A lattice $\mathcal{L}$ is said to be *bounded* if there are the elements 0 and 1 in $\mathcal{L}$ such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in \mathcal{L}$. Clearly, every finite lattice is bounded. Let $(\mathcal{L}, \wedge, \vee)$ be a lattice with a least element 0 and I be a non-empty subset of $\mathcal{L}$. Then I is said to be the *ideal* of $\mathcal{L}$, denoted by $I \trianglelefteq \mathcal{L}$,

1. For all $a, b \in I, a \vee b \in I$.
2. If $0 \leq a \leq b$ and $b \in I$, then $a \in I$.

In a lattice $(\mathcal{L}, \wedge, \vee)$ with a least element 0, an element a is called an *atom* if $a \neq 0$, and, for an element $x \in \mathcal{L}$, the relation $0 \leq x \leq a$ implies that either $x = 0$ or $x = a$. We denote the set of all atoms of $\mathcal{L}$ by $A(\mathcal{L})$. For basic facts about lattices, we refer the reader to [24].

Next, we recall the following terms regarding graph embedding. For the non-negative integers ℓ and k, let S_ℓ denote the sphere with ℓ handles, and N_k denote a sphere with k crosscaps attached to it. Note that every connected compact surface is homeomorphic to S_ℓ or N_k for some non-negative integers ℓ and k. The *genus* $\gamma(G)$ of a simple graph G is the minimum ℓ such that G can be embedded in S_ℓ. Similarly, *crosscap number (non-orientable genus)* $\tilde{\gamma}(G)$ is the minimum k such that G can be embedded in N_k. Note that the projective space is of crosscap one and the Klein bottle is of crosscap two. If $e = xy \in E(G)$, then the *contraction* of e in G, denoted as $[x, y]$ is the graph obtained from $G - xy$ by identifying vertices x and y to create a new vertex z incident with all edges of G that were incident with either x or y. We say H is a *minor* of G, if H can be obtained from G by deleting vertices, edges, and/or contracting edges. For a graph G, we denote $\tilde{G}$ for the subgraph $G - V'$ where $V' = \{v \in V | \deg(v) = 1\}$, and we call this graph the *reduction* of G. For details on the notion of the embedding of graphs in a surface, we recommend reading [25].

The following three results on the non-orientable embedding of graphs are used frequently in this paper. In what follows, we denote the complete graph with p vertices by K_p, the complete bipartite graph with parts of sizes p and q by $K_{p,q}$, the complete tripartite graph with parts of sizes p, q, and r by $K_{p,q,r}$, and the complete four-partite graph with parts of sizes p, q, r, and s by $K_{p,q,r,s}$.

Proposition 1 ([25,26])**.** *Let p, q, r, and s be positive integers greater than or equal to two. Then*

(a) $\tilde{\gamma}(K_p) = \begin{cases} \left\lceil \frac{(p-3)(p-4)}{6} \right\rceil & if \quad p \geq 3 \\ 3 & if \quad p = 7. \end{cases}$

(b) $\tilde{\gamma}(K_{p,q}) = \left\lceil \frac{(p-2)(q-2)}{2} \right\rceil$.

(c) $\tilde{\gamma}(K_{p,q,r}) = \left\lceil \frac{(p-2)(q+r-2)}{2} \right\rceil$ *except for* $K_{3,3,3}$, $K_{4,4,1}$ *and* $K_{4,4,3}$. *Further,* $\tilde{\gamma}(K_{3,3,3}) = 3$, $\tilde{\gamma}(K_{4,4,1}) = 4$ *and* $\tilde{\gamma}(K_{4,4,3}) = 6$.

(d). *If* $p \geq q + r$, *then* $\tilde{\gamma}(K_{p,q,r,s}) \geq \left\lceil \frac{(p-2)(q+r+s-2)}{2} \right\rceil$.
If $p \leq q + r$, *then* $\tilde{\gamma}(K_{p,q,r,s}) \geq \left\lceil \frac{(p+s-2)(q+r-2)}{2} \right\rceil$.

Proposition 2 (([16] Theorem 1.3) (Euler formula))**.** *Let $\phi : G \to N_k$ be a two-cell embedding of a connected graph G to the non-orientable surface N_k. Then $|V| - |E| + |F| = 2 - k$, where $|V|, |E|$, and $|F|$ are the number of vertices, edges, and faces that $\phi(G)$ has, respectively, and k is the crosscap of N_k.*

The following is an easy observation that will be used in the proof of the main theorem.

Observation 1. *Let G be a simple graph with $|E|$ edges embedded with $|F|$ faces. Then $\frac{2|E|}{|F|} \geq gr(G)$ where $gr(G)$ denotes the length of the shortest cycle in G.*

3. Basic Results and Notations

Before going into the classifications, we need to be familiar with the following notations and observations given by Parsapour and Javaheri in [20].

Notation: ([20]) Let $\mathcal{L}$ be a lattice and $A(\mathcal{L}) = \{a_1, a_2, \ldots, a_n\}$ be the set of all atoms. Let $i_1, i_2, \ldots, i_k$ be integers with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. The notation $U_{i_1 i_2 \ldots i_k}$ stands for the following set:

$$\Big\{ I \trianglelefteq \mathcal{L} : \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \subseteq I \text{ and } a_{i_j} \notin I \text{ for } i_j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\} \Big\}.$$

The next result provides the structure of $\mathbb{AG}(\mathcal{L})$.

Proposition 3. *Let $\mathcal{L}$ be a lattice with n atoms. Then $\mathbb{AG}(\mathcal{L})$ is a $2^n - 2$-partite graph.*

Proof. Let $|A(\mathcal{L})| = n$. For $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_{k'} \leq n$, if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$ respectively, are distinct, then $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Clearly, $V(\mathbb{AG}(\mathcal{L})) = \bigcup\limits_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} U_{i_1 i_2 \ldots i_k}$.

Therefore, for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, the set $U_{i_1 i_2 \ldots i_k}$ forms a partition of $V(\mathbb{AG}(\mathcal{L}))$. Since $0 \neq a_{i_1}$ belongs to every ideal in $U_{i_1 i_2 \ldots i_k}$, no pair of distinct vertices in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\mathbb{AG}(\mathcal{L})$. Note that the number of distinct $U_{i_1 i_2 \ldots i_k}$s is $2^n - 1$. This, together with the fact that every vertex in $U_{12\ldots n}$ is isolated in $\mathbb{AG}(\mathcal{L})$, implies that $\mathbb{AG}(\mathcal{L})$ is a $2^n - 2$-partite graph. □

According to the abovementioned result regarding the structure of $\mathbb{AG}(\mathcal{L})$, in order to identify the crosscap two r-partite graph or to classify the forbidden r-partite graphs of a non-orientable surface of order two for some $3 \leq r \in \mathbb{N}$, one may be interested in finding all crosscap two annihilating-ideal graphs. This is the main objective of this paper.

We shall also need the following notations:

Notations: Before proving our main results, the following points are assumed for convenience in notations and clarity in proofs. Let us take $|A(\mathcal{L})| = n$.

- To avoid repetition, we assume $|U_1| \geq |U_2| \geq \ldots \geq |U_n|$.
- We denote the vertices of the set $U_{i_1 i_2 \ldots i_k}$ by $\{I_{i_1 i_2 \ldots i_k}, I'_{i_1 i_2 \ldots i_k}, I''_{i_1 i_2 \ldots i_k}, \ldots\}$.
- For an integer p, an integer different from p will be denoted by p'.
- For the sake of convenience, we shall denote $U_{(i_1 i_2 \ldots i_k)^c} = U_{j_1 j_2 \ldots j_\ell}$ where $j_1, j_2, \ldots, j_\ell = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$ and the notation $U_{(i_1 i_2 \ldots i_k)^c}$ exists only when $U_{i_1 i_2 \ldots i_k} \neq \emptyset$.
- The edge between the two vertices I and J is denoted by (I, J).
- The notations $|F|$ and f_i denote the number of faces and number of i-gons in an embedding of G in N_k, respectively.
- There may be sets $U_{i_1 i_2 \ldots i_k}$ such that each vertex of $U_{i_1 i_2 \ldots i_k}$ is isolated, ends, or is adjacent to exactly two ends of an edge in $\mathbb{AG}(\mathcal{L})$. In such places, the vertices of $U_{i_1 i_2 \ldots i_k}$ do not affect the crosscap number of $\mathbb{AG}(\mathcal{L})$, which leads to ignoring the set $U_{i_1 i_2 \ldots i_k}$ from the corresponding embedding. This fact is used throughout the article and is sometimes not explicitly pointed out.
- For convenience in any drawing, we provide a particular type of N_2-embedding of $\mathbb{AG}(\mathcal{L})$. This means that instead of drawing graphs for the case U_{ij} with $1 \leq i \leq j \leq 3$, we assume $i = 1$ and $j = 2$ in figures. Additionally, the notation $\cdots$ is used to denote the possibility of embedding any number of vertices.

We show a few simple, but useful, properties of a crosscap on $\mathbb{AG}(\mathcal{L})$. We now state and prove the following lemma, which provides a subgraph and super-graph structure of $\mathbb{AG}(\mathcal{L})$.

Lemma 1. *Let $\mathcal{L}$ be a lattice, $|A(\mathcal{L})| = n$, and $n \geq k \in \mathbb{N}$. Let $\alpha_{i_1 i_2 \ldots i_k} = |U_{i_1 i_2 \ldots i_k}|$, $\lambda = \max\{\alpha_{i_1 i_2 \ldots i_k}\}$ for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Then*

(a). *$K_{\alpha_1, \alpha_2, \ldots, \alpha_n}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$.*
(b). *$K_{(2^n - 2)(\lambda)}$ is a super-graph of $\mathbb{AG}(\mathcal{L})$.*

Proof. Let H be the induced subgraph of $\mathbb{AG}(\mathcal{L})$, induced by the vertex subset $\bigcup_{i=1}^{n} U_i$. It is clear that no two distinct vertices in U_i are adjacent, and every vertex in U_i is adjacent to all of the vertices of U_j for $i \neq j$ in $\mathbb{AG}(\mathcal{L})$. Thus $H = K_{\alpha_1, \alpha_2, \ldots, \alpha_n}$.

The second part follows from the facts that $V(\mathbb{AG}(\mathcal{L})) = \bigcup U_{i_1 i_2 \ldots i_k}$; the number of vertex subsets $U_{i_1 i_2 \ldots i_k}$, except $U_{12\ldots n}$, in $V(\mathbb{AG}(\mathcal{L}))$ is $\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} = 2^n - 2$; and $\lambda = \max\{\alpha_{i_1 i_2 \ldots i_k}\}$. □

We are now in the position to provide a lower bound for the crosscap of $\mathbb{AG}(\mathcal{L})$. Applying Proposition 1c,d in the first part of the above lemma, we obtain the following result.

Theorem 1. *Let $\mathcal{L}$ be a lattice, $|A(\mathcal{L})| = n \geq 3$, and $|U_1| \geq |U_2| \geq \ldots \geq |U_n|$.*

(a). *If $n = 3$, then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq \left\lceil \frac{(|U_1|-2)(|U_2|+|U_3|-2)}{2} \right\rceil$. Moreover, the equality holds whenever $U_{ij} = \emptyset$ for all $1 \leq i \leq j \leq 3$.*

(b). *If $n \geq 4$, then* $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq \begin{cases} \left\lceil \frac{(|U_1|-2)(|U_2|+|U_3|+|U_4|-2)}{2} \right\rceil & if\ |U_1| \geq |U_2| + |U_3| \\ \left\lceil \frac{(|U_1|+|U_4|-2)(|U_2|+|U_3|-2)}{2} \right\rceil & if\ |U_1| < |U_2| + |U_3|. \end{cases}$

We now enter into the core part of the paper. We first observe that $\mathbb{AG}(\mathcal{L})$ is totally disconnected when $|A(\mathcal{L})| = 1$, and $\mathbb{AG}(\mathcal{L})$ contains K_7 as a subgraph when $|A(\mathcal{L})| \geq 7$. Further, according to Proposition 1a, the crosscap of K_7 is three. Thus, one obtains the following result, which provides a bound for the number of atoms in lattice $\mathcal{L}$ with $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Proposition 4. *Let $\mathcal{L}$ be a lattice. If the crosscap of the annihilating-ideal graph $\mathbb{AG}(\mathcal{L})$ is two, then $2 \leq |A(\mathcal{L})| \leq 6$.*

We start the characterization by analyzing the simple case that $|A(\mathcal{L})| = 2$. If $|A(\mathcal{L})| = 2$, then Theorem 2.6 [20] implies that $\mathbb{AG}(\mathcal{L}) \cong K_{|U_1|,|U_2|}$, and so

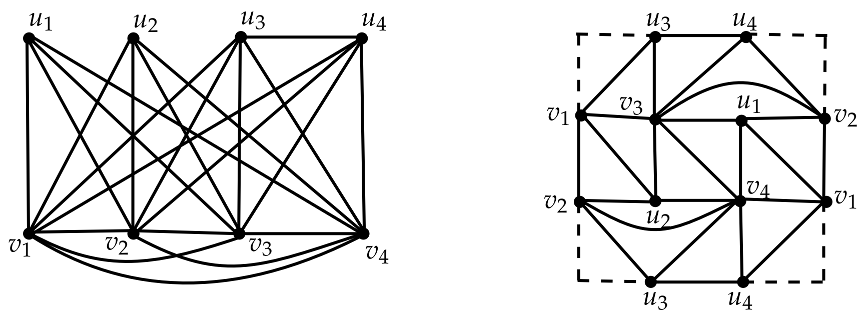
$$\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = \left\lceil \frac{(|U_1| - 2)(|U_2| - 2)}{2} \right\rceil$$

whenever $|U_1|, |U_2| \geq 2$. Now, a simple calculation has yielded the following result, which characterized lattice $\mathcal{L}$ with a crosscap two $\mathbb{AG}(\mathcal{L})$ in the case of $|A(\mathcal{L})| = 2$.

Theorem 2. *Let $\mathcal{L}$ be a lattice and $|A(\mathcal{L})| = 2$. Then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if and only if $|U_1| = |U_2| = 4$ or $|U_i| = 3$ and $|U_j| \in \{5, 6\}$ where $i, j \in \{1, 2\}$ with $i \neq j$.*

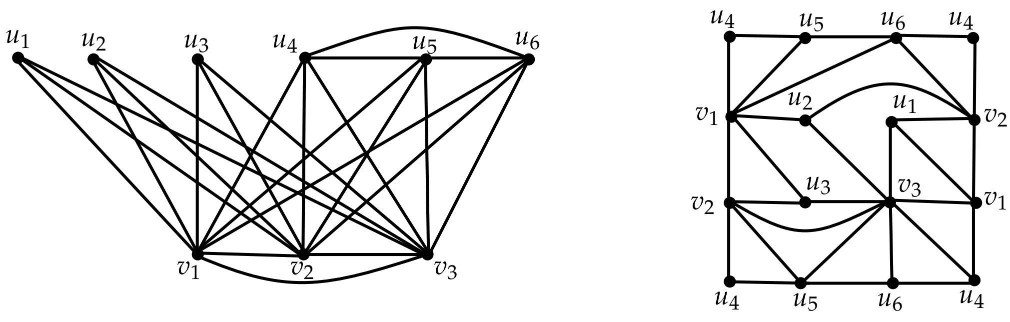
To finish this section we show two results that will be used to prove the main results. The graphs given in Figures 1 and 2 play a vital role in characterizing a lattice with crosscap two annihilating-ideal graphs, and, therefore, we draw the graph with its embedding in the first result.

Lemma 2. *For the graphs H_1 and H_2, as shown in Figures 1 and 2, we have $\tilde{\gamma}(H_1) = \tilde{\gamma}(H_2) = 2$.*



(**a**). The graph H_1 (**b**). An N_2-embedding of H_1

Figure 1. The graph H_1 and its N_2-embedding.

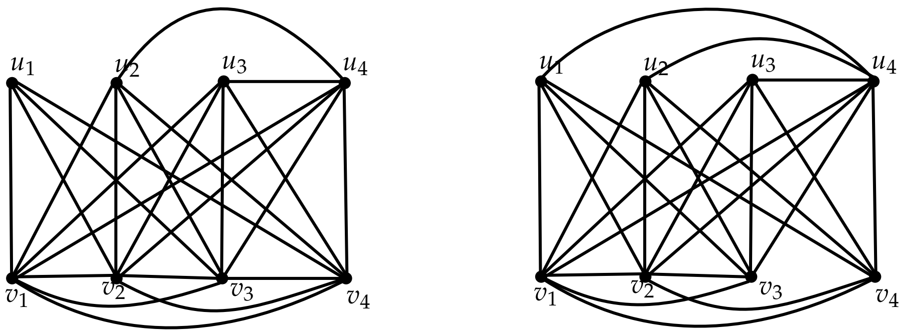


(**a**). The graph H_2 (**b**). An N_2-embedding of H_2

Figure 2. The graph H_2 and its N_2-embedding.

The graphs H_3 and H_4 given in Figure 3 play a vital role in our main theorems.

Lemma 3. *For the graphs H_3 and H_4, as shown in Figure 3, we have $\tilde{\gamma}(H_3) \geq 3$ and $\tilde{\gamma}(H_4) \geq 3$.*



(**a**). Graph H_3 (**b**). Graph H_4

Figure 3. The graphs H_3 and H_4.

Proof. (a). Consider the subgraph $H'_3 = H_3 - \{u_1\}$. Clearly $H'_3 \cong K_7 - e$ where $e = (u_2, u_3)$, and there are 13 faces in any N_2-embedding of H'_3 of which 12 are triangular, and 1 is rectangular. Now, we try to recover an N_2-embedding of H_3 by inserting u_1 with its edges. Since u_1 is adjacent to four vertices of H'_3, u_1 should be inserted into the rectangular face of H'_3. However, all vertices of H'_3 are adjacent to each other, except for u_2 and u_3, so the rectangular face of H'_3 must contain either u_2 or u_3, which is in contradiction to u_2 and u_3 not belonging to the neighborhood set of u_1. Therefore, $\tilde{\gamma}(H_3) \geq 3$.

(b). Apply a similar argument as in (a) for the subgraph $H'_4 = H_4 - \{u_1\} \cong K_7 - 2e$. Here, notice that the largest face in any N_2-embedding of H'_4 is a unique pentagon, and u_1 is adjacent to the five vertices v_1, v_2, v_3, v_4, and u_4. $\square$

4. The Case When $|A(\mathcal{L})| = 3$

Let us start the classification result with a lattice containing exactly three atoms. Note that the following theorem provides a class of multipartite graphs, which are embedded in the Klein bottle (refer to Example 1 for an illustration).

Theorem 3. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 3$, and let $1 \leq i \neq j \neq k \leq 3$. Then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if and only if one of the following conditions hold:*

(i). $|\bigcup_{n=1}^{3} U_n| = 9$*; there is U_i with $|U_i| = 6$ and $U_{jk} = \emptyset$.*

(ii). $|\bigcup_{n=1}^{3} U_n| = 8$*, and one of the following cases is satisfied:*

[a] *There is U_i with $|U_i| = 6$ and $|U_{jk}| = 1$.*

[b] *There exist U_i and U_j such that $|U_i| \in \{5,4\}$ and $|U_j| = 2$ with $U_{jk} = \emptyset$.*
[c] *There exist U_i and U_j such that $|U_i| = 4$ and $|U_j| = 3$ with $U_{ik} = U_{jk} = \emptyset$.*
[d] *There exist U_i and U_j such that $|U_i| = |U_j| = 3$ with $U_{ij} = U_{ik} = U_{jk} = \emptyset$.*

(iii). $|\bigcup_{n=1}^{3} U_n| = 7$, *and one of the following cases is satisfied:*
[a] *There is U_i with $|U_i| \in \{5,4\}$ and $|U_{jk}| = 1$.*
[b] *There exist U_i and U_j such that $|U_i| = |U_j| = 3$ with either $|U_{ik}| \in \{1,2\}$ and $U_{jk} = \emptyset$ or $U_{ik} = \emptyset$ and $|U_{jk}| \in \{1,2\}$.*
[c] *There exist U_i and U_j such that $|U_i| = 3$, $|U_j| = 2$ with $|U_{jk}| \in \{1,2\}$. Further, if $|U_{jk}| = 1$, then either $U_{ij} = \emptyset$ or $U_{ik} = \emptyset$ and, if $|U_{jk}| = 2$, then $U_{ij} = U_{ik} = \emptyset$.*

(iv). $|\bigcup_{n=1}^{3} U_n| = 6$, *and one of the following cases is satisfied:*
[a] *There is U_i with $|U_i| = 4$ and $|U_{jk}| = 2$.*
[b] *There is U_i with $|U_i| = 3$ and $|U_{jk}| \in \{2,3\}$.*

(v). $|\bigcup_{n=1}^{3} U_n| = 5$; *there is U_i with $|U_i| = 3$ and $|U_{jk}| \in \{3,4\}$.*

Proof. Assume that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. First of all, if $|\bigcup_{n=1}^{3} U_n| \leq 4$, then $\mathbb{AG}(\mathcal{L})$ is planar (see [19]). Suppose $|\bigcup_{n=1}^{3} U_n| \geq 10$. If $|U_2| \geq 2$, then by Theorem 1 we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq \left\lceil \frac{(|U_1|-2)(|U_2|+|U_3|-2)}{2} \right\rceil \geq 3$, which is a contradiction. Suppose $|U_2| = 1$. Then $|U_3| = 1$. Note that every vertex in U_{12}, U_{13}, and U_{23} is adjacent to all of the vertices of U_3, U_2, and U_1, respectively. So, if $U_{23} = \emptyset$, then clearly $\mathbb{AG}(\mathcal{L})$ is planar. If not, the vertices in U_1 are adjacent to all of the vertices of $U_2 \cup U_3 \cup U_{23}$. Since $|U_1| \geq 8$, $K_{8,3}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$ that has a crosscap of more than three, refer to Proposition 1a. Thus, $5 \leq |\bigcup_{n=1}^{3} U_n| \leq 9$.

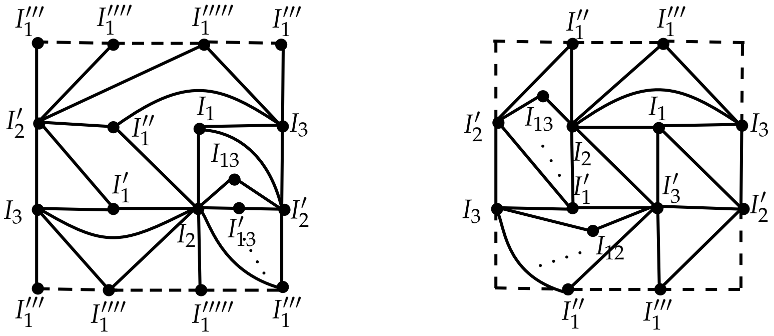
Case 1 Let $|\bigcup_{n=1}^{3} U_n| = 9$. Then, clearly, $|U_1| \leq 7$. If $|U_1| = 7$, then a slight modification to the discussion made in the above paragraph would show that $\mathbb{AG}(\mathcal{L})$ is planar whenever $U_{23} = \emptyset$ and the graph $\mathbb{AG}(\mathcal{L})$ contains $K_{7,3}$ as a subgraph when $U_{23} \neq \emptyset$. If $|U_1| = 6$, then $|U_2| = 2$ and $|U_3| = 1$. Now, if $U_{23} \neq \emptyset$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{6,4}$ as a subgraph, which is a contradiction. So, $U_{23} = \emptyset$. Here, all of the vertices in U_{12} are adjacent to a single vertex of U_3, and, therefore, the vertices in U_{12} do not affect the crosscap. In Figure 4a, we provide the canonical representation of the embedding of the resulting graph in N_2 so that, in this case, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Next, if $|U_1| = 5$ or 4, then $|U_2| + |U_3| \geq 4$, and so, by Theorem 1a, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Thus, $|U_1| = 3$, and, therefore, $|U_2| = |U_3| = 3$. Here, $K_{3,3,3}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$, and, therefore, according to Proposition 1c, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Case 2 Let $|\bigcup_{n=1}^{3} U_n| = 8$.

If $|U_1| = 6$, then $|U_2| = |U_3| = 1$. Clearly, by [19], $\mathbb{AG}(\mathcal{L})$ is planar in the case that U_{23} is empty. If $|U_{23}| \geq 2$, then the partite sets $X = U_1$ and $Y = U_2 \cup U_3 \cup U_{23}$ form $K_{6,4}$ as a subgraph in $\mathbb{AG}(\mathcal{L})$, which is a contradiction. Therefore, $|U_{23}| = 1$. In this case, the vertices in $U_{13} \cup U_{12}$ are all end vertices, and, therefore, it does not affect the crosscap. Thus, the resulting graph is $K_{6,3} \cup \{(I_2, I_3)\}$, which is a subgraph of a graph given in Figure 2a, and, therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Suppose $|U_1| \in \{5,4\}$. Then, according to Theorem 1a, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. If $U_{23} \neq \emptyset$, then the sets $X = U_1$ and $Y = U_2 \cup U_3 \cup U_{23}$ form $K_{5,4}$ as a subgraph of $\mathbb{AG}(\mathcal{L})$, and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $U_{23} = \emptyset$. Let $|U_2| = 2, |U_{12}| \geq 0$, and $|U_{13}| \geq 0$. For the embedding of $\mathbb{AG}(\mathcal{L})$ in N_2, in the case of $|U_1| = 5$, we can obtain help from Figure 4a because the number of vertices and edges of $\mathbb{AG}(\mathcal{L})$ is less than that of in Figure 4a. Further, Figure 4b provides an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ in the case of $|U_1| = 4$. Here, notice that the open neighborhood of each vertex in U_{13} is $\{I_2, I_2'\}$, and, in Figure 4a,b, there is a face in an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ that contains both I_2 and I_2' so that every vertex of U_{13} can be embedded in N_2 no matter what its cardinality may be. Let $|U_2| = 3$. This implies that $|U_1| = 4$. If $U_{13} = \emptyset$ (recall that $U_{23} = \emptyset$), then $\mathbb{AG}(\mathcal{L})$ is a subgraph of the graph H_1 in Figure 1, and, therefore, according to Lemma 2, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. If not, consider that the subgraph $\mathbb{AG}(\mathcal{L}) - \{(I_3, I_1), (I_3, I_1'), (I_3, I_1''), (I_3, I_1''')\}$ contains $K_{3,6}$. By Euler's formula, any embedding of $K_{3,6}$ in N_2 has nine faces. Further, by solving the equations

$2|E| = 4f_4 + 6f_6$ and $|F| = f_4 + f_6$, we have all the faces as rectangular faces in any N_2-embedding of $K_{3,6}$. Now we try to recover the embedding of $\mathbb{AG}(\mathcal{L})$ by inserting all edges (I_3, I_1), (I_3, I_1'), (I_3, I_1''), (I_3, I_1''') into the embedding of $K_{3,6}$. Since $deg_{K_{3,6}}(I_3) = 3$, the vertex I_3 is in the boundary of three rectangular faces of any N_2-embedding of $K_{3,6}$. In addition, note that, at the maximum, each rectangular face can adopt one edge incident with I_3. So, we cannot insert all four edges of I_3 into N_2 without crossing, which is a contradiction. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.



(**a**). $|\bigcup_{n=1}^{3} U_n| = 9, |U_1| = 6$ and $U_{23} = \emptyset$ (**b**). $|\bigcup_{n=1}^{3} U_n| = 8, |U_1| = 4, |U_2| = 2$ and $U_{23} = \emptyset$

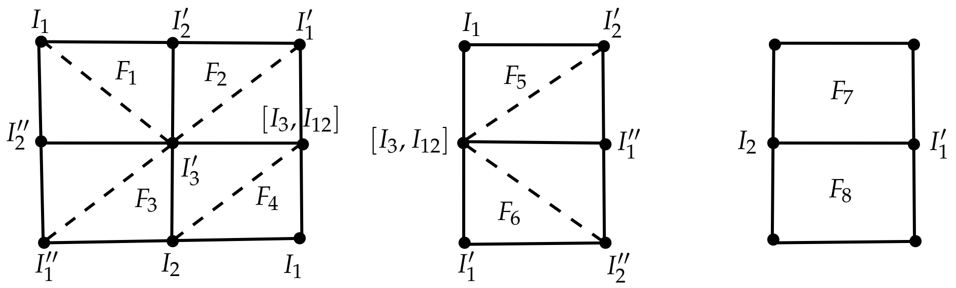
Figure 4. N_2-embedding of $\mathbb{AG}(\mathcal{L})$.

Suppose $|U_1| = 3$. If $U_{ij} = \emptyset$ for all $1 \leq i < j \leq 3$, then, by Proposition 1c, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Next, our claim is that $U_{ij} = \emptyset$ for all $1 \leq i < j \leq 3$.

Assume that $U_{12} \neq \emptyset$. Then the minor subgraph is

$$\mathbb{AG}(\mathcal{L}) - \{(I_1, I_3'), (I_1', I_3'), (I_1'', I_3'), (I_2, [I_3, I_{12}]), (I_2', [I_3, I_{12}]), (I_2'', [I_3, I_{12}])\} \cong K_{4,4}$$

with the partite sets $X = U_2 \cup \{[I_3, I_{12}]\}$ and $Y = U_1 \cup \{I_3'\}$. By Euler's formula, any N_2-embedding of $K_{4,4}$ has eight rectangular faces. Next, we attempt to obtain an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from any N_2-embedding of $K_{4,4}$. For this, we try to embed the six omitted edges of $\mathbb{AG}(\mathcal{L})$ into an arbitrary N_2-embedding of $K_{4,4}$. First, to embed the three edges $(I_1, I_3'), (I_1', I_3')$, and (I_1'', I_3'), three rectangular faces are required, denoted as F_1, F_2, and F_3, all of which contains I_3' (refer to Figure 5a). Since $deg_{K_{4,4}}(I_3') = 4$, exactly one more face should have I_3'; it is denoted as F_4. Intentionally, we label the diagonals of F_4 as the vertices I_2 and $[I_3, I_{12}]$ because F_4 can adopt one diagonal edge that can be used to embed the fourth edge $(I_2, [I_3, I_{12}])$. Finally, to embed the rest of the two edges $(I_2', [I_3, I_{12}])$ and $(I_2'', [I_3, I_{12}])$, two distinct faces are required, denoted by F_5 and F_6, which should have the vertex $[I_3, I_{12}]$. Note that, in any N_k-embedding, every edge of a graph is in exactly two faces. Since the edge $(I_1, [I_3, I_{12}])$ is in F_2 and the edge $(I_1', [I_3, I_{12}])$ is in F_4, the common edge between F_5 and F_6 must be $(I_1'', [I_3, I_{12}])$. Now, the choice for the unlabelled vertex of F_5 and F_6 is either I_1 or I_1'. Without a loss of generality, we label I_1 for F_5 and I_1' for F_6 (refer to Figure 5b). Since any N_2-embedding of $K_{4,4}$ has eight faces, there are two more faces, lets say F_7 and F_8, that have to be formed using all of the remaining vertices and edges of $K_{4,4}$. Notice that, in any N_2-embedding of $K_{4,4}$, each vertex is present in exactly four faces, and each edge is present in exactly two faces. Since the vertices $I_2 \in X$ and $I_1' \in Y$ are used twice in the faces $F_1, \ldots, F_6$, the faces F_7 and F_8 must share the edge (I_2, I_1') (refer to Figure 5c). Now, the choices for the third and fourth vertices of F_7 and F_8 are $I_2', I_2'' \in X$ and $I_1, I_1'' \in Y$, respectively. Clearly, we have to select distinct vertices for F_7 and F_8, in which one is from $\{I_2', I_2''\}$ and the other is from $\{I_1, I_1''\}$. A contradiction to this fact is that the edges (I_2', I_1) and (I_2'', I_1'') are used twice in the faces $F_1, \ldots, F_6$.



(**a**) (**b**) (**c**)

Figure 5. Representation of faces of N_2-embedding of $K_{4,4}$.

Assume that $U_{i3} \neq \emptyset$ for some $i \in \{1,2\}$. Then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{i3}, (I_i, I_3), (I'_i, I_3), (I''_i, I_3)\}$ contains $K_{4,4} - e$ with the partite sets $X = U_i \cup \{I_3\}$ and $Y = U_{i'} \cup \{I'_3\}$ where $i' \in \{1,2\} \setminus \{i\}$ and $e = (I_3, I'_3)$. By Proposition 2, any N_2-embedding of $K_{4,4} - e$ has one hexagonal and six rectangular faces. Note that the hexagonal face should have either I_3 or I'_3, and the vertex I_{i3} is adjacent to $\{I_{i'}, I'_{i'}, I''_{i'}\} \subset Y$. So, I_{i3} with its edges must be inserted into the hexagonal face, which implies that I_3 is in the hexagonal face. Since $deg_{K_{4,4}-e}(I_3)$
$= 3$, exactly two rectangular faces contain I_3 in which it is not possible to embed all of the three edges (I_i, I_3), (I'_i, I_3), and (I''_i, I_3), which is a contradiction. Thus, $U_{ij} = \emptyset$ for all $i, j \in \{1,2,3\}$.

Case 3 Let $|\bigcup_{n=1}^{3} U_n| = 7$.

Suppose $|U_1| \in \{5,4\}$. Clearly, $\mathbb{AG}(\mathcal{L})$ is either planar or projective when $U_{23} = \emptyset$ (refer to [19,20]), and $K_{5,4}$ is a subgraph of the contraction of $\mathbb{AG}(\mathcal{L})$ when $|U_{23}| \geq 2$. Therefore, $|U_{23}|$ will be one. Then, $\mathbb{AG}(\mathcal{L})$ is a subgraph of the graph given in Figure 4a when $|U_1| = 5$, and $\mathbb{AG}(\mathcal{L})$ is a subgraph of the graph given in Figure 4b when $|U_1| = 4$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Assume that $|U_1| = |U_2| = 3$. Then, $\mathbb{AG}(\mathcal{L})$ is projective when $U_{i3} = \emptyset$ for all $i = 1,2$, and the graph $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ as a subgraph when $|U_{i3}| \geq 3$ for some $i = 1,2$. Suppose $U_{13} \neq \emptyset$ and $U_{23} \neq \emptyset$. Now, the graph $\mathbb{AG}(\mathcal{L}) - \{I_3\}$ is isomorphic to $K_{4,4} - \{e\}$ with the bipartite sets $\{I_1, I'_1, I''_1, I_{13}\}$ and $\{I_2, I'_2, I''_2, I_{23}\}$ where $e = (I_{13}, I_{23})$. Note that $\tilde{\gamma}(K_{4,4} - \{e\}) = 2$, and there are seven faces in any N_2-embedding of $K_{4,4} - \{e\}$, of which six are rectangular, and one is hexagonal. Since $\tilde{\gamma}(K_{4,4}) = 2$ and every face in any N_2-embedding of $K_{4,4}$ is rectangular, the hexagonal face of any N_2-embedding of $K_{4,4} - \{e\}$ must have the vertices I_{13} and I_{23}. Now, we try to recover an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from an N_2-embedding of $K_{4,4} - \{e\}$ by inserting I_3 with its edges. Here, I_3 is adjacent to the six vertices $I_1, I'_1, I''_1, I_2, I'_2$, and I''_2. However, the hexagonal face of $K_{4,4} - \{e\}$ does not contain two of them so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, either $U_{13} = \emptyset$ or $U_{23} = \emptyset$. Now, with the help of Figure 6, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ when $1 \leq |U_{i3}| \leq 2$ for a unique $i \in \{1,2\}$.

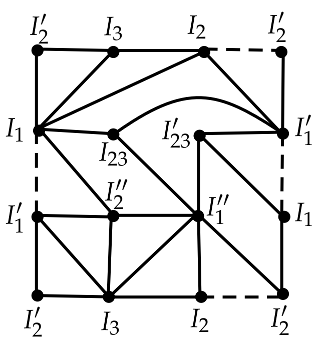


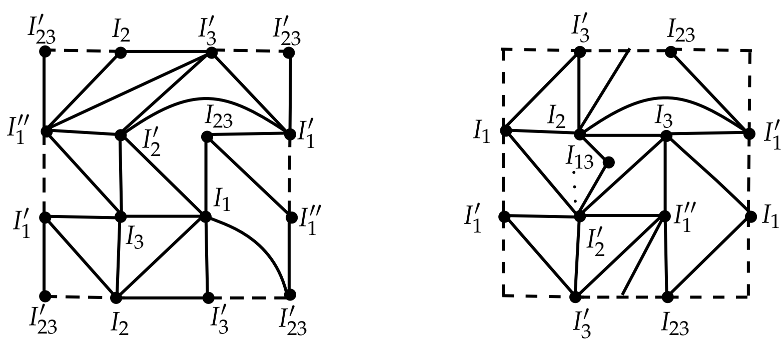
Figure 6. $|\bigcup_{n=1}^{3} U_n| = 7$ with $|U_1| = |U_2| = 3, U_{13} = \emptyset$ and $|U_{23}| = 2$.

Assume that $|U_1| = 3$ and $|U_2| = 2$. If $|U_{23}| \geq 3$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ as a subgraph, and, if $U_{23} = \emptyset$, then, by Theorem 2.4iii [20], $\mathbb{AG}(\mathcal{L})$ is projective. Suppose $|U_{23}| = 2$. If $U_{1j} \neq \emptyset$ for $j = 2$ or 3, then consider a subgraph $G_1 = \mathbb{AG}(\mathcal{L}) - \{I_{1j}, I'_{23}, e_1, e_2, e_3, e_4\}$ where $e_1 = (I_2, I_3), e_2 = (I_2, I'_3), e_3 = (I'_2, I_3)$, and $e_4 = (I'_2, I'_3)$. Clearly, G_1 contains $K_{3,5}$ with the partite sets $X = \{I_1, I'_1, I''_1\}$ and $Y = \{I_2, I'_2, I_3, I'_3, I_{23}\}$. Note that any N_2-embedding of $K_{3,5}$ has one hexagonal and six rectangular faces. Now, we try to recover an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from any N_2-embedding of $K_{3,5}$. Since I'_{23} is adjacent to all three vertices of X, the embedding of I'_{23} requires the hexagonal face of $K_{3,5}$ to have I_1, I'_1, and I''_1. Notice that each rectangular face may adopt at most one edge into it. So, to insert e_fs, for $1 \leq f \leq 4$, into any N_2-embedding of $K_{3,5}$, four rectangular faces with diagonals as the end vertices of each e_f are required. At last, to insert I_{1j}, a rectangular face with the diagonals $I_{j'}$ and $I'_{j'}$ for $j' \in \{2, 3\} \setminus \{j\}$ is required. Therefore, it requires one hexagonal face with five rectangular faces containing the vertices I_2, I'_2, I_3, and I'_3 in at least three different faces. Since the degree of I_2, I'_2, I_3, and I'_3 in $K_{3,5}$ is three, all four vertices are placed in exactly three faces of any N_2-embedding of $K_{3,5}$. So, the sixth rectangular face of $K_{3,5}$ could not be formed using the only left-out vertex in X (namely I_{23}), which is a contradiction. Thus, $U_{12} = U_{13} = \emptyset$, and an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ for this case is provided in Figure 7a.

Suppose $|U_{23}| = 1$. If $U_{1j} \neq \emptyset$ for $j = 2$ and 3, then the minor subgraph is

$$G_2 = \mathbb{AG}(\mathcal{L}) - \{I_{13}, e_1, e_2, e_3, e_4, e_5\} \cong K_{4,4} - \{e\}, \tag{1}$$

with the bipartite sets $\{I_1, I'_1, I''_1, I_3\}$ and $\{I_2, I'_2, [I'_3, I_{12}], I_{23}\}$ where $e_1 = (I_1, I_3), e_2 = (I'_1, I_3)$, $e_3 = (I''_1, I_3), e_4 = (I_2, [I'_3, I_{12}])$, $e_5 = (I'_2, [I'_3, I_{12}])$, and $e = (I_3, I_{23})$. Note that any N_2-embedding of $K_{4,4} - \{e\}$ has six rectangular faces and a hexagonal face, and the hexagonal face must have the vertices I_3 and I_{23}. Let us denote the six rectangular faces by $F_1, \ldots, F_6$ and the hexagonal face by F_7. Now, let us try to recover an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ by inserting the vertex I_{13} and the edges e_i for all $i = 1, \ldots, 5$. If we embed the edge e_4, the edge e_5, or the vertex I_{13} together with its edges into F_7, then we cannot insert the edges e_1, e_2, or e_3 into F_7. Since $deg_{G_2}(I_3) = 3$, the vertex I_3 is in exactly three faces of an N_2-embedding of G_2. So, in such cases, the edges e_1, e_2 and e_3 cannot be embedded in two rectangular faces which contains I_3. Therefore we have to add at least one of the edges e_1, e_2 or e_3 into F_7. For the best possibility, say e_1 and e_2 are embedded in F_7. Then, e_3 has to be embedded into one of the two rectangular faces that contains I_3, for example, F_1. Notice that there are two rectangular faces, say F_2 and F_3, that contain I_{23}, in which one should not embed any of e_4, e_5, or I_{13} with its edges. So, the edges e_4 and e_5 have to be embedded into different rectangular faces, say F_4 and F_5, respectively. Therefore, after embedding the edges from e_1 to e_5 nicely, we are left with the single rectangular face F_6 that could not be formed using the diagonal vertices I_2 and I'_2. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Hence, either $U_{12} = \emptyset$ or $U_{13} = \emptyset$. In this case, with the help of Figure 7b, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



(**a**). $U_{12} = U_{13} = \emptyset$ and $|U_{23}| = 2$ (**b**). $U_{12} = \emptyset$, $|U_{13}| \geq 0$ and $|U_{23}| = 1$

Figure 7. $|\bigcup_{n=1}^{3} U_n| = 7$ with $|U_1| = 3$ and $|U_2| = 2$.

Case 4 Let $|\bigcup_{n=1}^{3} U_n| = 6$. Suppose $|U_1| = 4$. If $|U_{23}| \geq 3$, then $K_{4,5}$ is contained in $\mathbb{AG}(\mathcal{L})$, and if $|U_{23}| = 1$, then $\mathbb{AG}(\mathcal{L})$ is projective. Therefore $|U_{23}| = 2$. Clearly, $\mathbb{AG}(\mathcal{L})$(except for the end vertices) is a subgraph of the graph H_1 given in Figure 1a, and so Lemma 2 implies $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

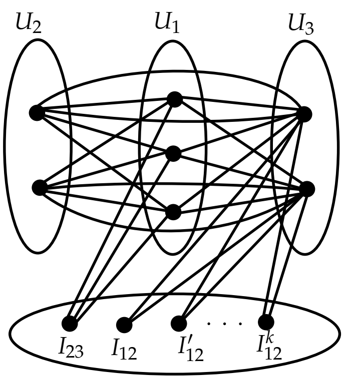
Suppose $|U_1| = 3$. Then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ when $|U_{23}| \geq 4$, and $\mathbb{AG}(\mathcal{L})$ is projective when $|U_{23}| \leq 1$. Thus, $2 \leq |U_{23}| \leq 3$. Then, $\mathbb{AG}(\mathcal{L}) - \{U_{13}\}$ is a subgraph of the graph H_2 (see Figure 2a), so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L}) - \{U_{13}\}) = 2$. Note that every vertex in U_{13} is adjacent to exactly two vertices of U_2 in $\mathbb{AG}(\mathcal{L})$. Therefore, replace the labels u_4 and u_5 with I_2 and I_2', respectively, in the N_2-embedding of H_2 provided in Figure 2b, and then label all of the other vertices accordingly. Now, we can insert any number of vertices of U_{13} into a face that contains both I_2 and I_2' so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Moreover, if $|U_1| = 2$, then $\mathbb{AG}(\mathcal{L})$ is either planar or projective (refer to [19,20]).

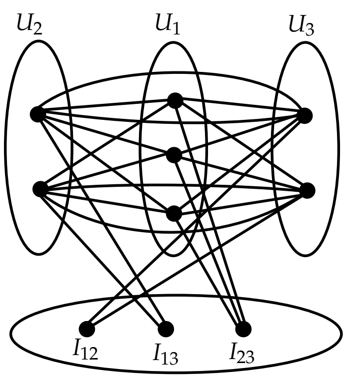
Case 5 Let $|\bigcup_{n=1}^{3} U_n| = 5$. Then $\mathbb{AG}(\mathcal{L})$ is planar or projective when $|U_1| = 2$. This implies that $|U_1| = 3$. If $|U_{23}| \geq 5$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$, and, if $|U_{23}| \leq 2$, then $\mathbb{AG}(\mathcal{L})$ is projective. Thus, $|U_{23}| = 3$ or 4. Then, clearly, $\mathbb{AG}(\mathcal{L})$ is a subgraph of the graph H_1, as in Figure 2a, so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. $\square$

All of the results proved in this paper have a similar structure to that of those given in the statement of Theorem 3. To familiarize readers with the connection between the multipartite graph and the statement of Theorem 3, we illustrate two four-partite graphs, G and H, with $\tilde{\gamma}(G) = 2$ and $\tilde{\gamma}(H) \neq 2$, respectively, in the following example.

Example 1. Consider Case (iii)[c] in Theorem 3. Let $|U_1| = 3, |U_2| = 2, |U_3| = 2$, and $|U_{23}| = 1$. If $|U_{12}| = k \in \mathbb{Z}^+$ and $U_{13} = \emptyset$, then the corresponding four-partite graph G is a crosscap two, which is given in Figure 8a. Additionally, if $|U_{12}| = 1$ and $|U_{13}| = 1$, then the crosscap of the corresponding four-partite graph H, given in Figure 8b, is not equal to two. It is worth mentioning that the four-partite graph H in Figure 8b is minimal with respect to $\tilde{\gamma}(H) \neq 2$; that is, there exists an edge e in H such that $\tilde{\gamma}(H - e) = 2$. Further, the graph H may be realized as one of the forbidden subgraphs for a crosscap two surface.



(**a**) A crosscap two 4-partite graph G



(**b**) A minimal 4-partite graph H with crosscap $\neq 2$

Figure 8. Four-partite graphs.

By using the proof of Theorem 3, we establish the following points, which will be used in the subsequent results.

Remark 1. If a graph G is isomorphic to $K_{6,3} \cup (K_4 - e)$ or $K_{4,5} - e$ where e is an edge, then $\tilde{\gamma}(G) \geq 3$.

5. The Case When $|A(\mathcal{L})| = 4$

Next, we fix the number of atoms as four. As mentioned in the introduction, for $1 \leq i \neq j \leq 4$, we denote $U_{(ij)^c} = U_{k\ell}$ where $k, \ell \in \{1,2,3,4\} \setminus \{i,j\}$, and the notation

$U_{(ij)^c}$ exists only when $U_{ij} \neq \emptyset$. Before going into the characterization of the crosscap two $\mathbb{AG}(\mathcal{L})$ with $|A(\mathcal{L})| = 4$, we provide modifications for Theorem 2.6 [20]. To be precise, the missing cases and the corresponding conditions for the projectiveness of $\mathbb{AG}(\mathcal{L})$ are given below.

(i) First of all, consider the missing case $|\bigcup_{n=1}^{4} U_n| = 4$. Then, $|U_i| = 1$ for all $1 \leq i \leq 4$. Clearly, $\mathbb{AG}(\mathcal{L})$ is planar whenever $\bigcup\limits_{U_{ij} \neq \emptyset} U_{(ij)^c} = \emptyset$. Therefore, $\bigcup\limits_{U_{ij} \neq \emptyset} U_{(ij)^c} \neq \emptyset$. If $|U_{ij} \cup U_{(ij)^c}| \geq 4$ with $U_{ij}, U_{(ij)^c} \neq \emptyset$, then the subgraph induced by the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup\limits_{k \neq i,j} U_k \cup U_{(ij)^c}$ contains $K_{4,4}$ or $K_{3,5}$ as a subgraph. This implies $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. Therefore, $2 \leq |U_{ij} \cup U_{(ij)^c}| \leq 3$ if $U_{ij}, U_{(ij)^c} \neq \emptyset$ for $1 \leq i \neq j \leq 4$.

Suppose $|U_{ij} \cup U_{(ij)^c}| = 3$ for some $U_{ij}, U_{(ij)^c} \neq \emptyset$ with $1 \leq i \neq j \leq 4$. If $U_{k\ell}, U_{(k\ell)^c} \neq \emptyset$ for $k\ell \neq ij$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{U_{ij} \cup U_{(ij)^c}\}$ contains $K_{3,3}$ with the partite sets $X = U_k \cup U_\ell \cup U_{k\ell}$ and $Y = \bigcup\limits_{m \neq k,\ell} U_m \cup U_{(k\ell)^c}$. Note that $\tilde{\gamma}(K_{3,3}) = 1$. Now, we try to embed all of the vertices of $U_{ij} \cup U_{(ij)^c}$ with their edges in any N_1-embedding of $K_{3,3}$. Since $|U_{ij} \cup U_{(ij)^c}| = 3$, either $|U_{ij}| = 2$ or $|U_{(ij)^c}| = 2$. Without a loss of generality, let $|U_{ij}| = 2$. Since the vertex $I_{(ij)^c} \in U_{(ij)^c}$ is adjacent to $I_{ij}, I'_{ij} \in U_{ij}$, all of the three vertices I_{ij}, I'_{ij}, and $I_{(ij)^c}$ must be embedded into a single face of the N_1-embedding of $K_{3,3}$, denoted as F_1. Now, draw the path $I_{ij} - I_{(ij)^c} - I'_{ij}$ into F_1 and then draw the edges $(I_{ij}, I_m), (I_{ij}, I_n), (I'_{ij}, I_m)$, and (I'_{ij}, I_n) where $m, n \notin \{i, j\}$. Now, the edges $(I_{(ij)^c}, I_i)$ and $(I_{(ij)^c}, I_j)$ cannot be embedded into F_1. Therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. Thus, $\bigcup\limits_{k\ell \neq ij, (ij)^c; U_{k\ell} \neq \emptyset} U_{(k\ell)^c} = \emptyset$.

Suppose $|U_{ij} \cup U_{(ij)^c}| = 2$ for all $U_{ij}, U_{(ij)^c} \neq \emptyset$ with $1 \leq i \neq j \leq 4$. Then, Figure 9 guarantees that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 1$.

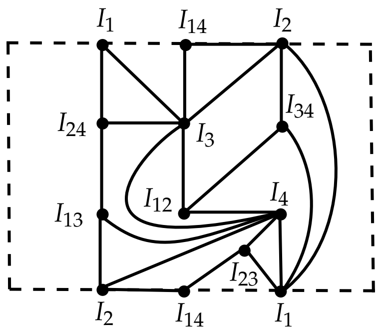


Figure 9. $|\bigcup_{n=1}^{4} U_n| = 4$ with $|U_{ij} \cup U_{(ij)^c}| \leq 2$ for all $U_{ij}, U_{(ij)^c} \neq \emptyset$.

(ii) Let $|\bigcup_{n=1}^{4} U_n| = 5$. Then, $|U_i| = 2$ for some $1 \leq i \leq 4$, and the condition for the projectiveness of $\mathbb{AG}(\mathcal{L})$ given in Theorem 2.6i [20] is that $|U_{jk}| = 1$ or 2, in which at most one of the U_{jk}s has exactly two elements for $1 \leq i \neq j \neq k \leq 4$. However, if $|U_{jk}| = 2$ with $U_{(jk)^c} \neq \emptyset$, then the sets $X = U_i \cup U_\ell \cup U_{(jk)^c}$ and $Y = U_j \cup U_k \cup U_{jk}$, where $\ell \notin \{i, j, k\}$, contain $K_{4,4}$ in $\mathbb{AG}(\mathcal{L})$ so that we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. In fact, if $|U_{jk}| = 2$ for some $j, k \neq i$, then $\bigcup\limits_{p,q \neq i; U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$. Otherwise, the sets $X = U_j \cup U_k \cup U_{jk} \cup [I_{pq}, I_{(pq)^c}]$ and $Y = U_1 \cup U_\ell$, where $\ell \notin \{i, j, k\}$, form $K_{5,3}$, so we can conclude that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. Further, if $|U_{jk}| \leq 1$ for all $j, k \neq i$, then $|\bigcup\limits_{p,q \neq i; U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$. For if $|U_{(pq)^c}| \geq 2$, then the sets $X = U_p \cup U_q \cup U_{pq}$ and $Y = U_i \cup U_r \cup U_{(pq)^c}$, where $r \notin \{i, p, q\}$, form $K_{3,5}$, and, if $|U_{(pq)^c}|, |U_{(p_1q_1)^c}| = 1$ for some $1 \leq p_1 \neq q_1 \leq 4$ with $p_1q_1 \neq pq$, then the sets $X = U_p \cup U_q \cup U_{pq} \cup \{[I_{p_1q_1}, I_{(p_1q_1)^c}]\}$ and $Y = U_i \cup U_r \cup U_{(pq)^c}$ form $K_{4,4} - \{e\}$ in $\mathbb{AG}(\mathcal{L})$ where $r \notin \{i, p, q\}$.

(iii) Let $|\bigcup_{n=1}^{4} U_n| = 6$. If there exists $|U_i| = 3$ for some $1 \leq i \leq 4$, then the statement of ([20] Theorem 2.6(ii)(a)) says that if $U_{jk\ell} = \emptyset$ for $1 \leq i \neq j \neq k \neq \ell \leq 4$, $|U_{jk}| \leq 1$, and at most one of the U_{jk}s has exactly one element, then $\mathbb{AG}(\mathcal{L})$ is projective. However, for instance, if $|U_{jk}| = 1$ with $U_{(jk)^c} = U_{i\ell} \neq \emptyset$, then the partite sets $X = U_i \cup U_\ell \cup U_{i\ell}$ and $Y = U_j \cup U_k \cup U_{jk}$ contain $K_{5,3}$ as a subgraph of $\mathbb{AG}(\mathcal{L})$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. Therefore, the condition $U_{(jk)^c} = \emptyset$ has to be added to the statement of ([20] Theorem 2.6iia).

As a result of the above remarks (i), (ii), and (iii), we modify the statement of ([20] Theorem 2.6) as follows.

Theorem 4. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 4$. Let $1 \leq i \neq j \neq k \neq \ell \leq 4$ and $1 \leq p \neq q \leq 4$. Then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 1$ if and only if one of the following conditions hold:*

(i). *$|\bigcup_{n=1}^4 U_n| = 4$; there exist two non-empty sets U_{ij} and $U_{(ij)^c}$ such that $2 \leq |U_{ij} \cup U_{(ij)^c}| \leq 3$. Moreover, if $|U_{ij} \cup U_{(ij)^c}| = 3$, then $\bigcup_{pq \neq ij,(ij)^c; U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$.*

(ii). *$|\bigcup_{n=1}^4 U_n| = 5$; there is U_i with $|U_i| = 2$, $|\bigcup_{p,q \neq i} U_{pq}| \leq 4$ in which at most one of the U_{pq}s has a maximum of two elements, and $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$. Moreover, if $|U_{pq}| = 2$, then $\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$, and, if $\bigcup_{p,q \neq i} U_{pq} = \emptyset$, then $U_{jk\ell} \neq \emptyset$.*

(iii). *$|\bigcup_{n=1}^4 U_n| = 6$, and one of the following is satisfied:*

[a] *There is U_i with $|U_i| = 3$. If $|U_{jk\ell}| = 1$, then $U_{jk} = U_{j\ell} = U_{k\ell} = \emptyset$ and if $U_{jk\ell} = \emptyset$, then $|U_{jk} \cup U_{j\ell} \cup U_{k\ell}| \leq 1$. Moreover, $U_{(pq)^c} = \emptyset$ whenever $U_{pq} \neq \emptyset$.*

[b] *There exist U_i and U_j such that $|U_i| = |U_j| = 2$ with $|U_{k\ell}| \leq 1$. Additionally, $U_{(pq)^c} = \emptyset$ whenever $U_{pq} \neq \emptyset$. Moreover, if $|U_{ik}|, |U_{i\ell}| \leq 1$ or $|U_{jk}|, |U_{j\ell}| \leq 1$, then $|U_{k\ell}| \leq 1$. Furthermore, if $|U_{ik}| = |U_{jk}| = 1$ or $|U_{i\ell}| = |U_{j\ell}| = 1$, then $U_{k\ell} = \emptyset$.*

(iv). *$|\bigcup_{n=1}^4 U_n| = 7$ and one of the following is satisfied:*

[a] *There is U_i with $|U_i| = 4$ and $U_{jk\ell} = U_{jk} = \emptyset$.*

[b] *There exist U_i and U_j such that $|U_i| = 3$ and $|U_j| = 2$. Additionally, $U_{k\ell} = \emptyset$, and $U_{jk\ell} = \emptyset$ whenever $U_{ik} = U_{i\ell} = U_{jk} = U_{j\ell} = \emptyset$.*

We are now in the position to state and prove the second result which classifies all lattices $\mathcal{L}$ with four atoms whose $\mathbb{AG}(\mathcal{L})$ has a crosscap two.

Theorem 5. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 4$. Let $1 \leq i \neq j \neq k \neq \ell \leq 4$ and $1 \leq p, q, r, s, t \leq 4$. Then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if and only if one of the following conditions hold:*

(i). *$|\bigcup_{n=1}^4 U_n| = 9$; there is U_i with $|U_i| = 6$ and $U_{jk} = U_{j\ell} = U_{k\ell} = U_{jk\ell} = \emptyset$.*

(ii). *$|\bigcup_{n=1}^4 U_n| = 8$, and one of the following cases is satisfied:*

[a] *There is U_i with $|U_i| = 5$ and $U_{jk} = U_{j\ell} = U_{k\ell} = U_{jk\ell} = \emptyset$.*

[b] *There exist U_i and U_j such that $|U_i| = 4$, $|U_j| = 2$ and $\bigcup_{pq \neq ij} U_{pq} = U_{jkl} = \emptyset$.*

[c] *There exist U_i and U_j such that $|U_i| = |U_j| = 3$ and $\bigcup_{pq \neq ij} U_{pq} = U_{ik\ell} = U_{jk\ell} = \emptyset$.*

[d] *There exist U_i, U_j, and U_k such that $|U_i| = 3, |U_j| = |U_k| = 2$, and $\bigcup U_{pq} = \bigcup_{pqr \neq ijk} U_{pqr} = \emptyset$ for $1 \leq p \neq q \neq r \leq 4$.*

(iii). *$|\bigcup_{n=1}^4 U_n| = 7$, and one of the following cases is satisfied:*

[a] *There is U_i with $|U_i| = 4$ and $|\bigcup_{p,q \neq i} U_{pq} \cup U_{jk\ell}| = 1$. Moreover, $U_{(pq)^c} = \emptyset$ whenever $|U_{pq}| = 1$ for $p, q \neq i$.*

[b] *There exist U_i and U_j such that $|U_i| = 3, |U_j| = 2$ and $|\bigcup_{p,q \neq i} U_{pq} \cup U_{jk\ell}| \leq 1$. Moreover, if $|\bigcup_{p,q \neq i} U_{pq} \cup U_{jk\ell}| = 1$, then $U_{(pq)^c} = \emptyset$ and $U_{ik} = U_{i\ell} = U_{ik\ell} = \emptyset$, and if $\bigcup_{p,q \neq i} U_{pq} \cup U_{jk\ell} = \emptyset$, then $|U_{ik} \cup U_{i\ell}| \in \{1, 2\}$.*

[c] *There exist U_i, U_j, and U_k such that $|U_i| = |U_j| = |U_k| = 2$ with $|\bigcup U_{pq}| \leq 2$, in which at most one of the $U_{p\ell}$s has exactly one element, and, also, at most two distinct sets'*

U_{rst}s are non-empty for all $rst \neq ijk$. Moreover, if $|U_{pq}| = 2$ or $|U_{p\ell}| = 1$ for $p, q \neq \ell$, then at most one of the U_{rst}s is non-empty.

(iv). *$|\bigcup_{n=1}^{4} U_n| = 6$, and one of the following cases is satisfied:*

[a] *There is U_i with $|U_i| = 3$, $|\bigcup_{p,q\neq i} U_{pq} \cup U_{jk\ell}| \in \{2,3\}$ in which $|U_{pq}| \leq 2$, and $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$. Moreover, if $|U_{pq}| \in \{1,2\}$ with $|U_{jk\ell}| = 2$, then $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$.*

[b] *There exist U_i and U_j such that $|U_i| = |U_j| = 2$ and $|U_{ij} \cup U_{k\ell}| \leq 3$ with $|U_{ij}|, |U_{k\ell}| \leq 2$. Additionally, if $|U_{ij}| = 2$, then $|U_{k\ell}| \leq 1$ and $\bigcup_{pq\neq ij,k\ell} U_{pq} = U_{ik\ell} = U_{jk\ell} = \emptyset$, and, if $|U_{ij}| = 1$, then $|U_{k\ell}| \leq 1$ and $|\bigcup_{pq\neq ij,k\ell} U_{pq}| \leq 1$. Moreover, in the case of $U_{ij} = \emptyset$, one of the following hold:*

[b1] *If $|U_{k\ell}| = 2$, then $|\bigcup_{pq\neq ij,k\ell} U_{pq}| \leq 2$ in which $|U_{pq}| \leq 1$ and $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$.*

[b2] *If $|U_{k\ell}| = 1$, then $|U_{rs}| \leq 3$ with $U_{(rs)^c} = \emptyset$ where $|U_{rs}| = \max_{pq\neq ij,k\ell} |U_{pq}|$ and $|\bigcup_{mn\neq ij,k\ell,rs,(rs)^c} U_{mn}| \leq 1$.*

[b3] *If $U_{k\ell} = \emptyset$, then $|\bigcup_{pq\neq ij,k\ell} U_{pq}| \leq 4$ in which at most three U_{pq}s are non-empty. Furthermore, if $|U_{pq}| \in \{2,3\}$, then $U_{(pq)^c} = \emptyset$.*

(v). *$|\bigcup_{n=1}^{4} U_n| = 5$; there exists U_i such that $|U_i| = 2$ and $1 \leq |\bigcup_{p,q\neq i} U_{pq}| \leq 6$ in which $|U_{pq}| \leq 4$. Moreover,*

[a] *If $|U_{pq}| = 4$, then $U_{(pq)^c} = \emptyset$, $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| \leq 1$, and $\bigcup_{U_{rs}\neq\emptyset} U_{(rs)^c} = \emptyset$.*

[b] *If $|U_{pq}| = 3$, then $U_{(pq)^c} = \emptyset$, $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| \leq 2$ and $U_{(rs)^c} = \emptyset$ whenever $|U_{rs}| = 2$.*

[c] *In the case of $|U_{pq}| = 2$, one of the following holds*

[c1] *If $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| = 4$, then $\bigcup_{U_{rs}\neq\emptyset} U_{(rs)^c} = \emptyset$.*

[c2] *If $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| \in \{2,3\}$, then $|\bigcup_{U_{rs}\neq\emptyset} U_{(rs)^c}| \leq 1$. In addition, $|\bigcup_{U_{rs}\neq\emptyset} U_{(rs)^c}| = 1$ whenever $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| = 2$ in which exactly two U_{rs}s are non-empty.*

[c3] *If $|\bigcup_{r,s\neq i;rs\neq pq} U_{rs}| \leq 1$, then either $U_{(pq)^c} = \emptyset$ with $1 \leq |\bigcup_{U_{rs}\neq\emptyset} U_{(rs)^c}| \leq 2$ or $U_{(rs)^c} = \emptyset$ with $|U_{(pq)^c}| \leq 1$.*

[d] *If $|U_{pq}| \leq 1$ for all $1 \leq p \neq q \neq i \leq 4$, then $2 \leq |\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 3$ in which at most two distinct $U_{(pq)^c}$s are non-empty.*

(vi). *$|\bigcup_{n=1}^{4} U_n| = 4$; there exist two non-empty sets U_{ij} and $U_{(ij)^c}$ such that $2 \leq |U_{ij} \cup U_{(ij)^c}| \leq 5$, and one of the following cases is satisfied:*

[a] *If $|U_{ij} \cup U_{(ij)^c}| = 5$, then either $|U_{ij}| = 4$ or $|U_{(ij)^c}| = 4$. Further, $\bigcup_{pq\neq ij,(ij)^c;U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$.*

[b] *If $|U_{ij} \cup U_{(ij)^c}| = 4$, then $|U_{pq} \cup U_{(pq)^c}| = 2$ whenever $U_{pq}, U_{(pq)^c} \neq \emptyset$ for $pq \neq ij$. Further, if $|U_{ij}| = |U_{(ij)^c}| = 2$, then at most one pair of $U_{pq}, U_{(pq)^c}$ is nonempty for all $pq \neq ij$.*

[c] *If $|U_{ij} \cup U_{(ij)^c}| = 3$, then $|U_{pq} \cup U_{(pq)^c}| \in \{2,3\}$ whenever $U_{pq}, U_{(pq)^c} \neq \emptyset$ for $pq \neq ij$. Further, if $U_{(rs)^c} \neq \emptyset$ for $1 \leq r \neq s \leq 4$ and $rs \neq pq, ij$, then $|U_{rs} \cup U_{(rs)^c}| \in \{2,3\}$ with $|(U_{pq} \cup U_{(pq)^c}) \cup (U_{rs} \cup U_{(rs)^c})| \in \{4,5\}$.*

Proof. Assume that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Then, by Theorem 1b, we have $|\bigcup_{n=1}^{4} U_n| \leq 9$. So, $4 \leq |\bigcup_{n=1}^{4} U_n| \leq 9$.

Case 1 Let $|\bigcup_{n=1}^{4} U_n| = 9$. Then, by Theorem 1b, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ implies $|U_1| = 6$. If $U_{ij} \neq \emptyset$ or $U_{ijk} \neq \emptyset$ for some $i \neq 1$, then the sets $X = U_1$ and $Y = V(\mathbb{AG}(\mathcal{L})) \setminus U_1$ contain $K_{6,4}$, which has a crosscap four. So, $U_{ij}, U_{ijk} = \emptyset$ for all $i \neq 1$. Here, remember that every vertex in U_{1jk} is an end vertex, and every vertex in U_{1j} is of degree two. Let G_{12} be the induced subgraph of $\mathbb{AG}(\mathcal{L})$ induced by the vertex subset $\bigcup_{n=1}^{4} U_n$. It is clear that $G_{12} \cong K_{6,1,1,1}$, and G_{12} is a subgraph of the graph H_2 given in Figure 2a with the labels $u_\ell \in U_1$ (for $\ell = 1, \ldots, 6$), $I_2 = v_1, I_3 = v_2$, and $I_4 = v_3$. By Figure 2b, the N_2-embedding of G_{12} contains three different faces with vertices I_2, I_3; I_3, I_4;, and I_2, I_4, respectively. So, any number of vertices in U_{1j} can be embedded into the N_2-embedding of G_{12} without edge-crossing. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Case 2 Let $|\bigcup_{n=1}^{4} U_n| = 8$.

Case 2.1 Suppose $|U_1| \in \{5, 4\}$. If $U_{ij} \neq \emptyset$ or $U_{ijk} \neq \emptyset$ for some $i \neq 1$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{5,4}$ as a subgraph, which is a contradiction. Therefore, $U_{ij} = \emptyset$ and $U_{ijk} = \emptyset$ for all $i \neq 1$. Now, if $|U_1| = 5$, then $\mathbb{AG}(\mathcal{L})$ is a subgraph of the annihilating-ideal graph in Case 1 with $|U_1| = 6$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Suppose $|U_1| = 4$. Here, $|U_2| = 2$. If $I \in \bigcup_{i \neq 1} U_{ij} \cup U_{234}$, then $\mathbb{AG}(\mathcal{L})$ contains a copy of $K_{4,5}$ where the partite sets are U_1 and $U_2 \cup U_3 \cup U_4 \cup \{I\}$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If $U_{1j} \neq \emptyset$ for some $j \in \{3, 4\}$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{5,4} - e$ as a subgraph with the partition sets $U_1 \cup U_{1j}$ and $U_2 \cup U_3 \cup U_4$ so that, by Remark 1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $\bigcup_{ij \neq 12} U_{ij} = \emptyset$ and $U_{234} = \emptyset$. In this case, one can retrieve an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from Figure 4b by changing the label I_3' to I_4 and its related edges such that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Case 2.2 Suppose $|U_1| = 3$. Let $|U_2| = 3$. If $U_{ij} \neq \emptyset$ or $U_{ijk} \neq \emptyset$ for $ij \neq 12$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{4,5} - e$, which is a contradiction. Therefore, $U_{ij} = \emptyset$ and $U_{ijk} = \emptyset$ for all $ij \neq 12$. In this case, the crosscap of $\mathbb{AG}(\mathcal{L})$ is same as the crosscap of $K_{3,3,1,1}$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Let $|U_2| = 2$ and $I \in \bigcup_{ijk \neq 123} U_{ij} \cup U_{ijk}$.

- In the case that $I \in U_{ij}$ for $ij \in \{12, 13\}$, the contraction of $\mathbb{AG}(\mathcal{L})$ induced by the partite sets $X = U_i \cup U_4$ and $Y = U_j \cup \{I_k, [I_k', I_{ij}]\}$, where $k \notin \{i, j, 4\}$, forms a copy of H_4.
- In the case that $I \in U_{ij}$ for $ij \in \{14, 23, 24, 34\}$, the graph $\mathbb{AG}(\mathcal{L})$ contains $K_{5,4}$ with the partite sets $U_i \cup U_j \cup U_{ij}$ and $U_k \cup U_\ell$ where $k, \ell \notin \{i, j\}$.
- In the case that $I \in \bigcup_{ijk \neq 123} U_{ijk}$, the contraction of $\mathbb{AG}(\mathcal{L})$ induced by $(\bigcup_{n=1}^{4} U_n \setminus \{I_\ell\}) \cup \{[I_\ell, I]\}$ forms H_4 where ℓ is the least integer in $\{1, 2, 3, 4\} \setminus \{i, j, k\}$.

Thus, $\bigcup_{ijk \neq 123} U_{ij} \cup U_{ijk} = \emptyset$, and, so, the crosscap of $\mathbb{AG}(\mathcal{L})$ is the crosscap of $K_{3,2,2,1}$, which is two.

Case 2.3 Suppose $|U_1| = 2$. Then, $K_{2,2,2,2}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$. Suppose $\tilde{\gamma}(K_{2,2,2,2}) = 2$. Then, by Euler's formula, the number of faces in an N_2 embedding of $K_{2,2,2,2}$ is 16 so that all the faces are triangular, which contradicts the fact that $K_{2,2,2,2}$ has no triangular embedding (see [27]). Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

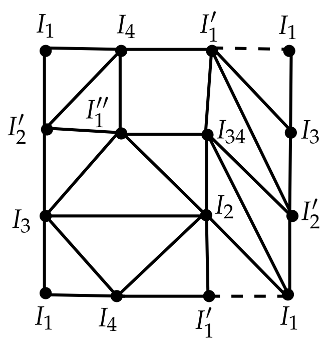
Case 3 Let $|\bigcup_{n=1}^{4} U_n| = 7$.

Case 3.1 Suppose $|U_1| = 4$. If $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \geq 2$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{4,5}$ with one partite set $X = U_1$, and, so, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Further, by Theorem 4iv, $\mathbb{AG}(\mathcal{L})$ is projective whenever $U_{ij} = U_{ijk} = \emptyset$ for all $i \neq 1$. Therefore, $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| = 1$, and let $I \in \bigcup_{i \neq 1} U_{ij} \cup U_{ijk}$. Now, if $U_{1j} = \emptyset$ for all $2 \leq j \leq 4$, then it is easy to verify that $\mathbb{AG}(\mathcal{L})$ is isomorphic to a subgraph of the graph H_1 (see Figure 1a). Therefore, by Lemma 2, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. So, let $U_{1j} \neq \emptyset$ for some $2 \leq j \leq 4$. Suppose $U_{k\ell} = \emptyset$ for $2 \leq j \neq k \neq \ell \leq 4$. Here, the open neighbor of each vertex in U_{1j} is I_k and I_ℓ in $\mathbb{AG}(\mathcal{L})$. Let G_{13} be the induced subgraph of $\mathbb{AG}(\mathcal{L})$ induced by the vertex subset $\bigcup_{n=1}^{4} U_n \cup \{I\}$. Clearly, G_{13} is a subgraph of the graph H_1 given in Figure 1a with the labels $u_\ell \in U_1$ (for $\ell = 1, \ldots, 4$), $v_1 = I_2, v_2 = I_3, v_3 = I_4$, and $v_4 = I$. Since $(I_3, I_4), (I_2, I_4), (I_2, I_3) \in E(\mathbb{AG}(\mathcal{L}))$, any number of vertices in U_{1j} (for $2 \le j \le 4$) can be embedded in the N_2-embedding of G_{13} without edge-crossing, and, therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Now, take $U_{k\ell} \neq \emptyset$ for $2 \le j \neq k \neq \ell \le 4$. Note that the set $U_{k\ell}$ is nothing but the singleton set $\{I\}$. Now, consider the subgraph $G_{14} = \mathbb{AG}(\mathcal{L}) - \{I_{1j}, (I_j, I_k), (I_k, I_\ell), (I_j, I_\ell), (I, I_j)\}$, which is isomorphic to $K_{4,4}$ with the partition sets $X = U_1$ and $Y = \{I_j, I_k, I_\ell, I\}$. Note that any N_2-embedding of G_{14} has eight rectangular faces so that each face shares exactly two vertices from X and Y. In $\mathbb{AG}(\mathcal{L})$, the vertex I_{1j} is adjacent to three vertices of Y, namely I_k, I_ℓ, and I. Therefore, one cannot insert I_{1j} with its edges into N_2 without crossing, which is a contradiction.

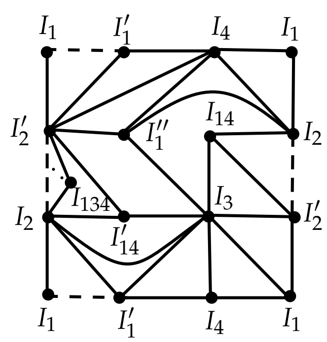
Case 3.2 Suppose $|U_1| = 3$. Then, $|U_2| = 2$. If $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \ge 2$, then it is easy to check that the contraction of $\mathbb{AG}(\mathcal{L})$ contains either $K_{4,5} - e$ or $K_{3,6} \cup (K_4 - e)$ as a subgraph, and, so, by Remark 1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Therefore, $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \le 1$.

Assume $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| = 1$. If $U_{ij} \neq \emptyset$, then $U_{(ij)^c} = \emptyset$; otherwise, the graph induced by the partition sets $X = U_1 \cup U_3$ and $Y = U_2 \cup U_4 \cup [I_{ij}, I_{(ij)^c}]$ form H_4 in $\mathbb{AG}(\mathcal{L})$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Further, if $I \in U_{13} \cup U_{14} \cup U_{134}$, then consider the graph $\mathbb{AG}(\mathcal{L}) - \{I, e_1, e_2, e_3, e_4, e_5\} \cong K_{4,4} - e$ with the bipartite sets $\{I_1, I_1', I_1'', I_j\}$ and $\{I_i, I_i', I_k, I_{ijk}\}$ where $e_1 = (I_1, I_j), e_2 = (I_1', I_j), e_3 = (I_1'', I_j), e_4 = (I_i, I_k), e_5 = (I_i', I_k)\}$, and $e = (I_j, I_{ijk})$. Now, a similar argument given for G_2 (refer to Equation 1) leads to $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Therefore, $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| = 1$ with $U_{13} = U_{14} = U_{134} = \emptyset$. In this case, with the help of Figure 10a, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Notice that in Figure 10a, we take $|U_{34}| = 1$.

Assume $\bigcup_{i \neq 1} U_{ij} \cup U_{ijk} = \emptyset$. If $|U_{1j}| \ge 3$ for some $j \in \{3, 4\}$, then the sets $X = U_2 \cup U_{j'}$ and $Y = U_1 \cup U_j \cup U_{1j}$, where $j' \in \{3, 4\} \setminus \{j\}$, form $K_{3,7}$. So, $|U_{1j}| \le 2$ for $j = 3, 4$. Suppose $|U_{13} \cup U_{14}| \ge 3$. Let $|U_{1j}| \ge 2$ and $|U_{1k}| \ge 1$ for $j, k \in \{3, 4\}$. Then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{1k}, (I_1, I_j), (I_1', I_j), (I_1'', I_j)\}$ contains $K_{3,6}$ with the partite sets $X = U_2 \cup U_k$ and $Y = U_1 \cup U_j \cup U_{1j}$. Since $deg_{K_{3,6}}(I_j) = 3$, I_j is contained in exactly three rectangular faces in any N_2-embedding of $K_{3,6}$. Since $\{I_1, I_1', I_1'', I_j\} \subset Y$, to embed the edges $(I_1, I_j), (I_1', I_j)$, and (I_1'', I_j), the vertices I_1, I_1', and I_1'' on the diagonals of the three rectangular faces that contain I_j, respectively, are required. Now, after embedding the three edges, I_j is in exactly six triangular faces, all of which were formed by using two vertices from Y and one vertex from X. Therefore, the vertex I_{1k} cannot be embedded because it is adjacent to I_j as well as two vertices from X. So, $|U_{13} \cup U_{14}| \le 2$. However, $\mathbb{AG}(\mathcal{L})$ is projective if $U_{13} \cup U_{14} = \emptyset$. Thus, $1 \le |U_{13} \cup U_{14}| \le 2$. Now, one can obtain help from Figure 10b to say that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



(**a**). $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| = 1$ and $U_{13} = U_{14} = U_{134} = \emptyset$



(**b**). $\bigcup_{i \neq 1} U_{ij} \cup U_{ijk} = \emptyset$ and $1 \le |U_{13} \cup U_{14}| \le 2$

Figure 10. $|\bigcup_{n=1}^{4} U_n| = 7$ with $|U_1| = 3$.

Case 3.3 Suppose $|U_1| = 2$.

Claim A: At most two distinct U_{ij}s are non-empty in which at most one U_{i4} is non-empty for $1 \leq i \neq j \leq 4$. Additionally, at most two distinct $U_{\ell mn}$s are non-empty for $\ell mn \neq 123$.

Assume on the contrary that at least three U_{ij}s are non-empty for $1 \leq i, j \leq 4$; say, $U_{i_1 i_2}, U_{i_3 i_4}$ and $U_{i_5 i_6}$ are non-empty. Let $p \in \{1,2,3\} \setminus \{i_1, i_2\}, q \in \{1,2,3\} \setminus \{p, i_3, i_4\}$ and $r \in \{1,2,3\} \setminus \{p, q, i_5, i_6\}$. If r exists, then the minor subgraph induced by the vertices $[I_p, I_{i_1 i_2}], I'_p, [I_q, I_{i_3 i_4}], I'_q, [I_r, I_{i_5 i_6}], I'_r$, and I_4 forms K_7 in $\mathbb{AG}(\mathcal{L})$, which is a contradiction. If r does not exist, then take r as $\{1,2,3\} \setminus \{p, q\}$ and form a minor of $\mathbb{AG}(\mathcal{L})$ with the partite sets $X = \{I_r, I'_r, I_4, I_{r4}\}$ and $Y = \{[I_p, I_{i_1 i_2}], I'_p, [I_q, I_{i_3 i_4}], I'_q\}$, which is isomorphic to either H_3 or H_4, as in Figure 3. So, by Lemma 3, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, only at most two distinct U_{ij}s can be non-empty for $1 \leq i \neq j \leq 4$. Further, if $U_{m4}, U_{n4} \neq \emptyset$ for some $1 \leq m \neq n \leq 4$, then the subgraph induced by the sets $X = U_m \cup U_{m4} \cup \{I_k\}$ and $Y = U_n \cup U_4 \cup \{[I'_k, I_{n4}]\}$, where $k \neq m$ or n, form H_4 which has a crosscap of at least three.

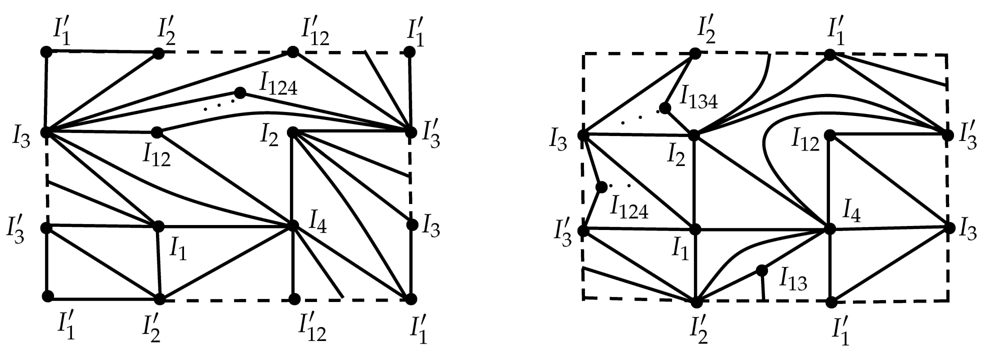
Note that all the vertices in U_{123} are end vertices in $\mathbb{AG}(\mathcal{L})$. If $U_{ijk}, U_{\ell mn}$, and U_{pqr} are non-empty for $ijk, \ell mn, pqr \neq 123$, then the minor subgraph induced by $\{[I_{(ijk)^c}, I_{ijk}], I'_{(ijk)^c}, [I_{(\ell mn)^c}, I_{\ell mn}], I'_{(\ell mn)^c}, [I_{(prq)^c}, I_{pqr}], I'_{(pqr)^c}, I_4\}$ is K_7, which is a contradiction. Therefore, at most two distinct $U_{\ell mn}$s are non-empty for $\ell mn \neq 123$.

Claim B: $|U_{ij}| \leq 2$ and $|U_{i4}| \leq 1$ for all $1 \leq i < j \leq 3$.

If $|U_{ij}| \geq 3$ for some $1 \leq i, j \leq 3$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{7,3}$ as a subgraph with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_k \cup U_4$ where $k \in \{1,2,3\} \setminus \{i, j\}$. Additionally, if $|U_{i4}| \geq 2$ for some $1 \leq i \leq 3$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{5,4}$ as a subgraph with the partite sets $X = U_i \cup U_4 \cup U_{i4}$ and $Y = U_j \cup U_k$ where $j, k \in \{1,2,3\} \setminus \{i\}$. Thus, $|U_{ij}| \leq 2$ and $|U_{i4}| \leq 1$ for all $1 \leq i < j \leq 3$.

Assume $|U_{ij}| = 2$ for some $1 \leq i, j \leq 3$. Suppose $U_{k\ell} \neq \emptyset$ for some $1 \leq k < \ell \leq 4$ and $k\ell \neq ij$. Let us take $j \notin \{k, \ell\} \cap \{i, j\}$. Then, $\mathbb{AG}(\mathcal{L})$ contains $K_{6,3} \cup (K_4 - e)$ with the partite sets $X = \{I_i, I'_i, I_j, [I'_j, I_{k\ell}], I_{ij}, I'_{ij}\}$ and $Y = U_m \cup U_4$ where $m \in \{1,2,3\} \setminus \{i, j\}$. So, by Remark 1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $U_{k\ell} = \emptyset$. In this case, the number of U_{ijk} cannot be more than one because here $\mathbb{AG}(\mathcal{L})$ contains $K_{6,3} \cup (K_4 - e)$. For the remaining cases, by Figure 11a, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Assume $|U_{ij}| \leq 1$ for all $1 \leq i, j \leq 3$. Suppose $|U_{k4}| = 1$ for some $1 \leq k \leq 3$. If there are two $U_{\ell mn}$s that are non-empty for $\ell mn \neq 123$, then it is not hard to verify that $\mathbb{AG}(\mathcal{L})$ contains a subgraph similar to the structure of H_3, which has a crosscap of at least three. For all the remaining cases, that is $|U_{ij}| = |U_{k4}| = 1$ with unique $U_{\ell mn} \neq \emptyset$ or $|U_{ij}| \leq 1$ and $|U_{pq}| \leq 1$ with at most two $U_{\ell mn}$s that are non-empty for $1 \leq i, j, k, p, q \leq 3$ and $\ell mn \neq 123$, one can use Figure 11b to obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



(**a**). $|U_{ij}| = 2$, $U_{k\ell} = \emptyset \; \forall \; k\ell \neq ij$ and at most one $U_{ijk} \neq \emptyset$ for $ijk \neq 123$

(**b**). $|U_{ij}|, |U_{pq}| \leq 1$ and at most two $U_{ijk} \neq \emptyset$ for $ijk \neq 123$ if $U_{i4} = \emptyset$

Figure 11. $|\bigcup_{n=1}^{4} U_n| = 7$ with $|U_1| = 2$.

Case 4 Let $|\bigcup_{n=1}^{4} U_n| = 6$.

Case 4.1 Suppose $|U_1| = 3$. Note that each vertex of U_{ij} for $i = 1$ is adjacent to exactly two vertices $I_{i'}$ and $I_{j'}$ for $i', j' \notin \{i, j\}$ and $(I_{i'}, I_{j'}) \in E(\mathbb{AG}(\mathcal{L}))$, so we do not want to bother about U_{1j} and U_{1jk} for all $2 \leq j < k \leq 4$. If $|U_{ij}| \geq 3$ for some $i \neq 1$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{4,5}$ as a subgraph with the partite sets $X = U_1 \cup U_k$ and $Y = U_i \cup U_j \cup U_{ij}$ where $k \in \{2, 3, 4\} \setminus \{i, j\}$, which is a contradiction. So, $|U_{ij}| \leq 2$ for all $i \neq 1$.

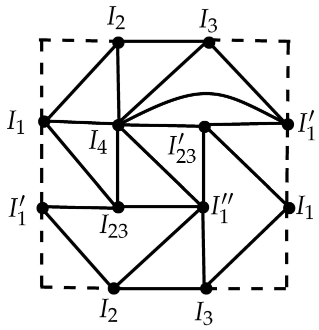
((**i**).Assume $|U_{ij}| = 2$ for some $i \neq 1$. If $U_{(ij)^c} \neq \emptyset$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_k \cup U_{(ij)^c}$ form $K_{4,5}$ in $\mathbb{AG}(\mathcal{L})$, and, if $U_{k\ell} \neq \emptyset$ for some $k \neq 1$ with $k\ell \neq ij$ or $U_{234} \neq \emptyset$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{4,5} - e$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If not, that is $U_{(ij)^c}, U_{k\ell}, U_{234} = \emptyset$ for all $k \neq 1$ with $k\ell \neq ij$, then by Figure 12a, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

(**ii**). Assume $|U_{ij}| \leq 1$ for all $i \neq 1$. If $U_{(i_1j_1)^c} \neq \emptyset$ and $U_{(i_2j_2)^c} \neq \emptyset$ for some $U_{i_1j_1} \neq \emptyset$ and $U_{i_2j_2} \neq \emptyset$, then the sets $X = U_{i_1} \cup U_{j_1} \cup U_{i_1j_1} \cup \{[I_{i_2j_2}, I_{(i_2j_2)^c}]\}$ and $Y = U_1 \cup U_m \cup U_{(i_1j_1)^c}$, where $m \neq i_1, j_1$, contains $K_{4,5} - e$ in $\mathbb{AG}(\mathcal{L})$. Additionally, if $|U_{(ij)^c}| \geq 3$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_m \cup U_{(ij)^c}$, where $m \neq i, j$, form $K_{3,7}$ in $\mathbb{AG}(\mathcal{L})$, which is a contradiction. So, at most one of the sets $U_{(ij)^c}$ is non-empty with $|U_{(ij)^c}| \leq 2$.

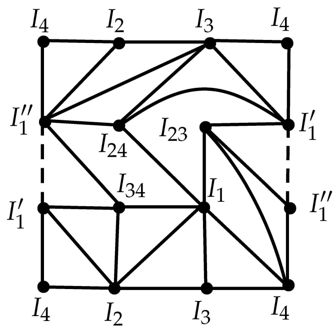
Let $|U_{(ij)^c}| = 2$. If $I \in \bigcup_{k\ell \neq ij} U_{k\ell} \cup U_{234}$, then the sets $X = \{I_i, I_j, I_{ij}\}$ and $Y = \{I_1, I_1', [I_1'', I], I_m, I_{(ij)^c}, I'_{(ij)^c}\}$, where $m \neq i, j$, form $K_{3,6} \cup (K_4 - e)$ so that, by Remark 1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $\bigcup_{k \neq 1; k\ell \neq ij} U_{k\ell} \cup U_{234} = \emptyset$. For this case, readers can verify the N_2-embedding of $\mathbb{AG}(\mathcal{L})$.

Let $|U_{(ij)^c}| = 1$. If $I, J \in \bigcup_{k \neq 1; k\ell \neq ij} U_{k\ell} \cup U_{234}$ with $|U_{k\ell}| \leq 1$, then the sets $\{I_i, I_j, I_m, I_1, [I_1', I], [I_1'', J], [I_{ij}, I_{(ij)^c}]\}$ form K_7. Therefore, $|\bigcup_{k \neq 1; k\ell \neq ij} U_{k\ell} \cup U_{234}| = 1$.

Let $\bigcup_{i \neq 1} U_{(ij)^c} = \emptyset$. Then, by Theorem 4iii[a], $\mathbb{AG}(\mathcal{L})$ is projective if $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \leq 1$. If $|\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \geq 4$, then $K_{3,7}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$ with the partite sets $X = U_1$ and $Y = V(\mathbb{AG}(\mathcal{L})) \setminus U_1$. So, in the case of $\bigcup_{i \neq 1} U_{(ij)^c} = \emptyset$, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $2 \leq |\bigcup_{i \neq 1} U_{ij} \cup U_{ijk}| \leq 3$ with $|U_{ij}| \leq 1$ (refer to Figure 12b).



(**a**). Unique $|U_{ij}| = 2$ for $i \neq 1$ and $U_{k\ell}, U_{234} = \emptyset\ \forall\ k \neq 1$ and $k\ell \neq ij$



(**b**). $|U_{ij}| = 1$ for all $i \neq 1$

Figure 12. $|\bigcup_{n=1}^{4} U_n| = 6$ with $|U_1| = 3$.

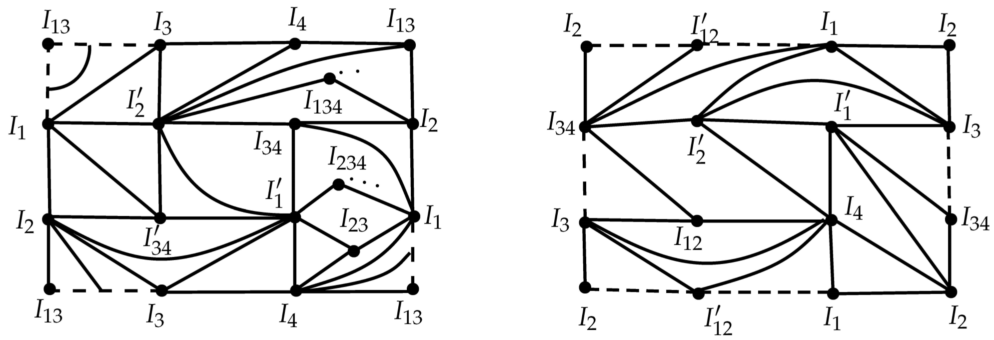
Case 4.2 Suppose $|U_1| = 2$. Then, $|U_2| = 2$ and $|U_3| = |U_4| = 1$. If $|U_{34}| \geq 3$, then the partite sets $X = U_1 \cup U_2$ and $Y = U_3 \cup U_4 \cup U_{34}$ form $K_{4,5}$ as a subgraph in $\mathbb{AG}(\mathcal{L})$, which is a contradiction.

Case 4.2.1 Assume $|U_{34}| = 2$. Then, $U_{(pq)^c} = \emptyset$ for all $U_{pq} \neq \emptyset$; otherwise, the sets $X = U_1 \cup U_2$ and $Y = U_3 \cup U_4 \cup U_{34} \cup \{[I_{pq}, I_{(pq)^c}]\}$ form $K_{4,5}$ in $\mathbb{AG}(\mathcal{L})$. In particular, $U_{12} = \emptyset$.

If $|U_{ij}| \geq 2$ for some $ij \neq 12, 34$ and $i < j$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{34}, I_{34}', (I_i, I_j), (I_i', I_j)\}$ contains $K_{3,5}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'}$ where $i' \in$

$\{1,2\} \setminus \{i\}$ and $j' \in \{3,4\} \setminus \{j\}$. Note that any N_2-embedding of $K_{3,5}$ has one hexagonal and six rectangular faces, and the vertices I_{34} and I'_{34} are adjacent to $I_i, I'_i, I_{i'}$ and $I'_{i'}$. So, to insert I_{34} and I'_{34} into an N_2-embedding of $K_{3,5}$, we require two faces, say F_1 and F_2, which contains $I_i, I'_i, I_{i'}$, and $I'_{i'}$. If either F_1 or F_2 is hexagonal, then the corresponding face may adopt one of the edges (I_i, I_j) or (I'_i, I_j). Let us take that the edge (I_i, I_j) is embedded. Now, to insert an edge (I'_i, I_j), a rectangular face containing I'_i and I_j as diagonals is required. However, no such rectangular face exists because the edges $(I'_i, I_{i'})$ and $(I'_i, I'_{i'})$ have been used twice in F_1 and F_2, which is a contradiction.

For all of the remaining cases, that is $|\bigcup_{ij \neq 12,34} U_{ij}| \leq 2$ with $|U_{ij}| \leq 1$ and $U_{(pq)^c} = \emptyset$ when $U_{pq} \neq \emptyset$ for $1 \leq p \neq q \leq 4$, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ (refer to Figure 13a).



(a). $|U_{34}| = 2$, $|\bigcup_{ij \neq 12,34} U_{ij}| \leq 2$ with $|U_{ij}| \leq 1$ and $U_{(ij)^c} = \emptyset$ when $U_{ij} \neq \emptyset$

(b). $|U_{34}| = 1$, $|U_{12}| = 2$ and $\bigcup_{ij \neq 12,34} U_{ij} \cup U_{134} \cup U_{234} = \emptyset$

Figure 13. $|\bigcup_{n=1}^{4} U_n| = 6$ with $|U_1| = 2$.

Case 4.2.2 Assume that $|U_{34}| = 1$. Let us take $ij \neq 12, 34$.

Let $|U_{12}| \geq 3$, then the subgraph of $\mathbb{AG}(\mathcal{L})$ induced by the sets $X = U_3 \cup U_4 \cup U_{34}$ and $Y = U_1 \cup U_2 \cup U_{12}$ contains $K_{3,7}$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Thus, $|U_{12}| \leq 2$.

Let $|U_{12}| = 2$. If $I \in \bigcup_{ij \neq 12,34} U_{ij} \cup U_{134} \cup U_{234}$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,6} \cup (K_4 - e)$, so that, by Remark 1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $\bigcup_{ij \neq 34,12} U_{ij} \cup U_{134} \cup U_{234} = \emptyset$, and in this case, by Figure 13b, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Let $|U_{12}| = 1$. If $|U_{ij}| \geq 2$, then the partite sets $X = U_{i'} \cup U_{j'}$ and $Y = \{I_i, I'_i, I_j, I_{ij}, I'_{ij}, [I_{34}, I_{12}]\}$ where $i' \in \{1,2\} \setminus \{i\}$ and $j' \in \{3,4\} \setminus \{j\}$ form a minor subgraph $K_{3,6} \cup (K_4 - e)$ in $\mathbb{AG}(\mathcal{L})$ so that, by Remark 1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If $U_{ij}, U_{k\ell} \neq \emptyset$ for $ij, k\ell \neq 12, 34$ where $\{i,j\} \cap \{k,\ell\} = j = \ell$, then the partite sets $X = \{I_i, [I'_i, I_{k\ell}], I_\ell, I_{ij}\}$ and $Y = \{I_k, I'_k, I_m, [I_{34}, I_{12}]\}$ where $m \notin \{i,j,k\}$ form $(H_4 \cup (u_2, u_3)) - (u_1, u_4)$. A slight modification of the proof for H_4 in Lemma 3 yields $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Further, minor changes to the labels in Figure 13a give $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $|\bigcup_{ij \neq 12,34} U_{ij}| \leq 1$.

Let $U_{12} = \emptyset$. Then $U_{(pq)^c} = \emptyset$ for all $U_{pq} \neq \emptyset$; otherwise, $\mathbb{AG}(\mathcal{L})$ contains $K_8 - 4e$, which is isomorphic to $(H_4 \cup (u_1, u_3)) - (v_1, v_2)$, so Lemma 3 gives us $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If $|U_{ij}| \geq 4$, then the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'}$ where $i' \in \{1,2\} \setminus \{i\}$ and $j' \in \{3,4\} \setminus \{j\}$ contain $K_{7,3}$ in $\mathbb{AG}(\mathcal{L})$, which is a contradiction. Suppose $|U_{ij}| \in \{2,3\}$. If $|U_{k\ell}| \geq 2$ for some $k\ell \neq ij$ where $\{k,\ell\} \cap \{i,j\} = k = i$, then the subgraph $G_{15} = \mathbb{AG}(\mathcal{L}) - \{I_{34}, I_{k\ell}, I'_{k\ell}, (I_i, I_j), (I'_i, I_j)\}$ contains $K_{5,3}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'}$ where $i', j' \notin \{i,j\}$. Note that any N_2-embedding of $K_{5,3}$ has one hexagonal and six rectangular faces. Further, in $\mathbb{AG}(\mathcal{L})$, I_{34} is adjacent to $I_i, I'_i, I_{i'}, I'_{i'}$, and, also, $I_{k\ell}, I'_{k\ell}$ are adjacent to $I_{i'}, I'_{i'}, I_j$. So, to embed the vertices $I_{34}, I_{k\ell}$, and $I'_{k\ell}$, one hexagonal and two rectangular faces containing both $I_{i'}$ and $I'_{i'}$ are required. In such a case, one cannot find two rectangular faces with the diagonal vertices I_i, I_j and I'_i, I_j. So, either the edge (I_i, I_j) or (I'_i, I_j) cannot be drawn without crossing, which is a contradiction. Thus, we obtain the result as in the statement-(iv)[b2].

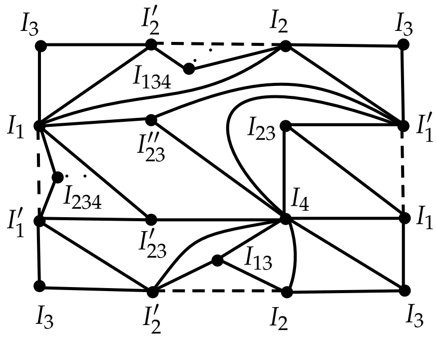
Case 4.2.3 Suppose $U_{34} = \emptyset$.

If $|U_{ij} \cup U_{(ij)^c}| \geq 4$ for some $ij \notin \{12, 34\}$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'} \cup U_{(ij)^c}$ where $i', j' \notin \{i, j\}$ form a complete bipartite graph whose crosscap is more than two.

Let $|U_{ij}| \in \{2, 3\}$ for some $ij \notin \{12, 34\}$. Then, clearly, $U_{(ij)^c}$ must be empty. Let $k\ell \notin \{12, 34, ij, (ij)^c\}$. If $|U_{ij} \cup U_{k\ell} \cup U_{(k\ell)^c}| \geq 5$, then the sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I_{k\ell}, I_{k\ell^c}]\}$ and $Y = U_{i'} \cup U_{j'}$ where $i' \in \{1, 2\} \setminus \{i\}$ and $j' \in \{3, 4\} \setminus \{j\}$ form $K_{6,3} \cup (K_4 - e)$ and, by Remark 1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $2 \leq |U_{ij} \cup U_{k\ell} \cup U_{(k\ell)^c}| \leq 4$. Now, there are at most three possibilities:

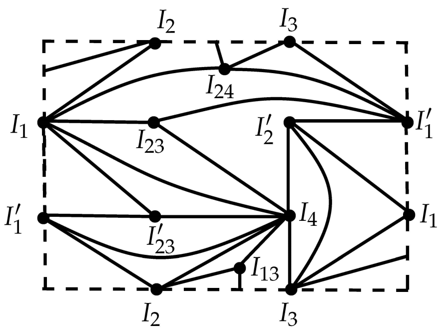
(i). $|U_{ij}| = 3$ and $|U_{k\ell}| = 1$; this case is pictured in Figure 14.
(ii). $|U_{ij}| = 2$ and $|U_{k\ell}| = |U_{(k\ell)^c}| = 1$; this case is pictured in Figure 15a.
(iii). $|U_{ij}| = |U_{k\ell}| = 2$; this case is pictured in Figure 15b.

Thus, in all these cases, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

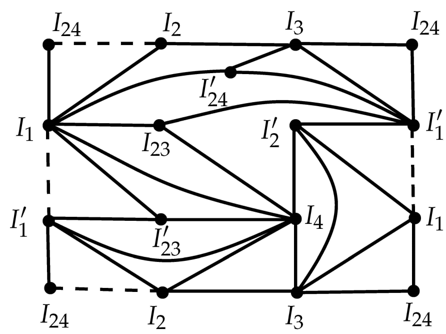


$U_{34} = \emptyset$ with $|U_{ij}| = 3$ and $|U_{k\ell}| = 1$ for some $ij, k\ell \neq 12, 34$

Figure 14. $|\bigcup_{n=1}^{4} U_n| = 6$ with $|U_1| = |U_2| = 2$.



(**a**). $U_{34} = \emptyset$ with $|U_{ij}| = 2$ and $|U_{k\ell}| = |U_{(k\ell)^c}| = 1$ for some $ij, k\ell \neq 12, 34$



(**b**). $U_{34} = \emptyset$ with $|U_{ij}| = |U_{k\ell}| = 2$ for some $ij, k\ell \neq 12, 34$

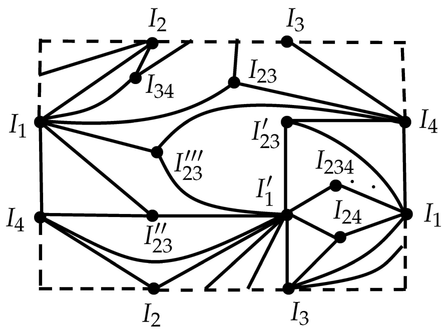
Figure 15. $|\bigcup_{n=1}^{4} U_n| = 6$ with $|U_1| = |U_2| = 2$.

Let $|U_{ij}| \leq 1$ for all $ij \notin \{12, 34\}$. Then, at least one $U_{ij} = \emptyset$ for $ij \notin \{12, 34\}$. Otherwise, the graph induced by $\{I_1, I'_1, I_2, I'_2, I_3, I_4, [I_{13}, I_{24}], [I_{14}, I_{23}]\}$ forms $K_8 - 3e$ in $\mathbb{AG}(\mathcal{L})$. Clearly, $\tilde{\gamma}(K_8 - 3e) \geq 3$ because the number of faces in the N_2-embedding of $K_8 - 3e$ is 17, which contradicts the well-known fact that $\frac{2|E|}{|F|}$ must be greater than the girth value (refer to Observation 1). Therefore, $|\bigcup_{ij \neq 12,34} U_{ij}| \leq 3$. Thus, by [20, Theorem 2.6iib)], we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $|\bigcup_{ij \neq 12,34} U_{ij}| = 3$.

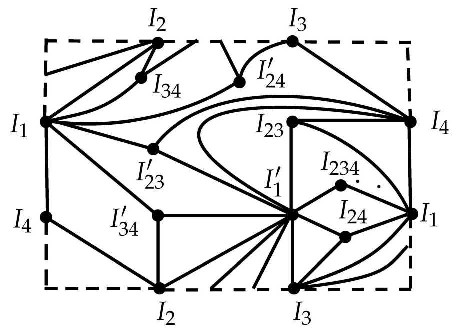
Case 5 Let $|\bigcup_{n=1}^{4} U_n| = 5$. Then, $|U_1| = 2$. If $U_{ij} = \emptyset$ for all $1 \le i < j \le 4$, then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \le 1$. Observe that we do not want to consider the sets U_{ij} for $i \ne 1$ whenever $U_{(ij)^c} = \emptyset$ because every vertex in U_{ij} is adjacent to I_i, I_j and $(I_i, I_j) \in E(\mathbb{AG}(\mathcal{L}))$. If $|U_{ij}| \ge 5$ for some $i \ne 1$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'}$ where $i', j' \notin \{i, j\}$ form $K_{3,7}$ in $\mathbb{AG}(\mathcal{L})$, which is a contradiction.

Case 5.1 Assume $|U_{ij}| = 4$ for some $i \ne 1$. Then, $U_{(mn)^c} = \emptyset$ whenever $U_{mn} \ne \emptyset$; otherwise, the sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I_{mn}, I_{(mn)^c}]\}$ and $Y = U_{i'} \cup U_{j'}$ where $i', j' \notin \{i, j\}$ form $K_{7,3}$ as a minor of $\mathbb{AG}(\mathcal{L})$. Similarly, $U_{(ij)^c} = \emptyset$; otherwise $K_{6,4}$ is a minor of $\mathbb{AG}(\mathcal{L})$. If $|U_{k\ell}| \ge 2$ for some $k \ne 1$ and $k\ell \ne ij$, then the subgraph $G_{16} = \mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I'_{k\ell}, (I_i, I_j)\}$ contains $K_{6,3}$ with the partition sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_{i'}$ where $i' \notin \{1, i, j\}$. Since $\{i, j\} \cap \{k, \ell\} \ne \emptyset$, let $\{i, j\} \cap \{k, \ell\} = i = k$. Clearly, $j \in \{2, 3, 4\} \setminus \{k, \ell\}$. Note that each face in any N_2-embedding of $K_{6,3}$ is rectangular, and the vertices $I_{k\ell}, I'_{k\ell}$ are adjacent to I_1, I'_1 and I_j. Therefore, to insert $I_{k\ell}$ and $I'_{k\ell}$, two rectangular faces that contain I_1, I'_1 and I_j are required. Next, to insert the edge (I_i, I_j), a rectangular face with the diagonals I_i and I_j is required. However, the edges (I_1, I_j) and (I'_1, I_j) have been used twice to form the first two rectangular faces. So, one cannot construct another rectangular face that contains I_i and I_j with a single left-out vertex of Y, which is a contradiction.

Therefore, for the remaining case, that is, $|U_{k\ell}| \le 1$ for all $k \ne 1$ and $k\ell \ne ij$ with $U_{(mn)^c} = \emptyset$ whenever $U_{mn} \ne \emptyset$, by using Figure 16a, one can have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



(**a**). $|U_{ij}| = 4$ for some $i \ne 1$ and $|U_{k\ell}| \le 1$ for all $k \ne 1$ and $k\ell \ne ij$



(**b**). $|U_{ij}| = 2$ and $U_{(ij)^c} = \emptyset$ for all $i \ne 1$

Figure 16. $|\bigcup_{n=1}^{4} U_n| = 5$ with $|U_1| = 2$.

Case 5.2 Assume $|U_{ij}| = 3$ for some $i \ne 1$. Let $p \notin \{1, i, j\}$. Clearly, $U_{(ij)^c} = \emptyset$; otherwise, the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_p \cup U_{(ij)^c}$ form $K_{5,4}$.

If $|U_{k\ell}| = 3$ for some $k \ne 1$ and $k\ell \ne ij$, then the subgraph $G'_{15} = \mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I'_{k\ell}, I''_{k\ell}, (I_i, I_j), (I_1, I_p), (I'_1, I_p)\}$ has a similar structure of G_{15} with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_p$, and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Suppose $|U_{k\ell}|, |U_{mn}| = 2$ for $k, m \ne 1$ and $k\ell, mn \ne ij$. Let $\{i, j\} \cap \{k, \ell\} = i = k$. Then, $G_{17} = \mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I'_{k\ell}, I_{mn}, I'_{mn}, (I_i, I_j)\}$ has $K_{5,3}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_\ell$. Any N_2-embedding of $K_{5,3}$ has one hexagonal and six rectangular faces. Notice that $I_{k\ell}, I'_{k\ell}$ are adjacent to I_1, I'_1, I_j, and I_{mn}, I'_{mn} are adjacent to I_1, I'_1, I_i. So, to embed $I_{k\ell}, I'_{k\ell}, I_{mn}$, and I'_{mn}, one hexagonal and two rectangular faces containing both I_1 and I'_1 are required. However, the edge (I_i, I_j) cannot be drawn without crossing, which is a contradiction. Therefore, $|\bigcup_{k \ne 1; k\ell \ne ij} U_{k\ell}| \le 3$ and $|U_{k\ell}| \ne 3$.

Suppose $|\bigcup_{k \ne 1; k\ell \ne ij} U_{k\ell}| = 3$. Since $|U_{k\ell}| \ne 3$ for all $k \ne 1$ and $k\ell \ne ij$, we have $|U_{k\ell}| = 2$ and $|U_{mn}| = 1$ for some $m \ne 1$ and $mn \ne ij, k\ell$. Next, we claim that $U_{(k\ell)^c} = U_{(mn)^c} = \emptyset$. If $U_{(k\ell)^c} \ne \emptyset$, then by letting $\{i, j\} \cap \{k, \ell\} = i = k$, $K_{7,3}$ can be formed by the sets $X = U_i \cup U_j \cup U_{ij} \cup U_{k\ell}$ and $Y = U_1 \cup \{[I_\ell, I_{(k\ell)^c}]\}$. If $U_{(mn)^c} \ne \emptyset$, then $\mathbb{AG}(\mathcal{L})$ has a similar structure to G_{15}, so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$.

Suppose $|\bigcup_{k\neq 1;k\ell\neq ij} U_{k\ell}| \leq 2$. As mentioned, $U_{(k\ell)^c} = \emptyset$ when $|U_{k\ell}| = 2$ for $k \neq 1$ and $k\ell \neq ij$. Suppose $|U_{k\ell}| = 1$ and $|U_{(k\ell)^c}| \geq 2$. Then, $\mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I_{(k\ell)^c}, I'_{(k\ell)^c}, (I_i, I_j), (I_1, I_\ell),$ *and*
$(I'_1, I_\ell)\}$ has $K_{5,3}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_\ell$. Note that any N_2-embedding of $K_{5,3}$ has one hexagonal and six rectangular faces, $I_{k\ell}$ is adjacent to $I_1, I'_1, I_j, I_{(k\ell)^c}, I'_{(k\ell)^c}$, and $I_{(k\ell)^c}, I'_{(k\ell)^c}$ are adjacent to $I_k, I_\ell, I_{k\ell}$. So, the three vertices $I_{k\ell}, I_{(k\ell)^c}, I'_{(k\ell)^c}$ together with the edges $(I_i, I_j), (I_1, I_\ell), (I'_1, I_\ell)$ cannot be embedded, and, also
, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|U_{k\ell} \cup U_{(k\ell)^c}| \leq 2$. Further, if $|U_{k\ell} \cup U_{(k\ell)^c}| = |U_{\ell m} \cup U_{(\ell m)^c}| = 2$ for $k\ell \neq ij$ and $\ell m \neq ij, k\ell$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$, which is a contradiction.

Thus, an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ can be retrieved from Figure 16a for $|\bigcup_{pq\neq ij} U_{pq}| \leq 3$ with $U_{(pq)^c} = \emptyset$ if $|U_{pq}| = 2$.

Case 5.3 Assume $|U_{ij}| = 2$ for some $i \neq 1$. Clearly, $|U_{(ij)^c}| \leq 1$; otherwise, the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_p \cup U_{(ij)^c}$ where $p \notin \{1, i, j\}$ form $K_{5,4}$.

If $|U_{k\ell}|, |U_{mn}| = 2$ for $k, m \neq 1$ and $k\ell, mn \neq ij$, then $U_{(ij)^c}, U_{(k\ell)^c}, U_{(mn)^c} = \emptyset$. Further, an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ in the case of $|U_{ij}| = |U_{k\ell}| = |U_{mn}| = 2$ is given in Figure 16b so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Suppose $|U_{k\ell}| = 2, |U_{mn}| \leq 1$ for $k, m \neq 1$ and $k\ell, mn \neq ij$. If $U_{(ij)^c}, U_{(k\ell)^c} \neq \emptyset$, then the sets $X = U_1 \cup U_p \cup U_{(ij)^c}$ and $Y = U_i \cup U_j \cup U_{ij} \cup \{[I_{(k\ell)}, I_{(k\ell)^c}]\}$ where $p \notin \{1, i, j\}$ form $K_{5,4} - e$ in $\mathbb{AG}(\mathcal{L})$ so that, by Remark 1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Further, since $|U_{k\ell}| = 2$, we have $|U_{(k\ell)^c}| \leq 1$. Therefore, $|U_{(ij)^c} \cup U_{(k\ell)^c}| \leq 1$. Suppose $|U_{(ij)^c} \cup U_{(k\ell)^c}| = 1$, say $U_{(ij)^c} \neq \emptyset$. Then, $U_{(mn)^c} = \emptyset$; otherwise, $X = U_1 \cup U_p \cup U_{(ij)^c}$ and $Y = U_i \cup U_j \cup U_{ij} \cup \{[I_{(mn)}, I_{(mn)^c}]\}$ where $p \notin \{1, i, j\}$ form $K_{4,5} - e$ in $\mathbb{AG}(\mathcal{L})$. So, $|U_{mn} \cup U_{(mn)^c}| \leq 1$. Suppose not, that is, $U_{(ij)^c}, U_{(k\ell)^c} = \emptyset$, then $|U_{(mn)^c}| \leq 1$; otherwise, $\mathbb{AG}(\mathcal{L}) - \{I_{ij}, I'_{ij}, I_{k\ell}, I'_{k\ell}, (I_m, I_n), (I_1, I_{m'}), (I'_1, I_{m'})\} \cong K_{5,3}$ with the partite sets $X = U_1 \cup U_{m'} \cup U_{(mn)^c}$ and $Y = U_m \cup U_n \cup U_{mn}$ where $m' \notin \{1, m, n\}$ is a similar structure to G_{17} which has a crosscap of at least three. So, $|U_{mn} \cup U_{(mn)^c}| \leq 2$.

Suppose $|U_{k\ell}|, |U_{mn}| \leq 1$ for $k, m \neq 1$ and $k\ell, mn \neq ij$. Then, by Theorem 4(ii), $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $|\bigcup_{k\neq 1;k\ell\neq ij} U_{k\ell}| = 2$ with $|\bigcup_{p\neq 1} U_{(pq)^c}| = 1$ or $|\bigcup_{k\neq 1;k\ell\neq ij} U_{k\ell}| = 1$ with $|U_{(ij)^c}| = 1$, $U_{(k\ell)^c} = \emptyset$ or $U_{(ij)^c} = \emptyset$, $|U_{(k\ell)^c}| \leq 2$ or $\bigcup_{k\neq 1;k\ell\neq ij} U_{k\ell} = \emptyset$ with $|U_{(ij)^c}| = 1$.

Hence, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $4 \leq |\bigcup_{i\neq 1} U_{ij} \cup U_{(ij)^c}| \leq 6$ with $|\bigcup_{i\neq 1} U_{(ij)^c}| \leq 1$ or $|\bigcup_{i\neq 1} U_{ij}| = 3$ with $|U_{ij} \cup U_{(ij)^c}| \leq 3$ and a unique $U_{(ij)^c} \neq \emptyset$ or $\bigcup_{i\neq 1} U_{ij} = 2$ with $|U_{(ij)^c}| = 1$.

Case 5.4 Assume $|U_{ij}| = 1$ for all $i \neq 1$. Then, $|U_{(ij)^c}| \leq 3$; otherwise, the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_{i'} \cup U_{(ij)^c}$ where $i' \notin \{1, i, j\}$ form $K_{3,7}$.

Suppose $|U_{k\ell}| = |U_{mn}| = 1$ for $k, m \neq 1$ and $k\ell, mn \neq ij$. If $U_{(ij)^c}, U_{(k\ell)^c}, U_{(mn)^c} \neq \emptyset$, then the sets $X = U_1 \cup U_2 \cup U_3$ and $Y = \{I_4, [I_{ij}, I_{(ij)^c}], [I_{k\ell}, I_{(k\ell)^c}], [I_{mn}, I_{(mn)^c}]\}$ form H_4 as a minor of $\mathbb{AG}(\mathcal{L})$, which is a contradiction. Assume that $|U_{(ij)^c}| = 3$. If $I \in U_{(k\ell)^c} \cup U_{(mn)^c}$, then $G_{18} = \mathbb{AG}(\mathcal{L}) - \{I, I_{k\ell}, I_{mn}, (I_i, I_j), (I_1, I_{i'}), (I'_1, I_{i'})\}$ contains $K_{6,3}$ with the partite sets $X = U_1 \cup U_{i'} \cup U_{(ij)^c}$ and $Y = U_i \cup U_j \cup U_{ij}$ and any N_2-embedding of $K_{3,6}$ has nine rectangular faces. Here, it is not hard to verify that all the left-out vertices and edges cannot be embedded into the nine rectangular faces so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $U_{(k\ell)^c} \cup U_{(mn)^c} = \emptyset$. Here, the graph $\mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I_{mn}\}$ is a subgraph of the graph in Figure 2a, and the suitable labels in Figure 2b give two different faces in the N_2-embedding of $\mathbb{AG}(\mathcal{L}) - \{I_{k\ell}, I_{mn}\}$ that contains the vertices $N(I_{k\ell})$ and $N(I_{mn})$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. Assume $|U_{(ij)^c}| \leq 2$. If $|U_{(ij)^c} \cup U_{(k\ell)^c}| \geq 4$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{(k\ell)^c}, I'_{(k\ell)^c}, I_{mn}, (I_i, I_j), (I_1, I_{i'}), (I'_1, I_{i'})\}$ has a similar structure to G_{15} so that we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Additionally, by Theorem 4ii, $\mathbb{AG}(\mathcal{L})$ is projective when $|\bigcup_{i\neq 1} U_{(ij)^c}| \leq 1$. For all of the remaining cases, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ can be verified by drawing the N_2-

embedding.
Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ when $2 \leq |\bigcup_{i\neq 1} U_{(ij)^c}| \leq 3$ with at least one of the sets' $U_{(ij)^c} = \emptyset$.

Suppose $|U_{k\ell}| = 1$ and $U_{mn} = \emptyset$ for $k, m \neq 1$ and $k\ell, mn \neq ij$. If $|U_{(ij)^c}| = 3$ and $U_{(k\ell)^c} \neq \emptyset$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{(k\ell)^c}, I_{k\ell}, (I_i, I_j), (I_1, I_{i'}), (I'_1, I_{i'})\}$ has a similar structure to G_{18}, and, if $|U_{(ij)^c}| = |U_{(k\ell)^c}| = 2$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{(k\ell)^c}, I'_{(k\ell)^c}, I_{k\ell}, (I_i, I_j), (I_1, I_{i'}), (I'_1, I_{i'})\}$ has a similar structure to G_{15} so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Further, $\mathbb{AG}(\mathcal{L})$ is projective if $|U_{(ij)^c} \cup U_{(k\ell)^c}| \leq 1$. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $|U_{(ij)^c} \cup U_{(k\ell)^c}| \in \{2, 3\}$.

Suppose $U_{k\ell}, U_{mn} = \emptyset$ for $k, m \neq 1$ and $k\ell, mn \neq ij$. Then, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $2 \leq |U_{(ij)^c}| \leq 3$.

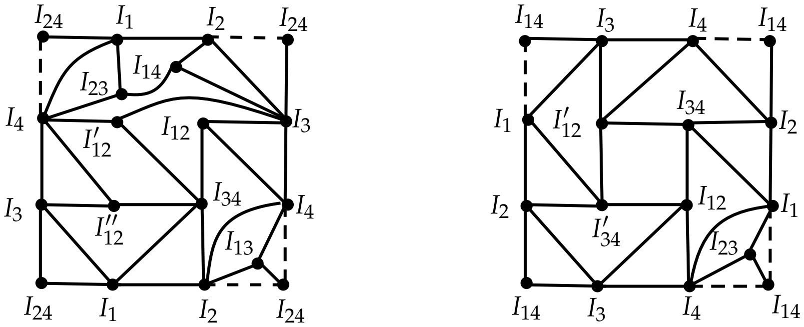
Case 6 Let $|\bigcup_{n=1}^{4} U_n| = 4$. Then, by Theorem 4(i), $|U_{ij} \cup U_{(ij)^c}| \geq 3$ for some $U_{ij}, U_{(ij)^c} \neq \emptyset$. Further, if $|U_{ij} \cup U_{(ij)^c}| \geq 6$ with $U_{ij}, U_{(ij)^c} \neq \emptyset$, then the subgraph induced by the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{k\neq i,j} U_k \cup U_{(ij)^c}$ contains one of the graph's $K_{3,7}$, $K_{4,6}$, or $K_{5,5}$ as a subgraph so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $3 \leq |U_{ij} \cup U_{(ij)^c}| \leq 5$ for some $U_{ij}, U_{(ij)^c} \neq \emptyset$.

(i) Suppose $|U_{ij} \cup U_{(ij)^c}| = 5$ for $U_{ij}, U_{(ij)^c} \neq \emptyset$. If either $|U_{ij}| = 3$ or $|U_{(ij)^c}| = 3$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{k\neq i,j} U_k \cup U_{(ij)^c}$ form $K_{4,5}$, which is a contradiction. So, either $|U_{ij}| = 4$ or $|U_{(ij)^c}| = 4$. With no loss of generality, assume that $|U_{ij}| = 4$. If $U_{k\ell}, U_{(k\ell)^c} \neq \emptyset$ for $k\ell \neq ij, (ij)^c$, then clearly $|\{i, j\} \cap \{k, \ell\}| = 1$ and $|\{m, n\} \cap \{k, \ell\}| = 1$ where $m, n \in \{1, 2, 3, 4\} \setminus \{i, j\}$. So, let us take $\{i, j\} \cap \{k, \ell\} = \{j\}$ and $\{m, n\} \cap \{k, \ell\} = \{m\}$. This implies that $(I_{k\ell}, I_i), (I_{(k\ell)^c}, I_m) \in E(\mathbb{AG}(\mathcal{L}))$. Then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_i, I_{k\ell}, I_{(k\ell)^c}\}$ contains $K_{5,3}$ with the partite sets $X = U_j \cup U_{ij}$ and $Y = U_m \cup U_n \cup U_{(ij)^c}$. Now, the path $I_i - I_{k\ell} - I_{(k\ell)^c}$ has to be embedded into a single face of any N_2-embedding of $K_{5,3}$. Further, the vertices I_i and $I_{(k\ell)^c}$ are adjacent to I_j and I_m. So, after embedding these four edges, the edge $(I_{k\ell}, I_n)$ cannot be embeded, which means $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $U_{(k\ell)^c} = \emptyset$ when $U_{k\ell} \neq \emptyset$ for all $k\ell \neq ij, (ij)^c$, and, in such cases, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

(ii) Suppose $|U_{ij} \cup U_{(ij)^c}| = 4$ for $U_{ij}, U_{(ij)^c} \neq \emptyset$. If $|U_{k\ell} \cup U_{(k\ell)^c}| \geq 3$ for $k\ell \neq ij$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{U_{k\ell} \cup U_{(k\ell)^c}\}$ contains a crosscap two graph $K_{5,3}$ or $K_{4,4}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{m\neq i,j} U_m \cup U_{(ij)^c}$. Since $|U_{k\ell} \cup U_{(k\ell)^c}| \geq 3$, we can take $|U_{k\ell}| \geq 2$. Notice that the path $I_{k\ell} - I_{(k\ell)^c} - I'_{k\ell}$ together with the edges $(I_{k\ell}, I_m), (I_{k\ell}, I_i), (I'_{k\ell}, I_m)$, and $(I'_{k\ell}, I_i)$ should be embedded into a single face of an N_2-embedding of $K_{5,3}$. Thereafter, the face cannot adopt the edges $(I_{(k\ell)^c}, I_j)$ and $(I_{(k\ell)^c}, I_n)$ where $n \notin \{i, j, m\}$, which implies that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|U_{k\ell} \cup U_{(k\ell)^c}| = 2$ for all $U_{k\ell}, U_{(k\ell)^c} \neq \emptyset$ with $k\ell \neq ij$ and $1 \leq i, j \leq 4$.

If $|U_{ij}| = 3$, then, by Figure 17a, we obtain $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. If not, then $|U_{ij}| = 2$. Suppose $|U_{k\ell} \cup U_{(k\ell)^c}| = |U_{mn} \cup U_{(mn)^c}| = 2$ for $U_{k\ell}, U_{(k\ell)^c}, U_{mn}, U_{(mn)^c} \neq \emptyset$ with $k\ell, mn \neq ij$. Then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{[I_{k\ell}, I_{(k\ell)^c}], [I_{mn}, I_{(mn)^c}]\}$ contains $K_{4,4}$ with the partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_{i'} \cup U_{j'} \cup U_{(ij)^c}$, where $i', j' \notin \{i, j\}$. Note that every face of any N_2-embedding of $K_{4,4}$ is rectangular, and the vertices $[I_{k\ell}, I_{(k\ell)^c}]$ and $[I_{mn}, I_{(mn)^c}]$ are adjacent to the four vertices $I_i, I_j, I_{i'}$, and $I_{j'}$. So, to embed the vertices $[I_{k\ell}, I_{(k\ell)^c}]$ and $[I_{mn}, I_{(mn)^c}]$, two distinct rectangular faces with boundaries $I_i, I_j, I_{i'}$, and $I_{j'}$ are required, which is a contradiction. Therefore, at least one $U_{(k\ell)^c} = \emptyset$ when $U_{k\ell} \neq \emptyset$ for $k\ell \neq ij$ and $1 \leq i \neq j \leq 4$. In this case, an N_2-embedding of $\mathbb{AG}(\mathcal{L})$ is given in Figure 17b.

(iii) Suppose $2 \leq |U_{ij} \cup U_{(ij)^c}| \leq 3$ for all $U_{ij}, U_{(ij)^c} \neq \emptyset$ with $1 \leq i \neq j \leq 4$. Then, by Theorem 4i, there exists $U_{k\ell}$ such that $U_{k\ell}, U_{(k\ell)^c} \neq \emptyset$ with $|U_{k\ell} \cup U_{(k\ell)^c}| = 3$ and $\bigcup_{mn\neq k\ell,(k\ell)^c; U_{mn}\neq\emptyset} U_{(mn)^c} \neq \emptyset$.



(**a**). $|U_{12}| = 3$ and $|U_{13} \cup U_{24}| = |U_{14} \cup U_{23}| = 2$ (**b**). $|U_{12}| = |U_{34}| = 2$ and $|U_{14} \cup U_{23}| = 2$

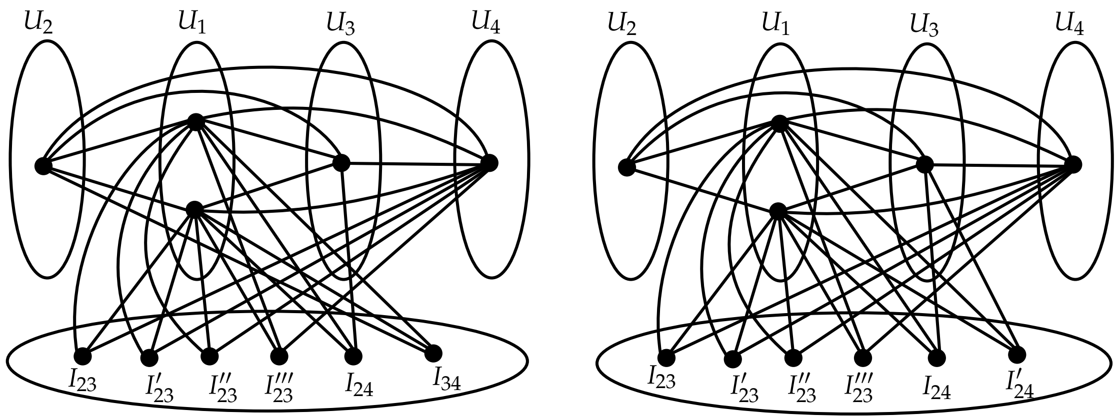
Figure 17. $|\bigcup_{n=1}^{4} U_n| = 4$ with $|U_{12} \cup U_{34}| = 4$.

Suppose $|U_{ij} \cup U_{(ij)^c}| = 3$ for all $1 \leq i \neq j \leq 4$. That is, $|U_{12} \cup U_{34}| = |U_{13} \cup U_{24}| = |U_{14} \cup U_{23}| = 3$. Without a loss of generality, we let $|U_{12}| = |U_{13}| = |U_{14}| = 2$. Now, consider the bipartite graph $G_{19} = \mathbb{AG}(\mathcal{L}) - \{(I_2, I_3), (I_2, I_4), (I_3, I_4), (I_2, I_{34}), (I_3, I_{24}), (I_4, I_{23})\}$ with the partite sets $X = U_1 \cup U_{12} \cup U_{13} \cup U_{14}$ and $Y = U_2 \cup U_3 \cup U_4 \cup U_{34} \cup U_{24} \cup U_{23}$. Note that $\tilde{\gamma}(G_{19}) = 2$ and the faces of any N_2-embedding of G_{19} have one of the following possibilities:

- Nine rectangular and two hexagonal faces;
- Ten rectangular faces and one octagonal face.

Since, in G_{19}, the only common neighbor for I_2 and I_{34} in X is I_1, no rectangular face has both I_2 and I_{34}. Therefore, the edge (I_2, I_{34}) should be embedded in a face of a length of more than four; so the edges are (I_3, I_{24}) and (I_4, I_{23}). Thus, we have to embed the three mutually disjoint edges of $\langle Y \rangle$ in either two hexagonal faces or one octagonal face. However, in any case, the faces may adopt at most two mutually disjoint edges of $\langle Y \rangle$, and, so, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. For the remaining cases, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. □

Remark 2. As an illustration, we consider the case (v)[a] in Theorem 5. Let $|U_1| = |U_2| = |U_3| = |U_4| = 1$ and $|U_{23}| = 4$. If $|U_{24}| = |U_{34}| = 1$, then the corresponding five-partite graph, as in Figure 18a, has a crosscap two. Additionally, if $|U_{24}| = 2$, then the crosscap of the corresponding five-partite graph, given in Figure 18b, is not equal to two. Moreover, the five-partite graph G in Figure 18b is minimal with respect to $\tilde{\gamma}(G) \neq 2$.



(**a**) A crosscap two 5-partite graph (**b**) A minimal 5-partite graph with crosscap $\neq 2$

Figure 18. Five-partite graphs.

6. Conclusions

The forbidden subgraphs for a crosscap two surface (a Klein bottle) are not known yet. In this regard, an open problem will be to determine a family of graphs that has a crosscap number two. This paper provides a class of r-partite graphs, where $2 \leq r \leq 5$, that can be both embedded and not embedded in a crosscap two surface. This was completed by using the classification of all lattices with at most four atoms whose annihilating-ideal graph has a crosscap two.

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