mathematics

Review

Nonassociative Algebras, Rings and Modules over Them

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Abstract: The review is devoted to nonassociative algebras, rings and modules over them. The main actual and recent trends in this area are described. Works are reviewed on radicals in nonassociative rings, nonassociative algebras related with skew polynomials, commutative nonassociative algebras and their modules, nonassociative cyclic algebras, rings obtained as nonassociative cyclic extensions, nonassociative Ore extensions of hom-associative algebras and modules over them, and von Neumann finiteness for nonassociative algebras. Furthermore, there are outlined nonassociative algebras and rings and modules over them related to harmonic analysis on nonlocally compact groups, nonassociative algebras with conjugation, representations and closures of nonassociative algebras, and nonassociative algebras and modules over them with metagroup relations. Moreover, classes of Akivis, Sabinin, Malcev, Bol, generalized Cayley–Dickson, and Zinbiel-type algebras are provided. Sources also are reviewed on near to associative nonassociative algebras and modules over them. Then, there are the considered applications of nonassociative algebras and modules over them in cryptography and coding, and applications of modules over nonassociative algebras in geometry and physics. Their interactions are discussed with more classical nonassociative algebras, such as of the Lie, Jordan, Hurwitz and alternative types.

Keywords: nonassociative algebra; module; structure; extension; ideal; radical; cohomology

MSC: Primary 17A30; 16D70; Secondary 17A36; 17A60; 17D10; 18E25; 18G60

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Citation: Ludkowski, S.V. Nonassociative Algebras, Rings and Modules over Them. *Mathematics* **2023**, *11*, 1714. https://doi.org/10.3390/math11071714

Academic Editor: Li Guo

Received: 16 February 2023
Revised: 23 March 2023
Accepted: 27 March 2023
Published: 3 April 2023

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1. Introduction

Nonassociative rings and algebras appear naturally in mathematics and attract great attention. This area is of great importance. We recall that it was W.R. Hamilton who first introduced a noncommutative ring in 1843 with his invention of the quaternion skew field [1]. A little later, J.T. Graves, in a letter to Hamilton dated 26 December 1843, and A. Cayley first studied a nonassociative ring, which is known today as the octonion algebra [2–4]. Nonassociative rings, algebras and modules over them have been intensively studied in recent years. This review is intended to describe main trends in this area.

This review is devoted to nonassociative rings, algebras and modules over them. Algebras over rings or fields are considered. Mostly rather new results on nonassociative algebras and their modules are reviewed below, though the main preceding results are recalled. The main actual and recent trends in this area are described.

Works are reviewed on radicals in nonassociative rings, nonassociative algebras related with skew polynomials, commutative nonassociative algebras and their modules, nonassociative cyclic algebras, rings obtained as nonassociative cyclic extensions, nonassociative Ore extensions of hom-associative algebras and modules over them, and von Neumann finiteness for nonassociative algebras. Classes of Akivis, Sabinin, Malcev, Bol, generalized Cayley–Dickson, and Zinbiel-type algebras are provided. Furthermore, there are outlined nonassociative algebras and rings and modules over them related to harmonic analysis on nonlocally compact groups, nonassociative algebras with conjugation, representations and closures of nonassociative algebras, nonassociative algebras and modules over them with metagroup relations, and near to associative nonassociative algebras and modules over them.

Then there are considered applications of nonassociative algebras and modules over them in cryptography and coding, and applications of modules over nonassociative algebras in geometry and physics. Their interactions are discussed with more classical nonassociative algebras, such as of the Lie, Jordan, Hurwitz and alternative types.

2. Principles of General Nonassociative Algebras and Rings

Remark 1. *The basic principles of general nonassociative algebras over fields or commutative associative rings are contained in [5] and references therein. They are nonassociative relative to multiplication. To avoid a misunderstanding, we recall, that if an element e (or b) in an algebra A over a commutative associative unital ring F is such that $ea = a$ (or $ab = a$) for each a in A, then e (or b correspondingly) is called a left (or right correspondingly) unit element (or identity). If A contains both a left unit e and a right unit b, then $e = b = eb$ is a two-sided unit element (identity).*

Proposition 1 (Section 2.3 in [5])**.** *Assume that B and J are solvable ideals of an algebra A. Then $B + J$ is a solvable ideal of A. Particularly, if A is finite-dimensional over a field F, then A has a unique maximal solvable ideal S. Moreover, the only solvable ideal of A/S is 0.*

Theorem 1 (Jacobson, Section 2.5 in [5])**.** *Assume that A is a finite-dimensional algebra over a field F of zero characteristic such that A is a direct sum $A = J_1 \oplus ... \oplus J_n$ of simple ideals J_l, and A contains a left (or right) identity. Then, every derivation D of A is inner.*

Theorem 2 (Albert, Section 5.1 in [5])**.** *If B is a finite power-associative division ring of characteristic $char(B) \notin \{2, 3, 5\}$, then B is a field.*

Theorem 3 (Albert, Section 5.4 in [5])**.** *Suppose that A is a finite-dimensional power-associative algebra over a field F of characteristic $char(F) \neq 2$ such that the following hold:*

(i) *There exists an (associative) trace form (x, y) defined on A;*
(ii) *$(e, e) \neq 0$ for each idempotent e in A;*
(iii) *$(x, y) = 0$ if $x \cdot y := (xy + yx)/2$ is nilpotent, where $x \in A$, $y \in A$.*

Then, the nilradical P of A coincides with the nilradical of A^+, and is the radical $A^\perp$ of the trace form (x, y). Moreover, the semisimple power-associative quotient algebra $G = A/P$ satisfies (i)–(iii) with (x, y) nondegenerate, and the following:

(iv) *$G = G_1 \oplus ... \oplus G_n$, where G_l is a simple algebra for each $l = 1, ..., n$;*
(v) *G is flexible;*
(vi) *G^+ is a semisimple Jordan algebra;*
(vii) *G_l^+ is a simple (Jordan) algebra for each $l = 1, ..., n$.*

Remark 2. *Varieties of algebras which may be nonassociative were studied in [6,7] and references therein. Identities and hyperidentities in varieties of algebras were investigated, for example, in [8–11] and references in them. It also is related with universal algebra [12–14]. Nonassociative lattice ringoids and their skew morphisms were studied in [15].*

Other relevant principles of general nonassociative algebras and rings are recalled in other sections devoted to specific classes of nonassociative algebras and rings.

3. Akivis Algebras

Definition 1. *Let A be a vector space over a field F endowed with an anticommutative bilinear operation $< x, y >$ and a trilinear operation $< x, y, z >$ satisfying the following identity:*

$<< x, y >, z > + << y, z >, x > + << z, x >, y >=< x, y, z > + < y, z, x > + < z, x, y > - < y, x, z > - < x, z, y > - < z, y, x >$ for each x, y and z in A. Then $(A, < \cdot, \cdot >, < \cdot, \cdot, \cdot >)$ is called an Akivis algebra.

Remark 3. *These algebras were first studied by M. A. Akivis as tangent algebras of local analytic loops [16]. If you take a nonassociative algebra B over the field F and put* $< x, y >= [x, y] = xy - yx$ *to be a commutator, and* $< x, y, z >= (x, y, z) = (xy)z - x(yz)$ *to be an associator, then the vector space B over F becomes an Akivis algebra under these operations. It also is denoted by* $Ak(B)$*. It was proved that for each Akivis algebra A over F, there exists a nonassociative algebra B over F such that there exists an isomorphic embedding of A into* $Ak(B)$ *[17,18]. Moreover, for the universal enveloping algebra* $U(A)$ *of the Akivis algebra over the field F, a unique algebra homomorphism exists* $\Delta : U(A) \to U(A) \bigotimes FU(A)$ *such that* $\Delta(b) = b \otimes 1 + 1 \otimes b$ *for each b in A. An element* $v \in U(A)$ *is called primitive if* $\Delta(v) = v \otimes 1 + 1 \otimes v$ *[19].*

Let $e_1, e_2, ...$ *be a linear basis of the Akivis algebra over the field F. It is useful to take the set of words* $V = \{e_i, e_ie_j, (e_ie_j)e_k : i \le j \le k\}$ *in the universal enveloping algebra* $U(A)$ *and put* $|e_i| = 1$*,* $|e_ie_j| = 2$*,* $|(e_ie_j)e_k| = 3$*. By* V^**, it is denoted the set of all nonassociative words in the alphabet V, including the unit 1 considered the empty word. Then by* V^0*, it is denoted the set of all words from* V^* *not containing subwords, such as* v_1v_2*, where* $v_1 \in V$ *and* $v_2 \in V$ *with* $|v_1| + |v_2| \le 3$*. The elements of* V^0 *are called* v^0*-words. It was proved in [17,18] that a basis of the algebra* $U(A)$ *is formed by the* v^0*-words. For an algebra D and a subset M in D,* $alg_D < M >$ *and* $idl_D < M >$ *denote the subalgebra and the ideal in D generated by M, correspondingly.*

Remark 4. *Assume that* $A = Ak < X >$ *is a free Akivis algebra over a field F with the set of free generators* $X = \{x_1, x_2, ...\}$*. Then a degree function d on A is defined such that* $d(x_i) = 1$ *for each* $i \ge 1$*, and d for any homogeneous elements u, v, w of A with* $[u, v] \ne 0$ *and* $(u, v, w) \ne 0$ *is such that* $d[u, v] = d(u) + d(v)$*,* $d(u, v, w) = d(u) + d(v) + d(w)$*. Therefore, it induces the decomposition* $A = A_1 \oplus A_2 \oplus ...$*, where* A_j *is the space of homogeneous elements of degree* $j \ge 1$*. This implies that the universal enveloping algebra* $U(A) = F\{X\}$ *is a free nonassociative algebra with the set of free generators X [18]. Recall that a variety of algebras is called Schreier if every subalgebra of a free algebra in this variety is also free [20].*

Theorem 4 ([21])**.** *The variety of Akivis algebras is Schreier.*

Theorem 5 ([21])**.** *The word problem is decidable for the variety of Akivis algebras.*

Theorem 6. *There exists a set* X_r *of all right-ordered (i.e., right-normed) words of the type* $u_m = u_m(x_{i_1}, ..., x_{i_m})$*, where* $u_m = u_{m-1}x_m$ *for each* $m \ge 2$ *with* $u_1 = x_{i_1}$*, where* $i_1 \le i2 \le ... \le i_m$*,* $1 \le m$*,* $u_0 = 1$*,* $x_{i_k} \in X$ *for each k (see also Remark 2).*

Definition 2 ([21,22])**.** *Suppose that D is a free unital nonassociative algebra over* $X \cup Y \cup \{z\}$*, where* $X = \{x_1, x_2, ...\}$ *and* $Y = \{y_1, y_2, ...\}$*. Take a unique homomorphism of unital algebras* $\Delta : D \to D \otimes D$ *such that* $v \mapsto \sum v_{(1)} \otimes v_{(2)}$*,* $X \cup Y \cup \{z\} \subseteq Prim(D)$*, where* $Prim(D)$ *denotes the set of all primitive elements of D. For some* $x_1, ..., x_m$ *in X and* $y_1, ..., y_n$ *in Y it is put* $u_m(x_1, ..., x_m) = u_m$ *with* $u_1 = x_1$*,* $u_2 = x_1x_2, ...,$ $u_m = u_{m-1}x_m$*,* $v_n(y_1, ..., y_n) = v_n$ *with* $v_1 = y_1$*,* $v_2 = y_1y_2$*, ...,* $v_n = v_{n-1}y_n$*. Then, let* $q_{0,0}(1, 1, z) = 0$*,* $q_{0,n}(1, v_n, z) = 0$*,* $q_{m,0}(u_m, 1, z) = 0$ *and recursively* $q_{m,n}(u_m, v_n, z)$ *are defined for each* $m \ge 1$ *and* $n \ge 1$ *such that* $q_{1,1}(x_1, y_1, z) = (x_1, y_1, z)$ *and* $(u_m, v_n, z) = \sum_{k,l}(u_{k,(1)}q(u_{m-k,(2)}, v_{n-l,(2)}, z))v_{l,(1)}$*, where* (u_m, v_n, z) *denotes the associator of* u_m*,* v_n *and z.*

Theorem 7 ([21])**.** *Assume that D is a unital algebra over a field F of zero characteristic, and* $\Delta : D \to D \bigotimes_F D$ *is a nontrivial homomorphism of algebras. Suppose that the algebra D is generated by a set M of* Δ*-primitive elements, and* $P(M)$ *is the minimal subspace of D containing M and closed with respect to primitive operations* $q_{m,n}$*. If* $\{e_1, e_2, ...\}$ *is a basis of* $P(M)$*. Then, the set of right-ordered words of the type* $u_m(e_{i_1}, ..., e_{i_m})$ *with* $0 \le m$ *and* $i_1 \le i_2 \le ... \le i_m$*,* $0 \le m$*, forms a basis of the algebra D.*

4. Nonassocative Algebras of Sabinin, Malcev, and Bol Types

Definition 3. *A vector space V over a field F is called a Sabinin algebra if there are multilinear operations on it $< x_1, ..., x_m; y, z >$ with $m \geq 0$ satisfying the following identities:*

$< x_1, ..., x_m; y, z >= - < x_1, ..., x_m; z, y >$;

$< x_1, ..., x_r, a, b, x_{r+1}, ..., x_m, y, z > - < x_1, ..., x_r, b, a, x_{r+1}, ..., x_m, y, z > + \sum_{k=0}^{r} \sum_{\alpha} < x_{\alpha_1}, ..., x_{\alpha_k}, < x_{\alpha_{k+1}}, ..., x_{\alpha_r}; a, b >, ..., x_m; y, z >= 0$;

$\sigma_{x,y,z}(< x_1, ..., x_r, x; y, z > + \sum_{k=0}^{r} \sum_{\alpha} < x_{\alpha_1}, ..., x_{\alpha_k}, < x_{\alpha_{k+1}}, ..., x_{\alpha_r}; y, z >, x >) = 0$

for each $x_1, .., x_m, y_1, ..., y_n$, y, z in V, where α denotes a bijection $\alpha : \{1, .., r\} \to \{1, ..., r\}$ with $\alpha_1 < ... < \alpha_k$, $\alpha_{k+1} < ... < \alpha_r$ for $0 \leq k \leq r$; $\sigma_{x,y,z}$ denotes the cyclic sum by x, y and z.

Remark 5. *Frequently, algebras satisfying the definition above and with a multioperator Φ such that*

$\Phi(x_1, ..., x_m; y_1, ..., y_n) = \Phi(x_{s(1)}, ..., x_{s(m)}; y_{q(1)}, ..., y_{q(n)})$ for each $x_1, .., x_m, y_1, ..., y_n$ in V, $s \in S_m$, $q \in S_n$, are also called Sabinin algebras, where S_n denotes the symmetric group of the set $\{1, ..., n\}$, where $m \geq 1$ and $n \geq 2$.

In particular, a Malcev algebra $(Y, [\cdot, \cdot])$ over a field F of the characteristic $char(F) \neq 2$ is a vector space Y with a skew-symmetric product $[,]$ such that $[J(x, y, z), x] = J(x, y, [x, z])$ for each x, y and z in Y, where $J(x, y, z) = [[x, y], z] - [[x, z], y] - [x, [y, z]]$ denotes the Jacobian of x, y and z. The Malcev algebras can also be generated with the help of tangent spaces of smooth Moufang loops.

Recall that a Lie triple system $(V, [\cdot, \cdot, \cdot])$ is a vector space V over a field V supplied with a trilinear operation $[\cdot, \cdot, \cdot]$ such that $[x, x, y] = 0$, $[x, y, z] + [y, z, x] + [z, x, y] = 0$ and $[x, y, [u, v, z]] = [[x, y, u], v, z] + [u, [x, y, v], z] + [u, v, [x, y, z]]$ for every u, v, x, y and z in V.

Then a (left) Bol algebra $(V, [\cdot, \cdot, \cdot], [\cdot, \cdot])$ consists of a Lie triple system $(V, [\cdot, \cdot, \cdot])$ supplied with a bilinear skew-symmetric operation $[x, y]$ satisfying the identities $[x, y, [z, u]] = [[x, y, z], u] + [z, [x, y, u]] + [z, u, [x, y]] + [[x, y], [z, u]]$ for every u, x, y and z in V. The left Bol algebra can also be generated with the help of the tangent space of a smooth left Bol loop B. The left Bol loop satisfies the left Bol identity $x(y(xz) = (x(yx))z$ for every x, y and z in B.

If there is the Malcev algebra $(Y, [\cdot, \cdot])$ over the field F of the characteristic $char(F) \notin \{2, 3\}$ and if we put $[x, y, z] = [[x, y], z] - \frac{1}{3}J(a, b, c)$, then it provides the Bol algebra $(Y, [\cdot, \cdot, \cdot], [\cdot, \cdot])$.

For a nonassociative algebra X over the field F of the characteristic $char(F) \neq 2$, the generalized alternative nucleus defined by the following is useful:

$N_{alt}(X) = \{x \in X : \forall y \in X, \forall z \in X, (x, y, z) = -(y, x, z) = (y, z, x)\}$, where $(x, y, z) = (xy)z - x(yz)$ denotes the associator.

Recall that for a Lie algebra L over the field F, a universal enveloping algebra arises as the quotient algebra $U(L) = T(L)/J$, where $T(L)$ is a tensor algebra of L considered as the vector space over F, where J is a two-sided ideal in $T(L)$ generated by all elements of the form $\{x \otimes y - y \otimes x - [x, y] : x \in L, y \in L\}$. There, $T(L) = F \oplus \bigoplus_{n=1}^{\infty} L^{\otimes n}$ is the standard tensor algebra of L with the usual associative tensor product $X \otimes Y = X \otimes_F Y$ of vector spaces X and Y over F, where $X^{\otimes n+1} = X \otimes X^{\otimes n}$ for each $n \geq 1$, $X^{\otimes 1} = X$, $X^{\otimes 0} = F$. The operations in L and $U(L)$ are related by $[x, y] = xy - yx$ for each x and y in L, where $U(L) \times U(L) \ni (x, y) \mapsto xy \in U(L)$ denotes multiplication on $U(L)$ (see, for example, Section 1.9 in [23]). For the Akivis algebra A, its universal enveloping algebra $U(A)$ was considered in the preceding section.

For the left Bol algebra $(V, [\cdot, \cdot, \cdot], [\cdot, \cdot])$ over the field F of the characteristic $char(F) \neq 2$, there exists a universal enveloping algebra $U(V)$ such that $V \subseteq U(V)$. The operations in V and $U(V)$ are related by $[x, y] = xy - yx$, $[x, y, z] = x(yz) - y(xz) - z(xy - yx)$ for each x, y and z in V according to [24]. Moreover, there is the embedding $V \subseteq N_{l,alt}(U(V))$, where $N_{l,alt}(U(V)) = \{x \in U(V) : \forall y \in U(V), \forall z \in U(V), (x, y, z) = -(y, x, z)\}$.

For the Malcev algebra $(Y, [\cdot, \cdot])$ over the field F of the characteristic $char(F) \notin \{2, 3\}$, there exists a universal enveloping algebra $U(Y)$ such that $Y \subseteq U(Y)$. The operations in Y and $U(Y)$ are related by $[x, y] = xy - yx$ for each x and y in Y, and $Y \subseteq N_{alt}(U(Y))$ [25,26]. Particularly, if the Malcev algebra $(Y, [\cdot, \cdot])$ is considered the left Bol algebra, then the universal envelopes of Y as Malcev and Bol algebras are isomorphic. Studies of these algebras also were accomplished with the help of co-algebras and nonassociative bi-algebras.

Definition 4. *Let X be a vector space over a field F, and let there be two maps $\Delta : X \to X \otimes X$ and $\epsilon : X \to F$ such that $(Id \otimes \epsilon)\Delta = Id$ and $(\epsilon \otimes Id)\Delta = Id$, where $X \otimes F$ and $F \otimes X$ are as usually isomorphic with X, where $X \otimes X = X \otimes FX$, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ or briefly $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. Then (X, Δ, ϵ) is called a co-algebra. The co-algebra (X, Δ, ϵ) is called coassociative if $(\Delta \otimes Id)\Delta = (Id \otimes \Delta)\Delta$. If $\tau\Delta = \Delta$, where $\tau(x \otimes y) = y \otimes x$ for each x and y in X, then the co-algebra is called cocommutative.*

If a co-algebra (Y, Δ, ϵ) possesses also a F-bilinear product p_Y, $Y \times Y \ni (x, y) \mapsto p_Y(x, y) \in Y$, briefly denoted by $p_Y(x, y) = xy$ such that $\Delta(xy) = \sum x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$ and $\epsilon(xy) = \epsilon(x)\epsilon(y)$ for each x, y in Y, then $(Y, \Delta, \epsilon, p_Y)$ is called a (nonunital) bialgebra over F. If, additionally, there exists a F-linear mapping $u : F \to Y$ such that $u(1_F) = 1 \in Y$, $1x = x = x1$, $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$, then $(Y, \Delta, \epsilon, p_Y, u)$ is called a unital bialgebra.

Let $(Y, \Delta, \epsilon, p_Y)$ be the bialgebra over F with bilinear operations $Y \times Y \ni (x, y) \mapsto x \setminus y \in Y$ and $Y \times Y \ni (x, y) \mapsto x/y \in Y$ called the left and right division such that $\sum x_{(1)} \setminus (x_{(2)}y) = \sum x_{(1)}(x_{(2)} \setminus y)$ and $\sum(yx_{(1)})/x_{(2)} = \sum(y/x_{(1)})x_{(2)}$ for each x and y in Y. Then $(Y, \Delta, \epsilon, p_Y, \setminus, /)$ is called an H-bi-algebra over F. Similarly $(Y, \Delta, \epsilon, p_Y, u, \setminus, /)$ is called a unital H-bi-algebra over F. A co-algebra is called connected, if the dimension of its coradical is one.

Such algebras exist as the following proposition demonstrates.

Proposition 2 ([22])**.** *Let $(Y, \Delta, \epsilon, p_Y, u)$ be a co-associative unital bi-algebra over F. If the co-algebra (Y, Δ, ϵ) is connected, then there exists a (unique) structure of the H-bi-algebra $(Y, \Delta, \epsilon, p_Y, u, \setminus, /)$ on $(Y, \Delta, \epsilon, p_Y, u)$.*

The operations Δ, $\setminus$ and $/$ possess identities on the H-bi-algebra according to the following proposition.

Proposition 3 ([22])**.** *Assume that $(Y, \Delta, \epsilon, p_Y, \setminus, /)$ is a co-associative H-bi-algebra over F. Then $\Delta(x \setminus y) = \sum x_{(2)} \setminus y_{(1)} \otimes x_{(1)} \setminus y_{(2)}$ and $\Delta(x/y) = \sum x_{(1)}/y_{(2)} \otimes x_{(2)}/y_{(1)}$.*

Remark 6. *For a co-associative and cocommutative co-algebra (X, Δ, ϵ) over a field F of a type $\mathcal{G}$ and any operation $\{f, k\} \in \mathcal{G}$ there exists a homomorphism of co-algebras $f : X^{\otimes k} \to X$, where $\mathcal{G}$ denotes a set of operations on X, where $X^{\otimes 0} = F$, $k \geq 0$. Then for any co-associative co-algebra V over F, the vector space $Hom_F(V, X)$ is a $\mathcal{G}$-algebra by putting $f(a_1, ..., a_n)(v) = \sum f(a_1(v_{(1)}), ..., a_n(v_{(n)}))$ if $n \geq 1$, $f(1)(v) = \epsilon(v)f(1)$ if $n = 0$, for every $a_1, ..., a_n$ in X, $v \in V$. On the co-algebra $X^{\otimes m}$ with $m \geq 1$, there are the distinguished maps $\epsilon_i(a_1 \otimes ... \otimes a_m) = \epsilon_i(a_1, ..., a_m) = \epsilon(a_1) \otimes ... \otimes \hat{\epsilon}(a_i) \otimes ... \otimes \epsilon_m(a_m)$, where $\hat{\epsilon}(a_i)$ means that this factor is omitted, where $1 \leq i \leq m$. This means that for each $n \leq m$, a homomorphism $l_{n,m}$ exists (which is called the linearizing map) from the term algebra $T(Y)$ on $Y = \{x_1, x_2, ..., x_n\}$ of type $\mathcal{G}$ into $Hom_F(X^{\otimes m}, X)$ sending x_i to ϵ_i for each i. If $l_{n,m}(p) = l_{n,m}(q)$ for some $n \leq m$ and p, q in $T(Y)$, then it is said that X satisfies the linearization of the identity $p \approx q$.*

Theorem 8 ([22])**.** *Assume that Υ is a set of identities and $p \approx q$ is a consequence of Υ. If the co-associative and cocommutative co-algebra (X, Δ, ϵ) over the field F of the type $\mathcal{G}$ satisfies the linearization of all identities in Υ, then X satisfies the linearization of the identity $p \approx q$.*

Theorem 9 ([22])**.** *If $(B, [\cdot, \cdot, \cdot], [\cdot, \cdot])$ is a left Bol algebra over a field F of characteristic $char(F) \neq 2$ and $U(B)$ is its universal enveloping algebra, then $U(B)$ satisfies the following $\sum x_{(1)}(y(x_{(2)}z)) = \sum(x_{(1)}(yx_{(2)}))z$ for each x, y, z in $U(B)$.*

Theorem 10. *If $(M, [\cdot, \cdot])$ is the Malcev algebra over the field F of characteristic $char(F) \notin \{2, 3\}$ and $U(M)$ is its universal enveloping algebra, then $\sum x_{(1)}(y(x_{(2)}z)) = \sum(x_{(1)}y)x_{(2)})z$ for each x, y, z in $U(M)$.*

Remark 7. *It was proved in [21] that the set of primitive elements of any bialgebra is closed under the usual commutator $[\cdot,\cdot]$ and $q_{m,n}(u_m, v_n, z)$ for each $m \geq 1$ and $n \geq 1$. Then, for a nonassociative algebra D over a field F of characteristic zero, it is possible to consider the operations*

$< y, z >= -[y, z]$,

$< x_1, ..., x_m; y, z >= -q_{m,1}(u_m, y, z) + q_{m,1}(u_m, z, y)$ with $u_m = u_m(x_1, ..., x_m)$ and $m \geq 1$,

$\Phi(x_1, ..., x_m; y_1, ..., y_n) = \frac{1}{m!n!}\sum_{\xi \in S_m, \eta \in S_n} q_{m,n-1}(u_m(x_{\xi(1)}, ..., x_{\xi(m)}), v_{n-1}(y_{\eta(1)}, ..., y_{\eta(n-1)}), y_{\eta(n)})$ for all $m \geq 1$ and $n \geq 2$. Let $\mathcal{Y}(D)$ denote the vector space D over F endowed with the operations $<, >$ and Φ. According to the work [21], $\mathcal{Y}(D)$ is a Sabinin algebra. Moreover, if D is a bi-algebra, then $Prim(D)$ is a Sabinin subalgebra of $\mathcal{Y}(D)$. Sabinin and Miheev [27,28] demonstrated that if there are a Lie algebra L, a subalgebra H and a vector space V with $L = H \oplus V$, then V can be supplied with the Sabinin algebra structure induced from the product on L.

Take a Sabinin algebra $(V, < \cdot, ..., \cdot >)$ over F and the quotient algebra $\tilde{S}(V) = T(V)/span_F(x[a, b]y + \sum x_{(1)} < x(2); a, b]y : x \in T(V), y \in T(V), a \in V, b \in V)$, where $\pi : T(V) \to \tilde{S}(V)$ is the quotient map. For $x \in T(V)$, it also is written shortly as $\bar{x}$ instead of $\pi(x)$.

Theorem 11. *(Poincare–Birkhoff–Witt). Let $\{b_j : j \in J\}$ be a totally ordered basis of V. Then $\{\overline{b_{j_1}...b_{j_n}} : j_1 \leq j_2 \leq ... \leq j_n, j_1 \in J, ..., j_n \in J, 0 \leq n\}$ is a basis of $\tilde{S}(V)$.*

Proposition 4. *$(\tilde{S}(V), \Delta, \epsilon, \cdot, u)$ is a co-associative, cocommutative connected unital bi-algebra and $V \subseteq Prim(\tilde{S}(V))$.*

Corollary 1. *There exist unique operations and / such that $(\tilde{S}(V), \Delta, \epsilon, \cdot, u, \ , /)$ is an H-bi-algebra.*

Theorem 12 ([22])**.** *Assume that $(V, < \cdot, ..., \cdot >, \Phi)$ is a Sabinin algebra over a field F of zero characteristic. Then there exists a unital algebra $U(V, < \cdot, ..., \cdot >, \Phi)$ and a monomorphism of Sabinin algebras $\mu : V \to \mathcal{Y}(U(V, < \cdot, ..., \cdot >, \Phi))$. Moreover, for any unital algebra D and any homomorphism of Sabinin algebras $\psi : V \to \mathcal{Y}(D)$, there exists a unique homomorphism of unital algebras $\bar{\psi} : U(V, < \cdot, ..., \cdot >, \Phi) \to D$ with $\psi = \bar{\psi} \circ \mu$.*

Corollary 2. *(Milnor–Moore). A cocommutative connected unital H-bi-algebra X over a field F of zero characteristic is isomorphic with the universal enveloping algebra $U(Prim(H))$ of the Sabinin subalgebra $Prim(H)$ in $\mathcal{Y}(X)$.*

5. Radicals in Nonassociative Rings

Remark 8. *Suppose that there is a universal class W of rings which may be nonassociative and $A \subseteq W$. Then, there exists the lower radical class LA determined by A in W [29]. Moreover, if A is a hereditary class (that is, if $K \in A$ and J is an ideal of K, then $J \in A$), then LA is also hereditary [30–33]. For $A \subseteq W$, $R_1(A)$ denotes the homomorphic closure of A. Then by induction, assume that β is an ordinal, $\beta > 1$, and the classes $R_\alpha(A)$ are defined for each $\alpha < \beta$. If β is not a limit ordinal, then $R_\beta(A) = \{K \in W : J \in R_{\beta-1}(A), K/J \in R_{\beta-1}(A), J < K\}$. If β is a limit ordinal, then it is put $R_\beta(A) = \{K \in W : K$ contains a chain $\{J_\gamma\}$ of ideals such that $\forall \gamma, J_\gamma \in \bigcup_{\alpha<\beta} R_\alpha(A), \& K = \bigcup_\gamma J_\gamma\}$. For the given W and A it is then put $R(A) = \bigcup_\delta R_\delta(A)$.*

Theorem 13 ([34])**.** *Let W be a universal class and let $A \subseteq W$. Then, A is a radical class in W if, and only if, the following conditions are satisfied: (i) A is homomorphically closed, (ii) $J \in A$, $K/J \in A$, then $K \in A$, (iii) the union of a chain of A-ideals of a W-ring K is again an A-ideal of K.*

Theorem 14 ([35])**.** *$R(A) = L(A)$.*

Theorem 15 ([30]). *Assume that $A \subseteq W$, where W is some universal class. Then, if A is hereditary, so is $L(A)$.*

Theorem 16 ([35]). *If A_1 and A_2 are homomorphically closed, hereditary classes of W-rings, then $L(A_1 \cap A_2) = L(A_1) \cap L(A_2)$.*

6. Nonassociative Algebras Related to Skew Polynomials

Remark 9. *Skew polynomial rings arise with the help of a unital associative ring S and a ring endomorphism σ of S. There is considered an additive map $\delta : S \to S$ such that $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for each x and y in S. That is, δ is a left σ-derivation of S. The skew polynomial ring $R = S[t;\sigma,\delta]$ consists of the set of all skew polynomials $f(t) = a_0 + a_1 t + ... + a_n t_n$, where $a_j \in S$ for each j, with term-wise addition and multiplication $ta = \sigma(a)t + \delta(a)$ for each $a \in S$ [36]. By induction this provides $at^n bt^m = \sum_{j=0}^{n} a(\Delta_{n,j} b)t^{m+j}$ for every a and b in S, where the map $\Delta_{n,j}$ is defined recursively $\Delta_{n,j} = \delta(\Delta_{n-1,j}) + \sigma(\Delta_{n-1,j-1})$, with $\Delta_{0,0} = id_S$, $\Delta_{1,0} = \delta$ and $\Delta_{1,1} = \sigma$. Particularly, if $\delta = 0$, then $\Delta_{n,n} = \sigma^n$. The usual ring of left polynomials is $S[t] = S[t;id,0]$. It is useful to put $Fix(\sigma) = \{a \in S : \sigma(a) = a\}$ and $Const(\delta) = \{a \in S : \delta(a) = 0\}$. A degree of a skew polynomial $f(t) = a_0 + a_1 t + ... + a_n t_n$ in R with $a_n \neq 0$ is defined as $deg(f) = n$, where $deg(0) = -\infty$. Therefore, $deg(gh) \leq deg(g) + deg(h)$ for each g and h in R. The equality is achieved if h has an invertible leading coefficient, or g has an invertible leading coefficient and σ is injective, or if S is a division ring. The skew polynomial f in R is called irreducible in R if it is not a unit and it has no proper factors, that is, if there do not exist g and h in R with $1 \leq deg(g) < \deg(f)$, $1 \leq deg(h) < deg(f)$ such that $f = gh$.*

Remark 10. *In this section, nonassociative algebras are unital over a unital commutative associative ring F. For the F-algebra A, associativity in A is estimated by the associator $(x,y,z) = (xy)z - x(yz)$. The left nucleus of A is defined as $N_l(A) = \{x \in A : (x,A,A) = 0\}$, the middle nucleus as $N_m(A) = \{x \in A : (A,x,A) = 0\}$ and the right nucleus as $N_r(A) = \{x \in A : (A,A,x) = 0\}$. Therefore, $N_l(A)$, $N_m(A)$ and $N_r(A)$ are associative subalgebras of A. Their intersection $N(A) = N_l(A) \cap N_m(A) \cap N_r(A)$ is the nucleus of A. This implies that $N(A)$ is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x, y, z is in $N(A)$. The commuter of A is $Comm(A) = \{x \in A : xy = yx \forall x \in A, \forall y \in A\}$ and the center of A is $\mathcal{C}(A) = N(A) \cap Comm(A)$ [5]. A nonassociative ring $A \neq 0$ (or an algebra $A \neq 0$ over a field F) is called a left (or right) division ring (or algebra correspondingly) if the left (or right correspondingly) multiplication operator L_a (or R_a correspondingly) is a bijective map for each $a \neq 0$ in A, where $L_a(x) = ax$, $R_a(x) = xa$ for each a and x in A. The nonzero ring (or the algebra) A is called a division ring (or algebra correspondingly) if it is the left and right division ring (or algebra correspondingly). This means that the division ring does not have zero divisors. If A is a finite-dimensional algebra over the field F, then A is a division algebra over F if and only if A has no zero divisors [5]. A nonassociative nonzero ring A has no zero divisors if and only if R_a and L_a are injective for each nonzero a in A. Notice that every algebra A is a right $N_r(A)$-module such that the left multiplication L_a is right $N_r(A)$-linear for each a in A.*

Assume that $f(t) \in R = S[t;\sigma,\delta]$ has an invertible leading coefficient $a_m \in S^\times = S - \{0\}$ with $m \geq 1$. Then for all $g(t) \in R$ of degree $l \geq m$, there exist uniquely determined $r(t)$ and $q(t)$ in R with $deg(r) < deg(f)$ such that $g(t) = q(t)f(t) + r(t)$. Moreover, if $\sigma \in Aut(D)$, then there exist uniquely determined $r_1(t)$ and $q_1(t)$ in R with $deg(r_1) < deg(f)$ such that $g(t) = f(t)q_1(t) + r_1(t)$ by Proposition 1 in [37]. So, there exist the remainder $mod_r f$ of right division by f and the remainder $mod_l f$ of left division by f. This means that the skew polynomials of degree less than m canonically represent the elements of the (left or right) $S[t;\sigma,\delta]$-modules $S[t;\sigma,\delta]/(S[t;\sigma,\delta]f)$ and $S[t;\sigma,\delta]/(fS[t;\sigma,\delta])$. Thus, $R_m = \{g \in S[t;\sigma,\delta] : deg(g) < m\}$ together with multiplication $g \cdot h = gh$ if $deg(g) + deg(h) < m$, $g \cdot h = gh \ (mod_r f)$ if $deg(g) + deg(h) \geq m$ is a unital nonassociative ring $S_f = (R_m, \cdot)$ also denoted by $R/(Rf)$. If $\sigma \in Aut(S)$, then R_m together with $g \times h = gh$ if $deg(g) + deg(h) < m$, $g \times h = gh \ mod_l f$ if $deg(g) + deg(h) \geq m$, is a unital nonassociative ring ${}_fS = (R_m, \times)$ also denoted by $R/(fR)$. As usual, if the context is specified, the notation $\cdot$ or $\times$ is frequently dropped, and juxtaposition *is utilized for multiplication in S_f or ${}_fS$. Certainly, S_f and ${}_fS$ are unital nonassociative algebras over the commutative subring $S_0 = \{a \in S : \forall h \in S_f, ah = ha\} = Comm(S_f)' \cap S$ of S, and $\mathcal{C}(S) \cap Fix(\sigma) \cap Const(\delta) \subseteq S_0$. Then, $S_f = S_{bf}$ for each invertible element b in S. Therefore, it suffices to consider monic polynomials in this construction. If f has degree 1, then S_f is isomorphic with S. If the skew polynomial f is reducible, then the ring S_f contains zero divisors. For $m \geq 2$, the algebra S_f is called a Petit algebra, though Petit considered only the case of division rings S [38,39]. Notice that the algebra ${}_fS$ is anti-isomorphic to S_f by Proposition 3 in [37]. Take a division algebra D with center F, $R = D[t;\sigma,\delta]$ with σ being any endomorphism of D and δ being any left σ-derivation. Let f be a skew polynomial in $R = D[t;\sigma,\delta]$ be monic of degree $m \geq 2$ [40]. The largest subalgebra of $R = D[t;\sigma,\delta]$ in which Rf is a two-sided ideal is the idealiser $J(f) = \{g \in R : fg \in Rf\}$ of Rf. The eigenring of f is then defined as the quotient $E(f) = J(f)/(Rf) = \{g \in R : deg(g) < m, \& fg \in Rf\}$.*

Theorem 17 ([37]). *If D, σ, δ, f, S_f are as in Remark 2, then $E(f)$ is the right nucleus of the algebra S_f.*

Theorem 18 ([38]). *Assume that a skew polynomial $f(t) = \sum_{j=0}^{m} a_j t^j$ in $R = D[t;\sigma,\delta]$ is monic of degree $m \geq 2$ (see Remark 2). Then,*

(i) $D \subseteq N_r(S_f)$ if and only if $f(t)b = \sigma^m(b)f(t)$ for each $b \in D$, if and only if $\sigma^m(b)a_k = \sum_{j=k}^{m} a_j \Delta_{j,k}(b)$ for all $b \in D$ and $k \in \{0, ..., m-1\}$.

(ii) Let σ be an automorphism of D of an infinite inner order. Then $D \subseteq N_r(S_f)$ implies that S_f is associative.

(iii) Let $\delta = 0$. Then $D \subseteq N_(S_f)$ if and only if $\sigma^m(b) = a_j \sigma^j(b) a_j^{-1}$ for all b in D and all $j \in \{0, ..., m-1\}$ with $a_j \neq 0$. Moreover, the algebra S_f is associative if and only if the skew polynomial $f(t)$ satisfies the identity above and $f(t) \in Fix(\sigma)[t] \subseteq Fix(\sigma)[t;\sigma]$.

(iv) Let $\sigma = id$. Then, $D \subseteq N_r(S_f)$ is equivalent to $\left(\genfrac{}{}{0pt}{}{ba_k = \sum_{j=k}^{m} =}{jka_j \delta^{j-k}(b)}\right)$ for all b in D, $k \in \{0, ..., m-1\}$. Furthermore, S_f is associative if and only if $f(t)$ satisfies the identity above and $f(t) \in Const(\delta)[t] \subseteq Const(\delta)[t;\delta]$.

(v) Let $\delta = 0$ and σ be an automorphism of D of finite inner order k, that is $\sigma^k = I_u$ for some $u \in D^\times$. Then, the polynomials $g \in D[t;\sigma]$ such that$D \subseteq N_r(S_g)$ are precisely those of the form $g(t) = s\sum_{j=0}^{n} b_j u^{n-j} t^{jk}$, where $n \in \mathbf{N}$, $b_n = 1$, $b_j \in F$ and $s \in D^\times$. Furthermore, the algebra S_g is associative if and only if $g(t)$ has the form provided above and $g(t) \in Fix(\sigma)[t] \subseteq Fix(\sigma)[t;\sigma]$.

Remark 11. *A skew polynomial f in $R = D[t;\sigma,\delta]$ is called right semi-invariant if for each $a \in D$, there exists $b \in D$ such that $f(t)a = bf(t)$. The latter is equivalent to $fD \subseteq Df$. Symmetrically, f is left semi-invariant if $Df \subseteq fD$ [41,42]. Then, f is right semi-invariant if and only if bf is right semi-invariant for each $b \in D^\times$. Moreover, if σ is an automorphism, then f is right semi-invariant if and only if it is left semi-invariant if and only if $fD = Df$ according to Proposition 2.7 in [41]. It is worth mentioning that right semi-invariant polynomials arise in a treatment of semi-linear transformations [43].*

If f is semi-invariant and also satisfies the following condition $f(t)t = (bt + a)f(t)$ for some elements a and b in D, then f is called right invariant. The latter is equivalent to $fR \subseteq Rf$. If f is right invariant, then Rf is a two-sided ideal in R. Vice versa, each two-sided ideal in R is generated by a right-invariant polynomial. This implies that R is not simple if and only if there exists a non-constant right-invariant skew polynomial f in R. In the particular case of σ being an automorphism, R is not simple if and only if there is a non-constant monic semi-invariant skew polynomial f in R if and only if δ is a quasi-algebraic derivation [42].

Choose a subring B of D. It is said that a skew polynomial f in $D[t;\sigma,\delta]$ is (right) B-weak semi-invariant if $fB \subseteq Df$. This means that any right semi-invariant polynomial is also B-weak semi-invariant for each subring B of D. If f is right B-weak semi-invariant and $f(t)t = (bt + a)f(t)$ for some a and b in B, then f is called a (right) B-weak invariant polynomial.

Theorem 19 ([38]). *Suppose that σ is an automorphism of D, B is a subring of D such that D is a free right B-module of finite rank and $f \in D[t;\sigma,\delta]$ is B-weak semi-invariant. Then S_f is a division algebra if and only if f is irreducible. In particular, if σ is an automorphism of D and f is right semi-invariant, then S_f is a division algebra if and only if f is irreducible.*

Theorem 20 ([38]). *Assume that a skew polynomial f in $R = D[t;\sigma,\delta]$ is irreducible. Then f is bounded if and only if S_f is free of finite rank as a $N_r(S_f)$-module. In this case, S_f is a division algebra.*

7. Commutative Nonassociative Algebras and Their Modules

Remark 12. *In this section, commutative nonassociative finite dimensional algebras are considered over a field K of the characteristic $char(K) \notin \{2,3\}$. Recall that an element c of an algebra A is called an idempotent if $c^2 = c$. Then the idempotent c induces the multiplication endomorphism $L_c \in End_K(A)$ such that $L_c x = cx$ for each x in A. A set consisting of all eigenvalues belonging to K of the operator L_c is called the Peirce spectrum of the idempotent c. It is frequently denoted by $\sigma(c)$. This implies that any eigenvalue λ in $\sigma(c)$ is a zero of the characteristic polynomial of L_c. For the idempotent c, the Peirce spectrum is nonvoid since $1 \in \sigma(c)$. Then, the idempotent c is called semi-simple if there exists a direct sum decomposition of A such that $A = \bigoplus_{\lambda\in\sigma(c)} A_c(\lambda)$, where $A_c(\lambda)$ denotes a λ-eigenspace of the operator L_c on A. In this decomposition, some $A_c(\lambda)$ are allowed to be trivial. This decomposition is sometimes called the Peirce decomposition of the algebra A relative to the idempotent c. For studying the multiplication structure of the λ-eigenspaces, a fusion law is used. The fusion law is a map $\star : \sigma(c) \times \sigma(c) \to 2^{\sigma(c)}$ such that $A_c(\lambda)A_c(\mu) \subseteq \bigoplus_{\nu\in\lambda\star\mu} A_c(\nu)$. It is assumed in suitable cases that this decomposition is minimal in an obvious sense.*

Take, for example, a Jordan algebra B. That is, $zy - yz = 0$ and $z((zz)y) - (zz)(zy) = 0$ for each y and z in B. There is the identity $2L_c^3 - 3L_c^2 + L_c = 0$ (see it, for example, on page 97 in [5]) for any idempotent c in B. That is, $f(L_c) = 0$, where $f(t) = (2t-1)(t-1)t$. Thus, the Peirce spectrum is $\sigma(c) = \{1, 0, 1/2\}$ and $B = B_c(0) \oplus B_c(1/2) \oplus B_c(1)$, $1 \star 1 = \{1\}$, $1/2 \star 1/2 = \{0, 1\}$, etc. The Peirce spectrum $\sigma(P, c)$ and the fusion law of the identity P can be calculated with the help of the first-order linearization $D^1(P; c, y)$ and the second-order derivation $D^2(P; c, x, y)$ [44,45]. In the latter work were considered polynomial identities with coefficients which may depend on indeterminates. It allowed to include in the consideration, for example, all train baric algebras A, where a commutative algebra is called baric if it carries a nontrivial K-homomorphism $\omega : A \to K$ [46,47]. Particularly, the Bernstein algebras satisfying $z^2z^2 - \omega(z)^2z^2 = 0$ are baric [48]. Then pseudo-composition algebras satisfying $z^3 - b(z,z)z = 0$ are baric, where b denotes a symmetric bilinear form. Additionally, general rank three algebras satisfying the identity $z^3 - a(z)z^2 - b(z)z = 0$ are also baric [49]. There are known baric train algebras of general rank and rank four identities [46,50]. For example, Hsiang algebras satisfy the identity $4zz^3 + z^2z^2 - 3b(z,z)z^2 - 2b(z^2,z)z = 0$, where b denotes an associating symmetric bilinear form, where a bilinear form $b(x,y)$ on an algebra A is called associating if $b(xy,z) = b(x,yz)$ for every x, y and z in A.

Recall that an algebra supplied with an associating non-degenerate symmetric bilinear form is called metrized [51]. Examples of the symmetric bilinear forms on algebras are the Killing form $tr(ad(x)ad(y))$ on a Lie algebra [23], and also the invariant trace form trL_{xy} of a formal real (Euclidean) Jordan algebra [52]. They also arise in a Norton–Griess algebra related to a monster sporadic simple group [53], or in some axial algebras [54]. Modules over axial algebras were studied in [55] and references therein. Frequently associating symmetric bilinear forms without the non-degeneracy condition are also considered, which may be related to the studying of radicals.

Sometimes, graphs, in particular, trees, for studying polynomials on a commutative nonassociative algebra A can be useful.

Then $\sigma(P,c) := \{t \in K : \rho_c(P,t) = 0\}$ is called the Peirce spectrum of P at c, where $\rho_c(P,t)$ denotes the Peirce polynomial [44]. Notice that $\sigma(c) \subseteq \sigma(P,c)$.

Proposition 5 ([44]). *An algebra is baric if and only if there exists a rank one nontrivial associating symmetric bilinear form.*

Theorem 21 ([44]). *Assume that A is a finite-dimensional commutative nonassociative algebra over a field K of characteristic $char(K) \notin \{2,3\}$ and A satisfies a nontrivial weighted polynomial identity $P(z) = 0$ in one nonassociative indeterminate z. Then,*

(i) $1/2 \in \sigma(P,c)$ for any nonzero idempotent c in A;

(ii) moreover, if c is semi-simple and λ is a simple root of the Peirce polynomial $\rho_c(P,t)$, then $A_c(\lambda)A_c(1/2) \subseteq A_c(\lambda)^{\perp} := \bigoplus_{\nu\in\sigma(c),\nu\neq\lambda} = A_c(\nu)$.

8. Nonassociative Cyclic Algebras

Remark 13. *Cyclic algebras considered in this section arise with the help of cyclic Galois extension K/F of degree n with Galois group $Gal(K/F) =< \sigma >$, where F is a field and K is its extension. Note that an associative cyclic algebra $(K/F,\sigma,c)$ of degree n over F with $c \in F^{\times}$, is constructed as an n-dimensional K-vector space $(K/F,\sigma,c) = K \oplus eK \oplus ... \oplus e^{n-1}K$ and with multiplication such that $e^n = c$, $be = e\sigma(b)$ for each $b \in K$. Therefore, $(K/F,\sigma,c)$ is the division algebra, if $c^s \notin N_{K/F}(K^{\times})$ for each s being a prime divisor of n, $1 \leq s \leq n-1$.*

If $c \in K - F$, there exists a unital nonassociative algebra $D = (K/F,\sigma,c)$ as the n-dimensional K-vector space $D = K \oplus eK \oplus ... \oplus e^{n-1}K$ and multiplication for every a and b in K, $0 \leq i < n$, $1 \leq j < n$, extended K-linearly to all elements in D such that $(e^i a)(e^j b) = e^{i+j}\sigma^j(a)b$ if $i+j < n$, $(e^i a)(e^j b) = e^{i+j-n}c\sigma^j(a)b$ if $i+j \geq n$ [56]. Then, D is called a nonassociative cyclic algebra of degree n. Notice that D has nucleus K and center F. The cyclic algebra D is not $(n+1)$-th power associative, for example, $(e^{n-1}e)e = \sigma(c)$ and $e(e^{n-1}e) = ec$. Furthermore, D is a division algebra, if $[K:F]$ is prime, or if 1, c, ..., c^{n-1} are linearly independent over F. Particularly, for $n = 2$, $(K/F,\sigma,c)$ is either an associative (if $c \in F$) or nonassociative (if $c \in K - F$) quaternion algebra over F [57]. Then $\{1,e,e^2,...,e^{n-1}\}$ is called the standard basis of $(K/F,\sigma,c)$. Notice that the algebra $(Mat_{n\times n}(K),\mathsf{m})$ of all square $n \times n$ matrices with entries in K is associative relative to the matrix multiplication m, so D as the nonassociative algebra cannot be embedded into $(Mat_{n\times n}(K),\mathsf{m})$.

Consider fields F and M and their cyclic field extension K such that $Gal(K/F) =< \sigma >$ and $[K:F] = n$, $Gal(K/M) =< \tau >$ and $[K:M] = m$. Assume that σ and τ commute. Let the intersection of fields be denoted by $F_0 = F \cap M$, and $D = (K/F,\sigma,c)$ be with reduced norm $N_{D/F}$. There exists a M-linear map extension $\tau : D \to D$ such that $\tau(x) = \tau(x_0) + e\tau(x_1) + ... + e^{n-1}\tau(x_{n-1})$ for each $x = x_0 + ex_1 + ... + e^{n-1}x_n$ in D, where $x_j \in K$ for each j. Particularly, if $c \in F_0$, then $\tau(xy) = \tau(x)\tau(y)$ for each x and y in D. For a matrix $X = (X_{i,j})$ with entries $X_{i,j}$ in D, $\tau(X)$ denotes the matrix with entries $(\tau(X))_{i,j} = \tau(X_{i,j})$ for each i and j.

In particular, nonassociative algebras of degree 4 were also studied in [57,58].

Definition 5 ([59]). *For any fixed $b \in F^{\times}$ and $c \in F_0$, let a right D-module $N = D \oplus fD \oplus \oplus f^{m-1}D$ be supplied with multiplication $(f^i x)(f^j y) = f^{i+j}\tau^j(x)y$ if $i+j < m$, $(f^i x)(f^j y) = f^{i+j-m}\tau^j(x)yb$ if $i+j \geq m$ for every x and y in D. This provides a so-called iterated algebra $It^m_R(D,\tau,b)$. Then $\{1,e,...,e^{n-1},f,fe,fe^2,...,f^{m-1}e^{n-1}\}$ is called the standard basis of the K-vector space $It^m_R(D,\tau,b)$.*

Remark 14. *Definition 1 implies that $It^m_R(D,\tau,b)$ is a nonassociative algebra over F_0 of dimension m^2n^2 with unit element $(1_D,0,...,0)$, where $1 = 1_D$ denotes the unit element in D. Therefore, the iterated algebra $It^m_R(D,\tau,b)$ contains a subalgebra D.*

Theorem 22 ([60]). *Assume that F and M are fields, $F_0 = F \cap M$ and K is a cyclic field extension of both F and M such that (i) $Gal(K/F) =< \sigma >$ and $[K:F] = n$, (ii) $Gal(K/M) =< \tau >$ and $[K:M] = m$, (iii) σ and τ commute. Assume that $D = (K/F,\sigma,c)$ is an associative cyclic division algebra over F of degree n, $c \in F_0$ and $b \in D^{\times}$. Then, $It^m_R(D,\tau,b) = S_f$, where $R = D[t;\tau^{-1}]$ is the skew-polynomial ring (see the preceding section) and $f(t) = t^m - b$.*

Corollary 3 ([60]). *Let the conditions of Theorem 1 be satisfied.*

(i) If $b \notin F_0$, then $N_l(It^m_R(D,\tau,b)) = N_m(It^m_R(D,\tau,b)) = D$ and $N_r(It^m_R(D,\tau,b)) = \{g \in S_f : fg \in Rf\}$.

(ii) *$It_R^m(D,\tau,b)$ is a division algebra if and only if $f(t)$ is irreducible in $D[t;\tau^{-1}]$.*

(iii) *$It_R^4(D,\tau,b)$ is a division algebra if and only if $b \neq z\tau(z)\tau^2(z)\tau^3(z)$ and $\tau^2(z_1)\tau^3(z_1)z_1 + \tau^2(z_0)z_1 + \tau^2(z_1)\tau^3(z_0) \neq 0$ or $\tau^2(z_0)z_0 + \tau^2(z_1)\tau^3(z_0)z_0 \neq b$ for every z, z_0 and z_1 in D.*

(iv) *Let m be prime and in the case of $m \notin \{2,3\}$, additionally let F_0 contain a primitive m-th root of unity. Then $It_R^m(D,\tau,b)$ is a division algebra if and only if $b \neq z\tau(z)...\tau^{m-1}(z)$ for each z in D.*

Corollary 4 ([60]). *Let the conditions of Theorem 1 be satisfied. Let m be prime and in the case of $m \notin \{2,3\}$, additionally let F_0 contain a primitive m-th root of unity.*

(i) *If $\tau(b^n) \neq b^n$, then $It_R^m(D,\tau,b)$ is a division algebra.*

(ii) *If $b \in F$ is such that $b^n \notin N_{D/F_0}(D^\times)$, then $It_R^m(D,\tau,b)$ is a division algebra. Particularly, for each $b \in F - F_0$ with $b^n \notin F_0$, $It_R^m(D,\tau,b)$ is a division algebra.*

Remark 15. *Suppose that F and M are two fields. Suppose also that F and M are linearly disjoint over F_0, where $F_0 = F \cap M$, M/F_0 and F/F_0 are cyclic Galois extensions of degrees n and m correspondingly with cyclic Galois groups $Gal(M/F_0) =< \sigma >$ and $Gal(F/F_0) =< \tau >$ correspondingly. Assume that K is a field such that $K = M \bigotimes_{F_0} F$, that is, $K = MF$ is a composite of M and F, with Galois group $Gal(K/F_0) =< \sigma >< \tau >$, where σ and τ are canonically extended on K. Put $D_0 = (M/F_0,\sigma,c)$ and $D_1 = (F/F_0,\tau,b)$ to be two cyclic algebras over F_0 with $c \in F_0^\times$ and $b \in F^\times$ such that D_0 is associative and D_1 is nonassociative.*

There exists the tensor product $A = D_0 \bigotimes_{F_0} D_1$, which is a nonassociative algebra over F_0. Therefore, K is a subfield of A of degree mn over F_0 and $K \subset N(A)$. There exists the standard basis $\{1, e, ..., e^{n-1}\}$ of D_0 as the M-vector space and the standard basis $\{1, f, ..., f^{m-1}\}$ of D_1 as the F-vector space. This implies that A as a K-vector space has the basis $\{1 \otimes 1, e \otimes 1, ..., e^{n-1} \otimes 1, 1 \otimes f, e \otimes f, ..., e^{n-1} \otimes f^{m-1}\}$. It was studied when the tensor product $A = D_0 \bigotimes_{F_0} D_1$ is a division algebra.

Theorem 23 ([60]). *If the conditions of Remark 3 are satisfied, then*

(i) *$A = It_R^m(D_0 \bigotimes_{F_0} F,\tau,b)$; (ii) if $D = D_0 \bigotimes_{F_0} F$ is a division algebra, then S_f is isomorphic with A, where $R = D[t;\tau^{-1}]$ and $f(t) = t^m - b$.*

Theorem 24 ([60]). *Let the conditions of Remark 3 be satisfied. Let m be prime and in case $m \notin \{2,3\}$, let, in addition, F_0 contain a primitive m-th root of unity. Then, A is a division algebra if and only if $b \neq z\tau(z)...\tau^{m-1}(z)$ for each z in D.*

The following theorem is more general.

Theorem 25 ([60]). *If the conditions of Theorem 3 are satisfied, then A is the division algebra if and only if the polynomial $f(t) = t^m - b$ is irreducible in $D[t;\tau^{-1}]$.*

A particular case of Theorem 3 is provided by the following.

Theorem 26 ([60]). *Let the conditions of Theorem 3 be satisfied. Let F_0 be of characteristic $char(F_0) \neq 2$. Let $(a,c)_{F_0}$ be a quaternion algebra over the field F_0, which is a division algebra over $F = F_0(\sqrt{s})$, and $(F_0(\sqrt{s})/F_0,\tau,b)$ be a nonassociative algebra of degree 4. Then their tensor product $(a,c)_{F_0} \bigotimes_{F_0} (F_0(\sqrt{s})/F_0,\tau,b)$ is a division algebra over F_0.*

Theorem 27 ([61]). *Let the field F possess no non-trivial m-th root of unity. Let $A = (K/F,\sigma,b)$ be a nonassociative cyclic algebra of degree m, where $b \in K^\times$ and b is not contained in any proper subfield of K. Then, every F-automorphism of the algebra A leaves K fixed and $Aut_F(A)$ is isomorphic with $ker(N_{K/F})$. In particular, all automorphisms of A are inner.*

Theorem 28 ([61]). (i) *Assume that $\tau \in Aut_{F_0}(D)$ commutes with σ, where D is a unital division ring and σ is a ring automorphism of D. Then τ can be extended to an automorphism*

$H \in Aut_{F_0}(A)$, if and only if there exists $k \in F^\times$ such that $\tau(b) = N_{F/F_0}(k)b$. In that case, the extension H of τ has the form $H = H_{\tau,k}$ with

$$H_{\tau,k}(\sum_{j=0}^{m-1} a_j t^j) = \tau(a_0) + \sum_{j=1}^{m-1} \tau(a_j)(\prod_{l=0}^{j-1} \sigma^l(k))t^j.$$

Moreover, $H_{\tau,k}$ is an automorphism of A if $\tau \in Aut_{F_0}(D)$ commutes with σ and if $k \in F^\times$ is such that $\tau(b) = N_{F/F_0}(k)b$. In particular, for $\tau \neq id$ and $b \notin Fix(\tau)$, $N_{F/F_0}(k) \neq 1$.

(ii) $id \in Aut(D)$ can be extended to an automorphism $H \in Aut_{F_0}(A)$ if and only if there is some $k \in F^\times$ such that $N_{F/F_0}(k) = 1$. In that case, the extension H of id has the form $H = H_{id,k}$ with

$$H_{id,k}(\sum_{j=0}^{m-1} a_j t^j) = a_0 + \sum_{j=1}^{m-1} a_j(\prod_{l=0}^{j-1} \sigma^l(k))t^j.$$

Moreover, $H_{id,k}$ is an automorphism of A if $k \in F^\times$ is such that $N_{F/F_0}(k) = 1$.

Proposition 6 ([61]). *Let the conditions of Theorem 7 be satisfied. Then, each automorphism $H_{id,k}$ of A is an inner automorphism of the form*

$$G_c(\sum_{j=0}^{m-1} a_j t^j) = (c^{-1} \sum_{j=0}^{m-1} a_j t^j)c$$

for some $c \in F^\times$ satisfying $k = \sigma(c)c^{-1}$.

9. Rings Obtained as Nonassociative Cyclic Extensions

Remark 16. *We recall that a nonassociative nontrivial ring A is called a right division ring if R_s is bijective for each nonzero s in A, where $R_s(x) = xs$ for each s and x in A. It was demonstrated in [62] that if D is a division ring and a polynomial f is irreducible, then $S_f = D[t;\sigma]/(D[t;\sigma]f$ is a right division algebra and has no zero divisors.*

Theorem 29 ([62]). *(i) Assume that B is a nonassociative ring with multiplication $\circ$. Suppose that conditions (α)–(γ) are satisfied:*

(α) B has an associative subring D which is a division algebra and B is a free left D-module of rank m, and there exists $t \in B$ such that $\{t^j : 0 \leq j < m\}$ is a basis of B over D, where $t^{j+1} = t \circ t^j$, $t^0 = 1$;

(β) For each nonzero b in D, there exist b_1 and b_2 in D with $b_1 \neq 0$ such that $t \circ b = b_1 \circ t + b_2$;

(γ) $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$ for every a, b and c in D, $i + j < m$, $k < m$.

Then B is isomorphic with S_f, where $f(t) \in D[t;\sigma,\delta]$, $f(t) = t^m - \sum_{j=0}^{m-1} b_j t^j$, where σ and δ are such that $t \circ b = \sigma(b) \circ t + \delta(b)$.

(ii) If B is a right division ring in (i), then f is irreducible.

Theorem 30 ([61]). *(i) Suppose that B is a nonassociative ring with multiplication denoted by $\circ$, a field K is a subring in B, and B is a free left K-vector space of dimension m. Suppose that conditions (α)–(ϵ) are satisfied:*

(α) there exists t in B such that $\{t^j : 0 \leq j < m\}$ is a basis of B over K, where $t^0 = 1$, $t^{j+1} = t \circ t^j$ for each $0 \leq j < m$;

(β) for each nonzero b in K, there exists $s \in K^\times$ such that $t \circ b = s \circ t$;

(γ) $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$ for every a, b and c in K, $i + j < m$ and $k < m$;

(δ) there exists d in $K^\times$ such that $t^m = d$;

(ϵ) the map $\sigma : K \to K$ such that $\sigma(b) = s$ has order m, and $Fix(\sigma)$ is a field $F = \{a \in K : t \circ a = a \circ t\}$, F contains a primitive m-th root of unity ω, and K/F is a finite cyclic Galois extension.

Then B is isomorphic with $S_f = (K/F, \sigma, d)$ with $f(t) = t^m - d \in K[t;\sigma]$.

(ii) If B is a right division ring in (i), then f is irreducible and B is isomorphic with a nonassociative cyclic extension $(K/F, \sigma, d)$ of K of degree m.

From Theorem 1 applied to nonassociative cyclic extensions of a central simple algebra D, it was deduced the following.

Theorem 31 ([61]). *(i) Assume that B is a nonassociative ring with multiplication denoted by $\circ$, and D is an associative subring in B, and B is a free left B-module of rank m. Assume that conditions (α)–(ϵ) are satisfied:*

(α) there exists t in B such that $\{t^j : 0 \leq j < m\}$ is a basis of B over D, where $t^0 = 1$, $t^{j+1} = t \circ t^j$ for each $0 \leq j < m$;

(β) for each nonzero b in K, there exists $s \in K^\times$ such that $t \circ b = s \circ t$;

(γ) $[a \circ t^i, b \circ t^j, c \circ t^k] = 0$ for every a, b and c in K, $i + j < m$ and $k < m$;

(δ) there exists d in $K^\times$ such that $t^m = d$;

(ϵ) the map $\sigma : K \to K$ such that $\sigma(b) = s$ has order m, and $Fix(\sigma)$ is a field $F = \{a \in D : t \circ a = a \circ t\}$, F contains a primitive m-th root of unity ω, and D has a structure of a central simple algebra over F.

Then, B is isomorphic with $S_f = (D, \sigma, d)$ with $f(t) = t^m - d \in D[t; \sigma]$.

(ii) If B is a right division ring and D is a central simple algebra in (i), then f is irreducible and B is isomorphic with a nonassociative cyclic extension (D, σ, d) of D of degree m.

10. Nonassociative Ore Extensions of Hom-Associative Algebras and Modules over Them

Remark 17. *A generalization of Lie algebras was studied in [63]. In them, the Jacobi identity was twisted by a vector space homomorphism. This was an origin of hom-associative algebras. It is necessary to note that hom-associative algebras may be nonassociative. In a hom-associative algebra A, the associativity condition is substituted with $\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c)$ for every a, b and c in A, where α is a linear map called a twisting map, particularly in the associative algebra $\sigma = id_A$.*

Later on, hom-co-algebras, hom-bi-algebras, and hom-Hopf algebras were studied in [64,65]. On the other side, Ore extensions arose as noncommutative polynomial rings [36]. Their nonassociative analogs for unital algebras were introduced in [66]. That construction was later generalized to non-unital, hom-associative Ore extensions in [67]. In the latter work, examples were provided of hom-associative versions of the first Weyl algebra, the quantum plane, and a universal enveloping algebra of a Lie algebra such that these algebras are formal deformations of their associative counterparts.

Definition 6. *Let R be an associative, commutative, and unital ring, let M be an R-module, let a binary operation $\cdot : M \times M \to M$ be bilinear, let $\alpha : M \to M$ be an R-linear map such that $\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c)$ for every a, b and c in M. Then, a triple $(M, \cdot, \alpha)$ is called a hom-associative algebra over R, where the map α is called a twisting map.*

If $A = (M, \cdot, \alpha)$ and $A_1 = (M_1, \cdot_1, \alpha_1)$ are two hom-associative R-algebras and if $f : M \to M_1$ is an R-linear map such that $f \circ \alpha = \alpha_1 \circ f$ and $f(a \cdot b) = f(a) \cdot_1 b$ for each a and b in M, then f is called a morphism from A into A_1. If f is bijective, then A and A_1 are isomorphic.

If N is a submodule of M such that N is closed under multiplication $\cdot$ and invariant under α, then it is said that the hom-associative algebra $(N, \cdot, \alpha|_N)$ is a hom-subalgebra of A.

By a right (or left) hom-ideal of a hom-associative R-algebra, A is implied an R-submodule J of A such that $\alpha(J) \subseteq J$ and $J \cdot A \subseteq J$ (or $A \cdot J \subseteq J$ correspondingly). If J is both a left and a right hom-ideal, it is called a hom-ideal.

A hom-associative ring is called a hom-associative algebra over the ring of integers.

If $S := (R, \cdot, \alpha)$ is a hom-associative ring, then the opposite hom-associative ring S^{op} of S is the hom-associative ring $(R, \cdot, \sigma)$ satisfying $r \cdot_{op} s := s \cdot r$ for every r and s in R.

Remark 18. *Assume that R is a unital nonassociative ring, $\sigma : R \to R$ and $\delta : R \to R$ are additive maps such that $\sigma(1) = 1$ and $\delta(1) = 0$. By $\mathbf{N}_0$, denote the set of all non-negative integers, and by $\mathbf{N}$ denote the set of all positive integers. Then as a set, a unital, nonassociative Ore extension $R[X;\sigma,\delta]$ of R consists of all formal sums $\sum_{j\in\mathbf{N}_0} a_j X^j$ which are called polynomials, where $a_j \in R$ for each j and card$\{j : a_j \neq 0\} < \aleph_0$. Then $R[X;\sigma,\delta]$ is supplied with addition and multiplication $\sum_{j\in\mathbf{N}_0} a_j X^j + \sum_{j\in\mathbf{N}_0} b_j X^j = \sum_{j\in\mathbf{N}_0} (a_j + b_j) X^j$ and $a_m X^m \cdot b_n X^n = \sum_{j\in\mathbf{N}_0} a_m \cdot \pi_j^m(b_n)) X^{j+m}$ for every m, n in $\mathbf{N}_0$ and a_j, b_j in R, where π_j^m denotes the sum of all $\binom{}{mj}$ possible compositions of j copies of σ and $m - j$ copies of δ in arbitrary order, $\pi_0^0 = id_R$, $\pi_j^m = 0$ for $j < 0$ or $j > m$. Then, for $1_R = 1$ there corresponds an identity element $1X^0$ in $R[X;\sigma,\delta]$; also, X is interpreted as an element $1X$ of $R[X;\sigma,\delta]$. There, two polynomials are supposed to be equal if and only if their corresponding coefficients are equal. Together with the distributivity of multiplication over addition, this makes $R[X;\sigma,\delta]$ a unital nonassociative noncommutative ring. Naturally, R is embedded into $R[X;\sigma,\delta]$ such that bX^0 corresponds to b in R.*

Recall that if R is a unital nonassociative ring, σ is a unital endomorphism and δ is an additive map on R satisfying $\delta(a \cdot b) = \sigma(a) \cdot \delta(b) + \delta(a) \cdot \sigma(b)$ for each a and b in R, then δ is called a σ-derivation. In particular, if $\sigma = id_R$, then δ is a derivation. Notice that if R is a unital nonassociative ring and δ is a σ-derivation on R, then $\delta(1) = 0$.

For the unital hom-associative ring R with twisting map α, the latter is extended α homogeneously to an additive map on $R[X;\sigma,\delta]$ by putting $\alpha(\sum_{j\in\mathbf{N}_0} a_j X^j) = \sum_{j\in\mathbf{N}_0} \alpha(a_j) X^j$, where $a_j \in R$ for each j. This is justified by the following.

Proposition 7 ([67])**.** *Suppose that R is a unital hom-associative ring with twisting map α, σ is a unital endomorphism and δ is a σ-derivation such that both σ and δ commute with α. If α is extended homogeneously to $R[X;\sigma,\delta]$, then $R[X;\sigma,\delta]$ is a unital hom-associative Ore extension with twisting map α.*

Definition 7. *Suppose that M is an additive group supplied with a group homomorphism $\alpha_M : M \to M$, which is called a twisting map. Take a non-unital, hom-associative ring R with twisting map α_R, multiplication written with juxtaposition. Suppose that there exists an operation $\cdot : M \times R \to M$, which is called scalar multiplication such that for every r_1 and r_2 in R, m_1 and m_2 in M*

$(m_1 + m_2) \cdot r_1 = m_1 \cdot r_1 + m_2 \cdot r_1$ (right-distributivity);
$m_1 \cdot (r_1 + r_2) = m_1 \cdot r_1 + m_1 \cdot r_2$ (left-distributivity);
$\alpha_M(m_1) \cdot (r_1 r_2) = (m_1 \cdot r1) \cdot \alpha_R(r_2)$ (hom-associativity).

Then, M is called a right R-hom-module and denoted by M_R. A left R-hom-module $_RM$ is defined similarly. Frequently, it is written briefly as M; if it does not matter whether it is a right or a left R-hom-module, it is called a R-hom-module. For several right (left) R-hom-modules, it is assumed that α_R is the same twisting map on R.

By a morphism from a right (or left) R-hom-module M to a right (or left correspondingly) R-hom-module M_1 is implied an additive map $f : M \to M_1$ such that $f \circ \alpha_M = \alpha_{M_1} \circ f$ and $f(m \cdot r) = f(m) \cdot r$ (or $f(r \cdot m) = r \cdot f(m)$ correspondingly) for each m in M and r in R. If the morphism f is also bijective, then M and M_1 are isomorphic.

Assume that M is a right (or left) R-hom-module, N is an additive subgroup of M and closed under scalar multiplication and invariant under α_M. Then, N is called a R-hom-submodule, where $\alpha_N = \alpha_M|_N$. It is denoted by $N \leq M$ or $M \geq N$, and in case N is a proper subgroup of M, by $N < M$ or $M > N$.

For a non-void subset S of a right (or left) R-hom-module M, the intersection N of all hom-submodules of M containing S is called the hom-submodule generated by S. In this case, S is called a generating set of N. If for the hom-submodule N in M, there exists a finite generating set S, then N is called finitely generated.

A family $\mathcal{G}$ of subsets of a set S satisfies the ascending chain condition if there is no properly ascending infinite chain $S_1 \subset S_2 \subset ...$ of subsets S_j in S belonging to $\mathcal{G}$.

Proposition 8 ([68])**.** *(Image and preimage under hom-module morphism.) Assume that $f : M \to M_1$ is a morphism of right (or left) R-hom-modules, $N \leq M$ and $N_1 \leq M_1$. Then $f(N)$ and $f^{-1}(N_1)$ are hom-submodules of M_1 and M correspondingly.*

Proposition 9. *(Intersection of hom-submodules). The intersection of a set of homsubmodules of a right (or left) R-hom-module M is a hom-submodule.*

Proposition 10 ([68])**.** *If M is a right (or left) R-hom-module, then the following conditions are equivalent:*

$(NM1)$ M satisfies the ascending chain condition on its hom-submodules;
$(NM2)$ Each non-void family of hom-submodules of M has a maximal element;
$(NM3)$ Each hom-submodule of M is finitely generated.

Corollary 5 ([68])**.** *Suppose that R is a nonunital hom-associative ring. Then the following conditions are equivalent:*

$(NR1)$ R satisfies the ascending chain condition on its right (or left) hom-ideals;
$(NR2)$ Each non-void family of right (or left) hom-ideals of R has a maximal element;
$(NR3)$ Each right (or left) hom-ideal of R is finitely generated.

Definition 8. *A right (or left) R-hom-module is called hom-noetherian if it satisfies the three equivalent conditions of Proposition 4 on its hom-submodules.*

A non-unital hom-associative ring R is called right (or left) hom-noetherian if it satisfies the three equivalent conditions of Corollary 1 on its right (or left) hom-ideals. If R satisfies the conditions on both its right and its left hom-ideals, it is called hom-noetherian.

Proposition 11 ([68])**.** *The hom-noetherian conditions are invariant under surjective morphisms of right (or left) R-hom-modules.*

Proposition 12 ([68])**.** *If M is a right (or left) R-hom-module, and $N \leq M$, then M is hom-noetherian if and only if M/N and N are hom-noetherian.*

Corollary 6. *The finite direct sum of hom-noetherian modules is hom-noetherian.*

Proposition 13 ([68])**.** *Assume that $R[X;\sigma,\delta]$ is a unital nonassociative Ore extension of a unital nonassociative ring R, where σ is a unital endomorphism and δ is a σ-derivation on R. Then, X^k belongs to the nucleus $N(R[X;\sigma,\delta])$ of $R[X;\sigma,\delta]$ for each k in $\mathbf{N}_0$.*

Proposition 14 ([68])**.** *Let R be a unital noetherian hom-associative ring with twisting map α, a unital endomorphism σ and a σ-derivation δ such that both commute with α. If we extend α homogeneously onto $R[X;\sigma,\delta]$, then $\sum_{j=0}^{m} X^j R$ (or $\sum_{j=0}^{m} RX^j$) is a hom-noetherian right (or left correspondingly) R-hom-module for each m in $\mathbf{N}_0$.*

Theorem 32 ([68])**.** *(Hilbert's basis theorem for hom-associative Ore extensions.) Suppose that R is a unital hom-associative ring with twisting map α, a unital endomorphism σ and a σ-derivation δ such that both commute with α. If we extend α homogeneously onto $R[X;\sigma,\delta]$, and if R is right (or left) noetherian, then $R[X;\sigma,\delta]$ also is right (or left correspondingly) noetherian.*

Corollary 7 ([68])**.** *(Hilbert's basis theorem for non-associative Ore extensions.) If R is a unital nonassociative ring, σ is an automorphism and δ is a σ-derivation on R, and if R is right (or left) noetherian, then $R[X;\sigma,\delta]$ also is right (or left correspondingly) noetherian.*

The latter corollary is the particular case of the preceding theorem with the trivial twisting map $\alpha = 0$. Examples of nonassociative Ore extensions of hom-associative algebras and modules over them were provided in [68].

11. Von Neumann Finiteness for Nonassociative Algebras

Remark 19. *If in a unital ring R each one-sided inverse also is two-sided, then R is called von Neumann finite (or Dedekind finite, or weakly 1-finite, or affine finite, or directly finite, or inverse symmetric). That is, $(ab = 1) \leftrightarrow (ba = 1)$ for each a and b in R. The ring R is called reversible if $(ab = 0) \leftrightarrow (ba = 0)$ for each a and b in R. Notice that the class of associative reversible rings is properly contained in the class of associative von Neumann finite rings, since $ab = 1$ for some a and b in an associative ring R implies $(ba - 1)b = 0$, $b(ba - 1) = 0$, $b^2a = b$, $ba = abba = ab = 1$. Particularly, if V is a finite dimensional vector space over a field F, then $End(V)$ is von Neumann finite, but it is not reversible. Below in this section, von Neumann finiteness and reversibility for nonassociative unital rings are considered.*

Recall that an F-algebra A is called alternative if $a^2b = a(ab)$ and $ab^2 = (ab)b$ for every a and b in A. This condition is equivalent to each subalgebra B of A generated by one or two elements being associative by Theorem 3.1 in [5]. Then, A is flexible if $a(ba) = (ab)a$ for every a and b in A. Algebra A is called quadratic if it is unital and the elements $1, b, b^2$ are linearly dependent for each b in A. Algebra A is called involutive if it is unital and an anti-automorphism exists σ of A such that $\sigma^2 = I_A$, $b + \sigma(b) \in F1$ and $b\sigma(b) \in F1$ for each b in A. Frequently, the notation $\sigma(b) = \bar{b}$ is used for the involution σ in A, and the scalars $Tr(b) = b + \bar{b}$ and $N(b) = b\bar{b}$ are called the trace and the norm of an element b in A, correspondingly.

In this section, algebras over fields are considered. Note that to each quadratic form $q : V \to F$ on a vector space V over a field F is associated a symmetric bilinear form $(x, y)_q = q(x + y) - q(x) - q(y)$. By the radical of q, it is implied the subspace $V^\perp = \{x \in V : (x, V)_q = 0\}$ of V. The form q is called non-degenerate if either $V^\perp = 0$ or $dim_F(V^\perp) = 1$ and $q(V^\perp) = 0$. Notice that the latter case is for $charF = 2$ only. It is said that a non-zero element v in V is isotropic if $q(v) = 0$; it is anisotropic if $q(v) \neq 0$. Then the form q is called isotropic (or anisotropic) if V contains (or does not contain correspondingly) an isotropic element. If a subspace U of V is such that $q(U) = 0$, then it is called totally isotropic.

Recall, that an algebra A possessing a non-degenerate quadratic form $n : A \to F$ such that $n(ab) = n(a)n(b)$ for each a and b in A is called Hurwitz. It is worth mentioning that the quadratic form n is uniquely determined by the Hurwitz algebra A structure. Furthermore, each non-zero algebra morphism between Hurwitz algebras is orthogonal. There exists a zero divisor in the Hurwitz algebra if and only if its quadratic form n is isotropic. In the latter case, it is said that A is split. It was found that there exist three isomorphism classes of split Hurwitz algebras, by one in each dimension 2, 4 and 8. They are embedded into each other. For example, the 4-dimensional split Hurwitz F-algebra is the 2×2 matrix algebra over F. Then the Hurwitz algebra A is commutative if and only if its $dim_F(A) \leq 2$. It is associative if and only if its dimension over F is not greater than 4. Each Hurwitz algebra is alternative [69].

For an involutive algebra B over a field F and a non-zero μ in F, $\mathcal{D}_\mu(B)$ denotes the Cayley–Dickson algebra obtained from B by the doubling procedure (i.e., smashed product). Recall that $\mathcal{D}_\mu(B) = B \oplus B$ as a vector space over F. It is supplied with multiplication such that $(a, b)(c, d) = (ac + \mu\bar{d}b, da + b\bar{c})$ for every a, b, c and d in B. The involution on $\mathcal{D}_\mu(B)$ is given by $\overline{(a, b)} = (\bar{a}, -b)$. Then $\mathcal{D}_\mu(B)$ is flexible if and only if B is flexible [70,71]. Certainly, B has an embedding into $\mathcal{D}_\mu(B)$ as $B \times 0$ such that there is an orthogonal decomposition of vector spaces $\mathcal{D}_\mu(B) = B \oplus B\mathbf{l}$, where $\mathbf{l} = (0, 1)$ is the doubling generator. If we begin from the field F with trivial involution, then an application by induction of the doubling procedure provides flexible involutive algebras of dimensions 2^n as vector spaces over F, where $n \geq 1$. The Cayley–Dickson algebras of the latter type are considered below in this section. In this case, if $char(F) \neq 2$, then the Cayley–Dickson algebras of dimension at most eight over F are the Hurwitz algebras over F [72].

Theorem 33 ([73])**.** *If $char(F) \neq 2$ and A is the Cayley–Dickson algebra of dimension 2^n with isotropic norm, then A has a totally isotropic subspace of dimension n.*

Theorem 34 ([74])**.** (a) *Each finite-dimensional alternative algebra is von Neumann finite.*

(b) *Each reversible alternative algebra is von Neumann finite.*

(c) *A Hurwitz algebra A is reversible if and only if either its quadratic form is anisotropic or $dim_F A \leq 2$.*

Theorem 35 ([74])**.** (a) *Each algebra without zero divisors, that is either flexible or quadratic, is von Neumann finite.*

(b) *Let $char(F) \neq 2$, let A be flexible and quadratic, and let the norm of A be nondegenerate on every 3-dimensional subalgebra of A. Then A is von Neumann finite and reversible.*

(c) *Assume that F is algebraically closed, $char(F) \neq 2$, and the algebra A is flexible and quadratic. Then A is reversible if and only if either of the following conditions holds: (i) A is commutative; (ii) $A = F1 \oplus V$, where V is an anti-commutative ideal in A, and the linear map $L_u : V \to V$ is nilpotent for each u in V, where $L_u v = uv$ for every u and v in A.*

Theorem 36 ([74])**.** *Assume that $char(F) \neq 2$. Then*

(a) *Each Cayley–Dickson algebra with anisotropic norm is von Neumann finite and reversible.*

(b) *A Cayley–Dickson algebra with isotropic norm is reversible if and only if its dimension is at most two.*

Theorem 37 ([74])**.** *Let A be an involutive algebra, and let $char(F) \neq 2$. Then*

(a) *the algebra A is von Neumann finite if and only if every 3-dimensional subalgebra of A is either commutative or associative;*

(b) *the algebra A is reversible if and only if every 3-dimensional subalgebra of A is commutative.*

12. Nonassociative Algebras, Rings and Modules over Them Related with Harmonic Analysis on Nonlocally Compact Groups

Remark 20. *Algebras related with harmonic analysis on locally compact groups are rather well investigated [75–80]. They arise from convolutions of functions and unitary representations of locally compact groups relative to Haar measures, which are either left or right invariant on groups. For locally compact groups, such algebras are associative. Recall that by the A. Weil theorem, if a topological group has a non-trivial Borelian measure quasi-invariant relative to left or right shifts of the entire group, then it is either locally compact or contains a dense locally compact subgroup. Furthermore, the compactification of a topological group may have no group structure. Therefore, the theory of non locally compact groups cannot be reduced to that of compact or locally compact groups. On the other hand, measures on nonlocally compact groups quasi-invariant relative to proper dense subgroups were constructed in [81–89].*

For nonlocally compact groups, algebras related to convolutions of functions or operators relative to quasi-invariant measures are nonassociative. They appear to be substantially different from that of locally compact groups. Families were considered of nonlocally compact completely regular groups $\{G_\alpha : \alpha \in \Lambda\}$ with embeddings $\theta^\beta_\alpha : G_\beta \to G_\alpha$ and with measures μ_α on G_α quasi-invariant relative to G_β for each $\alpha < \beta$ in a directed set Λ. Definitions and notation are provided in detail in [90,91]. Ideals in these algebras were studied in [92]. Operator valued functions for nonlocally compact groups and their normed spaces of different types were investigated in [93]. Norms of integral operators were estimated. They were used for studies of convolutions of functions having operator values and values in Banach spaces. Meta-centralizers of nonlocally compact group algebras were investigated in [91]. Representations of meta-centralizers with the help of families of generalized measures were studied. Then, with the help of them isomorphisms of group algebras were scrutinized.

Algebras of inverse homomorphism nonlocally compact group fine measured spectra were considered. Nonassociative noncommutative Hilbert algebras on spectra of nonlocally compact groups were investigated in [94]. Their weak semi-simplicity was studied. Regular maximal left ideals were scrutinized. An existence of fine measured spectra for nonlocally compact groups was scrutinized, and examples were provided. In [15], nonassociative ringoids related to cones in nonassociative algebras of nonlocally compact groups were investigated.

Definition 9 (Section 4 in [91]). *Let the algebra $\mathcal{E} := L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ be supplied with the multiplication $f \bar{\star} u = w$ such that*

(1) $w_\alpha(g) = (f_\beta \bar{\star} u_\alpha)(g) = \int_{G_\beta} f_\beta(h) u_\alpha(\theta^\beta_\alpha(h)g)\mu_\beta(dh)$ for every $f, u \in \mathcal{E}$ and $g \in G = \prod_{\alpha\in\Lambda} G_\alpha$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, $\beta = \phi(\alpha)$, $\alpha \in \Lambda$.

If a bounded linear transformation $T : \mathcal{E} \to \mathcal{E}$ satisfies Conditions $(2, 3)$,

(2) $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ for each $\alpha \in \Lambda$,

(3) $T(f \bar{\star} u) = f \bar{\star}(Tu)$

for each $f, u \in \mathcal{E}$, then T is called a left meta-centralizer.

Theorem 38 (Section 15 in [90]). *Topological group rings $L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$ and $L^\infty(L^1_{G_\beta}(G_\alpha, \nu_\alpha) : \alpha < \beta \in \Lambda)$ are isomorphic if and only if measures μ_α and ν_α are equivalent for each $\alpha \in \Lambda$.*

Theorem 39 (Section 16 in [90]). *Let $G = \prod_{\alpha\in\Lambda} G_\alpha$ and $H = \prod_{\alpha\in\Lambda} H_\alpha$ be two topological groups supplied with box topologies τ^b_G and τ^b_H, respectively, where topological groups G_α and H_α for each $\alpha \in \Lambda$ satisfy Conditions $1(1-4)$ in [90], measures μ_α on G_α and ν_α on H_α satisfy Conditions $2(1-4)$ in [90], and a directed set Λ has not a minimal element.*

1. If topological groups G_α and H_α for each $\alpha \in \Lambda$ are topologically isomorphic, then equivalent measures μ_α and ν_α exist so that topological algebras $L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha) : \alpha < \beta \in \Lambda)$ and $L^\infty(L^1_{H_\beta}(H_\alpha, \nu_\alpha) : \alpha < \beta \in \Lambda)$ are isomorphic and their isomorphism $\hat{T}$ satisfies properties $(1-3)$ below.

2. If a bijective surjective continuous mapping $\hat{T}$ of $L^\infty(L^1_{G_\beta}(G_\alpha) : \alpha < \beta \in \Lambda)$ onto $L^\infty(L^1_{H_\beta}(H_\alpha) : \alpha < \beta \in \Lambda)$ exists and $\hat{T}^{-1}$ is continuous such that

(1) a mapping $\hat{T} = (\hat{T}_\alpha f_\alpha : \alpha \in \Lambda)$ is linear so that $\hat{T}_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{H_\beta}(H_\alpha)$ for every $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$;

(2) $\hat{T}$ is positive, that is $f_\alpha \geq 0$ in $L^1_{G_\beta}(G_\alpha)$ if and only if $\hat{T}_\alpha f_\alpha \geq 0$ in $L^1_{H_\beta}(H_\alpha)$;

(3) $\hat{T}$ is a ring homomorphism, that is $\hat{T}(f \bar{\star} u) = (f \bar{\star} \hat{T} u)$ for each $f, u \in L^\infty(L^1_{G_\beta}(G_\alpha) : \alpha < \beta \in \Lambda)$,

then topological groups G_α and H_α are topologically isomorphic, and measures μ_α and ν_α are equivalent for each $\alpha \in \Lambda$.

Theorem 40 (Section 10 in [91]). *Let S be a bounded $\mathbf{F}$-linear mapping of $\mathcal{E}$ (see Subsections 1–3 in [91] and Definition 1 above) into itself such that $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$ with $S_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$. Then, the following statements (i) and (ii) are equivalent:*

(i) an operator S has the form

(1) $S = p\hat{U}_a$ for some marked elements $a \in G := \prod_{\alpha\in\Lambda} G_\alpha$ and $p = \{p_\alpha : |p_\alpha| = 1\ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, that is

(2) $S_\alpha f_\alpha(x) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$ for any $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ and each $x \in G_\alpha$, where

(3) $\hat{U}_{g_\beta} f_\alpha(x) = f_\alpha(\theta^\beta_\alpha(g_\beta)x)$ for each $g_\beta \in G_\beta$ and $x \in G_\alpha$;

(ii) (4) S is a left meta-centralizer and

(5) $\|S_\alpha f_\alpha\| = \|f_\alpha\|$ for every $f_\alpha \in L^1_{G_\beta}(G_\alpha)$ and $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$.

Definition 10 (Section 2.25 in [94]). *Let A be a nonassociative topological algebra over a field F and let A be a complete relative to its uniformity. We say that A is left approximate associator resolvable if there exists a dense subalgebra E in A over the same field F so that for each element c in E and each element a in A, there exists a dense A family $A_{a,c}$ for which a solution q in A satisfying the equality $a(bc) = (qb)c$ exists for each b in $A_{a,c}$, where q may generally depend on a, b, and c in A.*

Theorem 41 (Section 2.26 in [94]). *Suppose that A is a nonassociative topological algebra over a field F and that A is complete relative to its uniformity. If A is left approximate associator resolvable, then the left adverse of an element g of A exists if and only if it exists modulo every closed regular left ideal in A.*

13. Nonassociative Algebras with Conjugation

Remark 21. *Take a field k and its finite Galois field extension K with Galois group $G = Gal(K/k)$. There exists the induced norm map $n : K \to k$ such that $n(b) = \prod_{\sigma \in G} \sigma(b)$ for each b in K. The well-known Hilbert's theorem 90 in [95] asserts that if G is cyclic, then an element b in K satisfies $n(b) = 1$ if and only if for each automorphism σ which generates G, there exists a nonzero element c in K such that $\sigma(c)b = c$. It appears that Hilbert has proved this for a number field K and with G of prime order. In general it was proved by Speiser (see [96] and references therein). In terms of cohomology, it can be reformulated that $H^1(G, K^\times)$ is trivial for any finite Galois field extension K of k (see, for example, Chapter VI in [97]).*

It was observed later on that Hilbert 90 in degree two is valid for a large class of algebras, which are not necessarily commutative, distributive, or they may be nonassociative. In this section a unital k-algebra A is considered such that A is a left k-vector space with multiplication; k is contained in the center of A, and naturally, the additive and multiplicative structure of k is considered the restriction to k of the additive and multiplicative structure on A; $a(bc) = (ab)c$ for a, b and c in A, whenever at least one of a, b or c belongs to the field k.

It is said that the algebra A is (weak) right distributive if $(a + b)c = ac + bc$ for each a, b and c in A (with a in k correspondingly); (weak) left distributivity is defined symmetrically. Then it is said that A is (weak) distributive if it is both (weak) left and (weak) right distributive. The algebra A is called left (or right) alternative if $a(ab) = a^2b$ (or $(ab)b = ab^2$ correspondingly) for each a and b in A. Then, A is alternative if it is left and right alternative. Consider a self-inverse k-linear mapping $^- : A \to A$ such that its restriction to k is the identity map. It is called a conjugation. If the conjugation is a ring antiautomorphism of A, then it is called an involution. Thus, the latter means that $\overline{ab} = \bar{b}\bar{a}$ for each a and b in A. Then, mappings $N : A \to A$ and $T : A \to A$ such that $N(a) = \bar{a}a$ and $T(a) = a + \bar{a}$ for each a in A are called the norm and trace on A. It is said that an element b in A is imaginary if $T(b) = 0$. The norm is anisotropic if $N(b) \neq 0$ for each nonzero b in A. It is multiplicative if $N(ab) = N(a)N(b)$ for each a and b in A. It is symmetric if $N(\bar{b}) = N(b)$ for each b in A.

Examples of such algebras were provided in [98] with the help of doubling procedures of Cayley–Dickson and Conway–Smith. The Cayley–Dickson procedure was considered in the section above. The Conway–Smith procedure starts from a division algebra B over the field k: for each nonzero b in B, there exists a unique c in B such that $bc = 1$ and $cb = 1$, which is also denoted by $c = b^{-1}$. The Conway–Smith double $\mathcal{S}_d(B)$ of B with a fixed d in $k - \{0\}$ is the k-vector space $B \times B$ with the same conjugation as in the Cayley–Dickson double. Multiplication on $\mathcal{S}_d(B)$ is

$(a,b)(c,d) = (ac + d\overline{b\bar{\bar{d}}}, \overline{\bar{b}\bar{c}} + \overline{\bar{b}\,\overline{\bar{a}\,\overline{\overline{b^{-1}}\,\bar{d}}}})$ if $b \neq 0$;

$(a,b)(c,d) = (ac, \bar{a}d)$, if $b = 0$, for every a, b, c and d in A. Therefore, $(1,0)$ is the unit element in $\mathcal{S}_d(B)$ by multiplication, and $(0,1)$ is a nonzero imaginary element.

Theorem 42 ([98]). *Let A be a weak distributive left alternative k-algebra with a conjugation such that the trace is k-valued and the norm is anisotropic and k-valued. Let A posses a nonzero imaginary element. Then an element b in A satisfies $N(b) = 1$ if and only if there is a nonzero c in A satisfying $\bar{c}b = c$.*

Proposition 15 ([98]). *Let A be a weak distributive algebra over k and let A possess a nonzero imaginary element. If an element b in A satisfies $N(b) = 1$, then there is a nonzero element c in A such that $\bar{c}b = c$.*

Proposition 16 ([98]). *If an algebra A over a field k is weak right distributive and the trace and norm on A are k-valued, then A is left alternative if and only if $\bar{a}(ab) = n(a)b$ for each a and b in A.*

14. Representations and Closures of Nonassociative Algebras

Remark 22. *Frequently, there is a situation such that an algebra A is a subalgebra of a finite dimensional algebra B over a (possibly larger) base field. In this case, it is said that the algebra A is (finite dimensional) representable. For associative algebras as B, it is usually considered a matrix algebra. Note that matrix algebras are usually not suitable in the nonassociative theory. Indeed, the matrix algebra over a field is associative, but the matrices over an alternative algebra are not necessarily alternative. Nevertheless, representable algebras compose a broader class than finite dimensional algebras. For example, any linear Lie or Jordan algebras are representable. If the algebra A is represented in the finite dimensional algebra B, then its closure in B under the Zariski topology is often studied such that it is an algebra in the same variety. The codimension sequence $c_n(A)$ for the Zariski closure in the variety of Lie algebras was estimated in [99,100]. It was studied that for Jordan PI-algebras, the codimensions can grow superexponentially [101]. It was illustrated by the Grassmann envelope of the Kantor double.*

Recall that by an algebraic structure, it is implied a collection of sets $A_1, A_2, \ldots, A_l$ supplied with a signature. The latter means that a set $W = \{w_{m,j}, m \in \mathbf{N}, 0 \leq j \leq t_m\}$ exists of operators $w_{m,j} : A_{j_1} \times \ldots \times A_{j_m} \to A_{j_{m+1}}$, where a sequence $1 \leq j_1 \leq l, \ldots, 1 \leq j_{m+1} \leq l$ depends on the operator $w_{m,j}$, and a set Id of universal relations is provided. Each relation can be written as a formula $\xi(x) = \eta(x)$, where ξ and η are terms in the operations $w_{m,j}$, where x denotes a set of indeterminates. In particular, there may be $l = 2$, $A_1 = B$, where B satisfies the operations and universal relations of a commutative associative algebra. Then, A_2 can be taken as having multiplication distributing over addition and satisfying the axioms of a B-algebra. There may be more operations. In this section there are taken $A_1 = F$, $A_2 = A$, where F is a commutative associative unital ring, and where A is a F-module. There are 0-ary products $0_F := w_{0,0}$ and $1_F := w_{0,1}$ of F, one 0-ary product $0_A := w_{0,0}$ of A, a 1-ary product $w_{1,0} : A \to A$ being negation $b \mapsto -b$ for each $b \in A$, and four binary operators corresponding to the module structure:

$(f_1, f_2) \mapsto f_1 + f_2$ for addition in F;

$(f_1, f_2) \mapsto f_1 f_2$ for multiplication in F;

$(b_1, b_2) \mapsto b_1 + b_2$ for addition in A;

$(f, b) \mapsto fb$ for scalar multiplication from $F \times A$ into A

for every f_1 and f_2 in F, b_1 and b_2 in A. The latter operators occur together with the universal relations from algebra. It is convenient to denote by $\bar{W}$ the set of operators obtained from W by deleting the specified above operators.

Then, a degree $deg_i(w_{m,j})$ is provided to each operator $w_{m,j}(x_1, \ldots, x_m) \in \bar{W}$ such that $w_{m.j}(b_1, \ldots, fb_i, \ldots, b_m) = f^{deg_i(w_{m,j})}(b_1, \ldots, b_i, \ldots, b_m)$ for each $1 \leq i \leq m$, $b_1, \ldots, b_m$ in A and $f \in F$. Then, by a W-homomorphism, it is implied a mapping $\phi = (\phi_1, \phi_2)$ with $\phi_1 : F \to F'$ and $\phi_2 : A \to A'$ such that $\phi(w_{m,j}(b_1, \ldots, b_m)) = w_{m,j}(\phi(b_1), \ldots, \phi(b_m))$. This provides a variety V_W of W-algebras considered with their W-homomorphisms. Henceforth, it is assumed that $\phi_= 1_F$. Each homomorphism $\phi : A \to A'$ induces a congruence Φ such that $(b_1, b_2) \in \Phi$ if and only if $\phi(b_1) = \phi(b_2)$. In the considered case, A as an additive group is Abelian. Therefore, each congruence Φ on A induces an ideal J_Φ such that $b \in J_\Phi$ if and only if $(b, 0) \in \Phi$. On the other side, an ideal J defines a congruence Φ_J such that $(a, b) \in \Phi_J$ if and only if $a - b \in J$. This implies that a quotient algebras modulo an ideal exists. For the given signature W, the variety V_W is considered. Then, A is called an $(W; Id; F)$-algebra. It is said that A is generated by a subset S if A does not posses the proper $(W; Id; F)$-subalgebra containing S. Then A is called affine, if W is finite and A is generated by a finite subset.

Then, each formal letter x_i is defined to be a W-formula. If $\phi_1, \ldots, \phi_m$ are W-formulas and $w_{m,j} \in W$, then $\phi(x_1, \ldots, x_k) = w_{m,j}(\phi_1, \ldots, \phi_m)$ also is a W-formula. Thus, W-formulas are defined inductively. Particularly, a formula with each operator $w_{m,j}$ taken from $\bar{W}$ provides a so-called word formula. Notice that a free $(W; F)$-algebra denoted by $F\{x; W\}$ exists. Its elements are the $(W; Id; F)$-formulas. There, each x_i is considered a formal element, which is called indeterminate. Thus, the free W-algebra $F\{x_1, \ldots, xn; W\}$ in the indeterminates $x_1, \ldots, x_n$ exists by putting $x = \{x_1, \ldots, x_n\}$. That is, $F\{x; W\}$ is spanned by the word formulas.

Furthermore, to each extra universal relation, a $(W; Id; F)$-formula can be posed. The latter is called a $(W; Id; F)$-polynomial identity, or briefly W-PI. This means that the W-PI of the

$(W;Id;F)$-algebra A is the $(W;F)$-polynomial $f \in F\{x;W\}$ with coefficients in F such that f vanishes identically for any substitution in A. Generally, congruences and polynomial identities are considered in the universal algebra theory such that $(f,g) \in F\{x;W\} \times F\{x;W\}$ with $f(b) = g(b)$ for each substitutions into A. In the considered case, the algebra A with regard to addition $(A,+)$ is the Abelian group; hence, (f,g) can be replaced by $f-g$. By $id(A)$, it is denoted the set of PIs of the algebra A.

The T_W-congruence of a family of pairs of polynomials $\{(f_j,g_j) : j \in P\}$, $P \subset \mathbf{N}$, in a $(W;Id;F)$-algebra A is the $(W;Id;F)$-ideal arising from a congruence supplied by all substitutions of the pairs $\{(f_j,g_j) : j \in P\}$ in A. The T_W-ideal of a family of polynomials $\{f_j : j \in P\}$ means the $(W;Id;F)$-ideal corresponding to the T_W-congruence of the pairs $\{(f_j,0) : j \in P\}$. Vice versa, each element of the T_W-ideal J is a PI of the quotient $(W;Id;F)$-algebra $F\{x;W;Id\}/J$. Note that the $(W;Id;F)$-algebra $F\{x;W;Id\}/J$ is relatively free: for each $(W;Id;F)$-algebra A with $id(A) \supseteq J$ and each b_1, b_2,...,b_m in A, there exists a natural homomorphism $F\{x;W;Id\}/J \to A$ mapping $x_i \mapsto a_i$ for each $i = 1,2,...$.

If for a $(W;Id;F)$-algebra A an embedding exists into a finite dimensional $(W;Id;K)$-algebra A_K over a (commutative associative) field K such that A_K is a faithful F-algebra, then A is called representable.

Assume that K is an algebraically closed field such that $K \supseteq F$ and there exists a representation $\rho : A \to B$ of a $(W;Id;F)$-algebra A into a finite dimensional $(W;Id;K)$-algebra B. Then, the Zariski closure $cl_Z(\rho(A))$ is the closure of $\rho(A)$ relative to the Zariski topology of B over K. Recall that a W-formula $\phi(x_1,...,x_n)$ is $(W;Id;F)$-multilinear if ϕ is F-linear by x_j for each $1 \le j \le n$. If each operator in $\bar{W}$ is $(W;Id;F)$-multilinear, then the variety of $(W;Id;F)$-algebras is called $(W;Id;F)$-multilinear. Below in this section, A_K denotes the $(W;K)$-subalgebra of B generated by A, and the variety V_W is assumed to be multilinear. For more details and examples, see [102,103].

Assume that $S = \{s_1,...,s_m\} \subseteq A$, and assume that for each (W,F)-polynomial $f(x_1,...,x_n) \in F\{x\}$, f is a PI of A if and only if $f(s_{j_1},...,s_{j_n}) = 0$ for every $1 \le j_i \le m$ and $1 \le i \le n$. Then S is called is a test set of A. Then the minimal number m in a family of test sets of A is called the PI-generic rank of A.

If each operator is $(W;Id;F)$-multilinear and each universal relation (ϕ,ψ) is homogeneous (that is, $deg\phi = deg\psi$), then a variety of $(W;Id;F)$-algebras is said to be $(W;Id;F)$-multilinear. By $F\{x_1,...,x_n;W;Id\}$, the relatively free (W,Id)-algebra $F\{x_1,...,x_n;W\}/Id$ is denoted. If the $(W;Id;F)$-multilinear variety of $(W;Id;F)$-algebras is $\mathbf{N}$-graded, then it induces a $\mathbf{N}$-grading of $F\{x_1,...,x_n;W;Id\}$ as a direct sum of vector spaces.

Let $\mu(n)$ denote the dimension of Q_n over F, where Q_n is the subspace of degree n elements of $F\{x_1,...,x_n;W;Id\}$. The considered variety is locally finite if each $\mu(n)$ is finite. It is assumed that the variety V_W is multilinear and locally finite. Then, the codimensions of the $(W;Id;F)$-algebra A is defined as $c_n(A) := dim_F(Q_n/(id(A) \cap Q_n))$.

For a finite dimensional affine $(W;K)$-algebra B presented in the form $B = \sum_{j=1}^{r} Kb_j$, the decomposition of operators exists:

$w_{m,j}(b_{l_1},...,b_{l_m}) = \sum_{l=1}^{r} \gamma_{l_1,...,l_m,l} b_l$,

where $\gamma_{l_1,...,l_m,l} \in K$ is a so-called structure constant of the algebra B for every l_1,...,l_m, l.

Proposition 17 ([102])**.** *If the variety V_W is multilinear, then a test set S for a Zariski closed algebra A can be chosen as the union of the finite component with finitely many elements from each infinite component.*

Theorem 43 ([102])**.** *Each Zariski closed $(W;Id;F)$-algebra A over an arbitrary field F has finite PI-generic rank, which is bounded by the size of the finite component of A plus the number of infinite components of A.*

Theorem 44 ([102])**.** *Assume that A is a representable $(W;Id;F)$-algebra over a field F, where $B \supseteq A$ is a finite dimensional algebra over an extension field K of F. Assume also that the PI-generic rank of A is $m \in \mathbf{N}$, $r = dim_K B$, and the number of structure constants of B is $q \in \mathbf{N}$. Then the codimension of A is estimated by $c_n(A) \le r(2n)^{mr+q-1} m^n$ for each $n > 1$.*

Other results on codimension theory for nonassociative algebras are contained in [104] and references therein. Braided categories in relation with nonassociative algebras are considered in [105] and references therein. A Gröbner–Shirshov basis for the universal enveloping right-symmetric algebra of a Lie algebra was investigated in [106]. There are specific features in comparison with associative rings and tri-algebras [107,108].

15. Nonassociative Algebras and Modules over Them with Metagroup Relations

Remark 23. *Generalized Cayley–Dickson algebras play very important roles in mathematics and quantum field theory [5,70,109–111]. These algebras are nonassociative such that a multiplicative law of their canonical generators provides nonassociative metagroups instead of a group [112]. Nonassociative algebras and modules over them with metagroup relations were studied. Their structure and cohomologies were investigated. Definitions and notations are given of nonassociative metagroup algebras, modules over nonassociative algebras with metagroup relations, graded over metagroups algebras and modules, and their homological complexes in [113–117]. Smashed and twisted wreath products of metagroups, constructions of metagroups and their examples are provided in [113–118]. Nonassociative smashed tensor products and splitting extensions of modules and algebras with metagroup relations were scrutinized in [116]. Homotopisms and homologisms of homological complexes were studied over nonassociative algebras with metagroup relations in [114]. Torsions of homological complexes and modules were investigated over nonassociative algebras with metagroup relations in [115]. Functors for categories with metagroup relations and satellites of functors were investigated in [117]. Moreover, an exactness of satellite sequences and diagrams was studied.*

Definition 11. *Let G be a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G satisfying the conditions:*

(1) For each a and b in G, there is a unique $x \in G$ with $ax = b$.

(2) A unique $y \in G$ exists, satisfying $ya = b$, which are denoted by
$x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$ correspondingly.

(3) There exists a neutral (i.e., unit) element $e_G = e \in G$:
$eg = ge = g$ for each $g \in G$.

The set of all elements $h \in G$ commuting and associating with G.

(4) $Com(G) := \{a \in G : \forall b \in G,\ ab = ba\}$.

(5) $N_l(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ab)c = a(bc)\}$.

(6) $N_m(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ba)c = b(ac)\}$.

(7) $N_r(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}$.

(8) $N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$;
$\mathcal{C}(G) := Com(G) \cap N(G)$ is called the center $\mathcal{C}(G)$ of G.

We call G a metagroup if a set G possesses a single-valued binary operation and satisfies conditions (1)–(3) and

(9) $(ab)c = t_3(a,b,c)a(bc)$
for each a, b and c in G, where $t_3(a,b,c) \in \mathbf{\Psi}$, $\mathbf{\Psi} \subset \mathcal{C}(G)$,
where t_3 shortens a notation $t_{3,G}$, where $\mathbf{\Psi}$ denotes a (proper or improper) subgroup of $\mathcal{C}(G)$.

Then, G will be called a central metagroup if in addition to (9), it satisfies the condition

(10) $ab = t_2(a,b)ba$
for each a and b in G, where $t_2(a,b) \in \mathbf{\Psi}$.

Particularly, $Inv_l(a) = Div_l(a,e)$ is a left inversion, and $Inv_r(a) = Div_r(a,e)$ is a right inversion.

In view of the nonassociativity of G in general, a product of several elements of G is specified as usual by opening "(" and closing ")" parentheses. For elements $a_1,...,a_n$ in G, we shall denote shortly by $\{a_1,...,a_n\}_{q(n)}$ the product, where a vector $q(n)$ indicates an order of pairwise multiplications of elements in the row $a_1,...,a_n$ in braces in the following manner. Enumerate positions: before a_1 by 1, between a_1 and a_2 by 2,..., by n between a_{n-1} and a_n, by $n+1$ after a_n. Then, put $q_j(n) = (k,m)$ if there are k opening "(" and m closing ")" parentheses in the ordered product at the j-th position *of the type* $)...)(...($*, where* k *and* m *are nonnegative integers,* $q(n) = (q_1(n), ..., q_{n+1}(n))$ *with* $q_1(n) = (k, 0)$ *and* $q_{n+1}(n) = (0, m)$.

Traditionally, S_n *denotes the symmetric group of the set* $\{1, 2, ..., n\}$*. Henceforth, maps and functions on metagroups are supposed to be single valued if something else is not specified.*

Let $\psi : G \to G$ *be a bijective surjective map satisfying the following condition:* $\psi(ab) = \psi(a)\psi(b)$ *for each* a *and* b *in* G*. Then* ψ *is called an automorphism of the metagroup* G*.*

Theorem 45 (Section 6 in [113])**.** *Let* A *be a nonassociative metagroup algebra over a commutative associative unital ring* $\mathcal{T}$*. Then, an algebra* B *over* $\mathcal{T}$ *exists such that* B *contains* A *and each* $\mathcal{T}$*-homogeneous derivation* $d : A \to A$ *is the restriction of an inner derivation of* B*.*

Theorem 46 (Section 7 in [113])**.** *Suppose that* A *is a nonassociative metagroup algebra of finite order over a commutative associative unital ring* $\mathcal{T}$*, and* M *is a finitely generated two-sided* A*-module. Then,* M *is semisimple if and only if its cohomology group is null* $H^n(A, M) = 0$ *for each natural number* $n \geq 1$*.*

Theorem 47 (Section 1 in [119])**.** *Suppose that* $A = \mathcal{T}[G]$ *is a nontrivial nonassociative metagroup algebra over a commutative associative unital ring* $\mathcal{T}$ *such that* $\mathbf{\Psi}1 \subseteq (G1) \cap \mathcal{T}e$*, where* $(G1) \cup \mathcal{T}e \subset A$*. Then* $H^1_{\mathcal{T}}(A, M) = 0$ *for each two-sided* A*-module* M *if and only if* A *is a separable* $\mathcal{T}$*-algebra.*

Theorem 48 (Section 10 in [116])**.** *Assume that* B *and* D *are* A*-algebras, where* $A = \mathcal{T}[G]$ *is a metagroup algebra,* $\mathcal{T}$ *is an associative commutative unital ring. Assume also that* D *is a subalgebra of* B*. Then the following conditions are equivalent:*

(i) $B = D \oplus Y$*, where* Y *is a* (D, D)*-bisubmodule in* B*.*

(ii) *For each* A*-algebra* C *and each* (C, D)*-bimodule* X*, a homomorphism* $\nu_X : X \to (X^B)_D$ *is a splitting injective* A*-exact homomorphism of* (C, D)*-bimodules.*

Theorem 49 (Section 6 in [115])**.** *Assume that* $\mathcal{C}$ *and* ${}^1\mathcal{C}$ *are* G*-graded* B*-complexes of* G*-graded* B*-bimodules and* G*-graded left* B*-modules, respectively, and that* $\mathcal{B}(\mathcal{C})$ *and* $\mathcal{B}({}^1\mathcal{C})$ *are projective. Then the canonical homomorphism* $\hat{h}(\mathcal{C}, {}^1\mathcal{C}) : H(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{C} \bigotimes_B {}^1\mathcal{C})$ *has a* G*-epigeneric retraction.*

Theorem 50 (Section 3 in [117])**.** *Assume that there exists an exact sequence*

$${}^{i_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2}M_0$$

in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *with* $s \in \{\mathsf{eg}, \mathsf{e}\}$*. If* T *is an additive covariant (or contravariant) half-exact functor, then there exists an exact sequence*

$$... \to S^{n-1}T({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_1}M) \xrightarrow[S^nT({}^{i}_{i_1}\mathbf{f})]{} S^nT({}^{i}M)$$

$$\to S^nT({}^{i_2}M) \to S^{n+1}T({}^{i_1}M) \to ..., \text{ (or}$$

$$... \to S^{n-1}T({}^{i_1}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_2}M) \xrightarrow[S^nT({}^{i_2}_{i}\mathbf{f})]{} S^nT({}^{i}M)$$

$$\to S^nT({}^{i_1}M) \to S^{n+1}T({}^{i_2}M) \to ... \text{ correspondingly).}$$

16. Near to Associative Nonassociative Algebras and Modules over Them

Remark 24. *For a nondegenerate alternative algebra* A*, regular in the Neumann sense and with a semigroup identity, necessary and sufficient conditions were studied for* A *to be a ring with a single-valued addition [120]. There were studied generalizations of alternative algebras, such as hom-alternative algebras and hom-prealternative bialgebras (see [121,122] and references therein). Radicals of algebras close to associative were studied in [123]. For Zinbiel and q-Zinbiel algebras, identities and varieties were studied in [124] and references therein.*

One of the important classes of near to associative nonassociative algebras compose Cayley–Dickson algebras, which were first investigated by Dickson [70]. As a particular case, they include Cayley algebras. Generalized Cayley–Dickson algebras were studied in [125]. In particular, relations *of Cayley–Dickson algebras with loops and appearing in them identities were studied in [126]. Relations with analytic geometry over them were also outlined there. Then colour algebras were studied with the help of Cayley-Dickson algebras in [127]. Algebras over the Steenrod algebra were investigated in [128].*

For alternative algebras the Peirce decomposition plays an important role [5]. For alternative and Jordan algebras, their derivations also were investigated. Particularly, octonion orthocomplemantable modules were investigated in [129].

Theorem 51 (Artin, Section 3.1 in [5])**.** *If an algebra A is alternative, then the subalgebra generated by any two elements of A is associative.*

Theorem 52 (Zorn, Section 3.7 in [5])**.** *If A is an alternative finite-dimensional algebra, then its radical is the set S of all properly nilpotent elements of A.*

Theorem 53 (Section 3.10 in [5])**.** *If A is a nontrivial finite-dimensional semisimple alternative algebra, then A has a unit element.*

Theorem 54 ([5,130])**.** *If an algebra A is semi-simple alternative finite dimensional over a field of characteristic zero, then each derivation of A is inner.*

Theorem 55 ([131])**.** *If an algebra B is semi-simple Jordan finite dimensional over a field of characteristic zero, then each derivation of B is inner.*

Moreover, explicit forms of derivations were provided there for these algebras.

Remark 25. *It is worthwhile to compare this with derivations of associative algebras. For a unital separable C^*-algebra A over the complex field $\mathbf{C}$ each derivation is inner if and only if A is a direct sum of C^*-algebras which are either homogeneous of finite degree or simple [132]. For non-separable C^*-algebras, conditions were studied for which non-inner derivations exist [132]. Automorphisms and derivations of nonassociative analogs of infinite dimensional C^*-algebras were investigated in [112]. There were found specific features of the nonassociative case in comparison with the associative case of the C^*-algebras. They were studied with the help of infinite dimensional Cayley–Dickson algebras over $\mathbf{R}$ and metagroups. For infinite dimensional Cayley–Dickson algebras, their completions and homomorphisms were investigated in [133]. They were studied over Banach associative commutative unital rings, particularly, also over fields.*

The Cayley–Dickson algebras were useful for the development of noncommutative and nonassociative mathematical analysis over them [134]. The (super)differentiability of functions defined on domains of the real Cayley–Dickson algebra was investigated. A noncommutative version of the Cauchy–Riemann conditions was studied. The noncommutative analogue of the Cauchy integral was scrutinized. Criteria for functions of Cayley–Dickson variables to be analytic were investigated. The Cayley–Dickson algebra analogues of the Cauchy, Hurewicz, Mittag–Löffler, Rouche, and Weierstrass theorems and the argument principle were proven. This was applied to the study of zeros of polynomials of Cayley–Dickson variables. Certainly, there, specific features appear. There exist polynomials $P_n(z)$ of degree $n \geq 2$ of the Cayley–Dickson variable $z \in \mathcal{A}_r$ such that $V = \{z : P_n(z) = 0\}$ may contain connected components V_j, which are manifolds of dimension greater than zero if $r \geq 3$, where $\mathcal{A}_r$ denotes the standard Cayley–Dickson algebra over the real field $\mathbf{R}$ of dimension 2^r as the vector space over $\mathbf{R}$. Examples of analytic and special functions including the beta and gamma functions of Cayley–Dickson variables were studied. Noncommutative nonassociative algebraic analysis over Cayley–Dickson algebras is based on noncommutative nonassociative word algebras over Cayley–Dickson algebras [135].

Moreover, functions of several Cayley–Dickson variables were investigated. Integral representation theorems for them were proven. With the help of these theorems, solutions of $\bar{\partial}$-equations were investigated. Integral formulas of the Martinelli–Bochner, Leray, and Koppelman type used in complex analysis were scrutinized in a new generalized form for functions of Cayley–Dickson variables [136]. Then, the specific class of pseudoconformal functions of octonion variables was *studied. Their normal families were investigated. For their family to be normal, four criteria were proven [137].*

A new method of studies of Diophantine equations with the help of Cayley–Dickson algebras was presented in [138]. It was based on investigations of special meromorphic functions of Cayley–Dickson variables. Then new classes of quasi-conformal and quasi-meromorphic mappings were studied in [139]. Residues and the argument principle for quasi-meromorphic mappings were investigated. It was proven that the family of all quasi-conformal diffeomorphisms of a domain is a topological group G relative to composition of mappings. Particular conditions on them were studied, for which G is a finite-dimensional Lie group over $\mathbf{R}$*. Relations between integral transformations of functions of octonion variables and quasi-conformal functions were scrutinized. It also included studies of noncommutative analogs of the Mellin transformations. Applications were outlined to solutions of problems of complex analysis and number theory in [139].*

Recall that loop algebras and Kac–Moody algebras over the complex field $\mathbf{C}$ *became already classical (see, for example, [140] and references therein). They are related to meromorphic functions in an open domain* U *with one singular marked point* $z_0 \in U$ *in* $\mathbf{C}$*. Their generalizations, such as affine and wrap quasi-algebras over Cayley–Dickson algebras, were investigated in [141]. For this purpose, residue operators of functions of Cayley–Dickson variables were studied. They were utilized for a construction of such quasi-algebras. Their structure was scrutinized. It is worth mentioning that meromorphic functions of the Cayley–Dickson variable may have singularities in a closed subset* W *of codimension not less than* 2 *such that* W *may be of dimension greater than zero. This implies that winding around* W *may exist in any plane containing* $\mathbf{R}$*. This means that in such a case, winding surfaces around* W *appear such that the loop interpretation is already lost [142]. Therefore, analogs of loop algebras over the Cayley–Dickson algebras were called wrap quasi-algebras.*

Operator theory of bounded and unbounded operators in Hilbert spaces over the octonion algebra (i.e., Hilbert octonion bimodules) was investigated (see [143–147] and references therein). In them, theorems were proven on spectral representations of projection-valued graded measures of normal quasilinear operators, which can be unbounded. Appearing there, graded projection-valued measures in the general case may be noncommutative and nonassociative. Furthermore, nonassociative analogs of C^**-algebras were scrutinized in [148].*

Theorem 56. *(Wedderburn principal theorem for alternative algebras 3.18 in [5]). If* A *is a finite-dimensional alternative algebra over a field* F *with radical* J *such that the quotient algebra* A/J *is separable, then* A *is the direct sum* $A = H \oplus S$*, where* S *is a subalgebra of* A *isomorphic with* A/J*.*

Remark 26. *Alternative bimodules. Assume that* A *is an alternative algebra over a field* F*, and* $M = {}_F M$ *is a vector space over* F *such that there exist two* F*-bilinear compositions* $w_{2,3} : A \times M \to M$ *and* $w_{2,4} : A \times M \to M$ *with a shortened notation* $w_{2,3}(a, x) = ax$ *and* $w_{2,4}(a, x) = xa$ *for each* $a \in A$*,* $x \in M$*, satisfying*

$(a, a, x) = (x, a, a) = 0$ *for each* $a \in A$*,* $x \in M$*, and*

$(a, x, b) = -(x, a, b)$*,* $(b, a, x) = -(b, x, a)$ *for every* a *and* b *in* A*,* $x \in M$*, where* $(a, b, c) = (ab)c - a(bc)$ *denotes the associator with one argument in* M *and two arguments in* A*. Then* $M = {}_A M_A$ *is called an alternative* A*-bimodule (or bimodule over* A*).*

Then, the vector space direct sum $B = A \oplus M$ *can be supplied with an* F*-algebra structure. For example, there exists multiplication on* B *such that* $(a + x)(b + y) = ab + (xb + ay)$ *for every* a *and* b *in* A*,* x *and* y *in* M*. In this case,* B *is called the split null extension or semidirect sum of* A *and* M*. This implies that the* A*-bimodule* M *is alternative if and only if* B *is the alternative algebra. Notice that* M *is an ideal of* B *and* $M^2 = 0$*, if* M *is the alternative* A*-bimodule [5].*

For the alternative algebra A *over* F *and its alternative* A*-bimodule* M*, there exist the* F*-linear operators* S_b *and* D_b *on* M *such that* $S_b x = bx$ *and* $D_b x = xb$ *for each* $x \in M$ *and* $b \in A$*. Therefore,* $S_b^2 = S_{b^2}$*,* $D_b^2 = D_{b^2}$*,* $[D_a, S_b] = -S_a \circ S_b + S_{ab}$ *and* $D_{ba} - D_a \circ D_b = -[D_b, S_a]$ *for every* a *and* b *in* A *since* A *and* M *are alternative, where* $[D_a, S_b] = D_a \circ S_b - S_b \circ D_a$*,* $D_a \circ S_b$ *denotes the composition of* F*-linear operators such that* $(D_a \circ S_b)x = D_a(S_b x)$ *for each* $x \in M$*. Thus, there exists the pair of* F*-linear mappings* $S : A \ni b \mapsto S_b$ *and* $D : A \ni b \mapsto D_b$ *from* A *into* $L(M, M)$*, where* $L(M, M)$ *denotes the associative* F*-algebra of all* F*-linear mappings from* M

into M. Since S and D are not homomorphisms of algebras for the nonassociative A, then (S, D) is not, strictly speaking, a (bi)representation. Though, by analogy with the associative case in [5,71], such a terminology is used implying the complicated equations provided above, it would be better to say a pseudo-representation instead of a representation in such cases. For comparison, if G is an associative F-algebra and a vector space N over F is a G-bimodule $N = {}_G N_G$, then $S_{ab} = S_a \circ S_b$ and $D_{ba} = D_a \circ D_b$ for each a and b in G since N satisfies $(ab)x = a(bx)$ and $x(ba) = (xb)a$ for each $x \in N$, b and a in G. Therefore, $S : G \to L(N, N)$ and $D : G^{op} \to L(N, N)$ are homomorphisms, where G^{op} denotes the opposite algebra of G. That is, for (G, N), the pair (S, D) is the birepresentation [149,150].

Particularly, in other notation, $D = R$ and $S = L$ for the alternative algebra A over F, where $L_b a = ba$ and $R_b = ab$ for each a and b in A. Certainly, A also has the structure of the alternative A-bimodule ${}_A A_A$. If A is a subalgebra of an alternative algebra B over F, and if J is an ideal of B, then the pair (L, R) of F-linear mappings on B induces (L, R) from A into $L(J, J)$. Then the Lie multiplication algebra $\mathcal{L}(A)$ of the alternative F-algebra A is isomorphic with $\mathcal{L}(A) = R(A) + L(A) + [L(A), R(A)]$ if $char(F) \neq 2$.

Proposition 18 (The second Whitehead lemma for alternative algebras, 3.22 in [5])**.** *Suppose that A is a finite-dimensional separable alternative algebra over a field F, and M is a finite-dimensional over F alternative A-bimodule. If f is a F-bilinear mapping from A into M such that $F(a, a, b) = F(b, a, a) = 0$ for each a and b in A, where $F(a, b, c) = f(a, b)c + f(ab, c) - af(b, c) - f(a, bc)$, then a F-linear mapping $g : A \to M$ exists such that $f(a, b) = ag(b) + g(a)b - g(ab)$ for each a and b in A.*

Proposition 19. *(The first Whitehead lemma for alternative algebras, pages 89-90 in [5]) If A is a finite-dimensional separable alternative algebra over a field F of characteristic $char(F) \notin \{2, 3\}$, M is a finite-dimensional over F alternative A-bimodule, and $B = A \oplus M$ is the split null extension, f is a one-cocycle of A into M (that is, f is a F-linear mapping of A into M such that $f(ab) = f(a)b + af(b)$ for each a and b in A). Then there exist b in the nucleus $N(B)$ of B, x_j in A, z_j in M such that $f(a) = [a, b] + a \sum_j D_{x_j, z_j}$ for each a in A, where $D_{x,z} = R_{[x,z]} - L_{[x,z]} - 3L_x, R_z]$.*

Theorem 57 (page 90 in [5])**.** *Assume that A is a finite-dimensional alternative algebra over a field F of zero characteristic with Wedderburn decomposition $A = H \oplus S$ and B is a semisimple subalgebra of A. Then a (nilpotent) derivation D of A exists into the radical of the multiplication algebra $\mathcal{M}(A)$ such that the automorphism $g = exp(D)$ of A maps B onto a subalgebra of H.*

Many other results on the structure of alternative and Jordan algebras, their radicals, modules over them, and representations are described in [71] and references therein.

Theorem 58 (Section 4 in Chapter 11, Section 3 in [71])**.** *Let A be an alternative algebra, $J(A)$ be its Zevlakov radical, P_1 be a set of all irreducible right alternative representations of A, P_2 be a set of all regular representations of P_1. Then $J(A) = \bigcap_{\rho \in P_1} Ker_\rho(A) = \bigcap_{\rho \in P_2} Ker_\rho(A)$.*

17. Applications of Nonassociative Algebras and Modules over Them in Cryptography and Coding

Definition 12. *If for a linear $[n, k, d]_q$-code k is the maximum possible dimension of a linear code over the finite field $\mathbf{F}_q$ with length n and distance d, then this code is called linearly optimal. $n(k, q)$ (or $m(k, q)$) denotes the maximum length of an MDS code with combinatorial dimension k over an alphabet consisting of q elements (or linear MDS code over the field $\mathbf{F}_q$ correspondingly, where $q = p^n$, p is a prime number, and n is a natural number).*

This definition implies that $m(k, q) \leq n(k, q)$. By ϕ is denoted the Euler totient function.

Proposition 20 ([151])**.** *If n and k are positive integers, q is a primary number such that $n > m(k + 1, q)$, then any linear $[n, n - k, k]_q$-code is linearly optimal.*

Remark 27. *For constructing linear over the field $\mathbf{F}_q$ codes with extremal properties, it is possible to use the following. One can take a finite loop $L = \{l_1, ..., l_n\}$ and a loop algebra $A = \mathbf{F}_q L$. Then, for each left ideal $J \leq_A A$ the code $\mathcal{C} = \mathcal{C}(J)$ is defined as the set of all words $(a_1, ..., a_n)$ in $\mathbf{F}_q^n$ such that $\sum_j a_j l_j \in J$. Such codes are called loop codes. Each loop algebra (or a quasigroup algebra) contains two trivial MDS codes: $[n, 1, n]$-code $\mathcal{C}(J_0)$ which corresponds to the left ideal $J_0 = \mathbf{F}_q(\sum_{l \in L} l)$ and $[n, n-1, 2]$-code $\mathcal{C}(S)$ corresponding to the fundamental ideal $S(A) = \{\sum_{l \in L} b_l l : \sum_{l \in L} b_l = 0; \forall l \in L, b_l \in \mathbf{F}_q\}$, which is the left and right annihilators of the ideal J_0. For example, the chains of linear $[k, k-3, 3]_q$-codes over the finite field $\mathbf{F}_q$ were constructed in [152], where $k = 2q$ or $k = 2q - 2$. It appears that such codes are linearly optimal.*

Theorem 59 ([152])**.** *Let $P = \mathbf{F}_q$ with $q = p^n$. Let L be a loop of order $2q - 2$ containing a cyclic group H of order $q - 1$. Let an element b in $L - H$ exist, satisfying the following three conditions:*

$\forall l \in L - H, \exists h_l \in H, \forall h \in H, lh = b(h_l h)$;

$\forall l \in L - H, \exists s_l \in H, \forall h \in H, l(bh) = s_l h$;

$\forall a \in H, \exists h_a \in H, \forall h \in H, a(bh) = h_a h$. If $char(P) \neq 2$, then the lattice of left ideals of loop algebra PL of L over P contains $\phi(q-1)$ structures of the following form: $\mathcal{L}_i \subseteq \mathcal{M}_i^-$, $\mathcal{M}_i^+ \subseteq \mathcal{N}_i$, where $i \in \{1, ..., q-1\}$, and $\mathcal{C}(\mathcal{M}_i)$ are linearly optimal $[2q-5, 2q-3, 3]_q$-codes. Moreover, all ideals occurring in these structures are pairwise different.

If $char(P) = 2$, then the lattice of left ideals of PL contains $\phi(q-1)$ chains of the following form: $\mathcal{L}_i \subseteq \mathcal{M}_i \subseteq \mathcal{N}_i$, where $i \in \{1, ..., q-1\}$, and $\mathcal{C}(\mathcal{M}_i)$ are linearly optimal $[2q-5, 2q-3, 3]_q$-codes. Moreover, all ideals occurring in these chains are pairwise different.

Other applications of nonassociative algebras to cryptography and coding are provided, for example, in [59,60,153–157] and references therein. There also are useful partially pseudo-ordered (K-ordered) rings, which can be nonassociative [158].

18. Applications of Modules over Nonassociative Algebras in Geometry and Physics

Recall that the Witt algebra W_n is the Cartan-type Lie algebra. It arises from vector fields on the n-dimensional torus with Laurent polynomial coefficients. This algebra is related with the Lie algebra of derivations of Laurent polynomial algebra with n variables. Modules over simple generalized Witt algebras were investigated in [159]. Another direction of investigations was nonassociative geometry in quasi-Hopf representation categories. In [160] were studied noncommutative and nonassociative algebras A and bimodules over them using the representation category of a quasitriangular quasi-Hopf algebra. Their applications to noncommutative and nonassociative gravity and string theory were discussed there. Nonassociative algebras were used for investigations of slave-Boson decomposition in supercondactivity [161], also for studies of nonassociative quantum mechanics [162,163].

Principles of noncommutative geometry of Stein manifolds analogues over Cayley–Dickson graded algebras were investigated in [136]. Then groups of pseudoconformal diffeomorphisms of octonion manifolds were scrutinized. Their structure was elucidated: for compact octonion manifolds, they have a structure of finite-dimensional Lie groups. Examples were provided. There appeared many characteristic features of noncommutative nonassociative geometry over the octonion algebra $\mathbf{O}$ in comparison with commutative geometry over $\mathbf{R}$ or $\mathbf{C}$ [137].

Applications of the Cayley–Dickson algebras to problems of hydrodynamics and semiconductors were provided in [164–168]. There were studied multidimensional noncommutative Laplace direct and inverse transforms over octonions and Cayley–Dickson algebras in [169]. Their applications were investigated to solutions of partial differential equations including that of elliptic, parabolic and hyperbolic type. There also were studied partial differential equations of higher order with variable coefficients with or without boundary conditions with the help of multidimensional noncommutative Laplace direct and inverse transforms in [169]. Furthermore, nonassociative algebras are widely used in particle physics (see [170] and references therein). Unification theories in physics and Yang–Baxter PDEs analysis are based on nonassociative algebras, including quasi-Hopf algebras (see [105,171,172] and references therein). For gauge theory, nonassociative algebras were utilized in [173]. Classical aspects of nonassociative binary systems and nonassociative geometry are provided in [174–177] and references therein. The Green–Schwarz superstring was investigated in [178] with the help of nonassociative algebras. Quasi-Hopf twist deformations and nonassociative quantum mechanics were investigated in [179]. De Sitter space representation of a curved space-time was studied with the help of the Cayley–Dickson algebra in [180]. Grand unification theory was investigated in [181] with the help of the octonion algebra. It also was applied to Yang–Mills fields. Applications of Lie algebras to partial differential equations and networks were investigated in [182,183].

19. Conclusions

The material reviewed above on nonassociative algebras, rings and modules over them can be used for further research in this area. More concrete directions for further activity can be found in the cited above literature. Other useful ideas are provided in [184–192]. This will be important not only for the development of algebra but also interactions of different branches of mathematics and applications in other sciences. As it was demonstrated above, nonassociative algebras, rings and modules over them play a very important role in cryptography, physics, hydrodynamics, partial differential equations, quantum mechanics, etc.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Hamilton, W.R. *Elements of Guaternions*; Chelsea Pub. Co: New York, NY, USA, 1969.
2. Graves, J.T. On a Connection between the general theory of normal couples and the theory of complete quadratic functions of two variables. *Phil. Mag.* **1845**, *26*, 315–320. [CrossRef]
3. Cayley, A. On Jacobi's elliptic functions, in reply to the Rev.; and on quaternions. *Phil. Mag.* **1845**, *26*, 208–211.
4. Hamilton, W.R. Note, by Sir W. R. Hamilton, respecting the researches of John T. Graves, Esq. *Trans. Royal Irish Acad.* **1848**, *21*, 338–341.
5. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Dover Publishing Inc.: New York, NY, USA, 1995.
6. Plotkin, B. Varieties of algebras and algebraic varieties. *Israel J. Math.* **1996**, *96*, 511–522.
7. Berzins, A.; Plotkin, B.; Plotkin, E. Algebraic geometry in variaties of algebras with the given algebra of constants. *J. Math. Sci. N. Y.* **2000**, *102*, 4039–4070. [CrossRef]
8. Malcev, A.I. *Algebraic Systems*; Nauka: Moscow, Russia, 1970.
9. Movsisyan, Y.M. Hyperidentities and related concepts. *Armen. J. Math.* **2017**, *9*, 146–222. [CrossRef]
10. Movsisyan, Y.M. Hyperidentities in algebras and varieties. *Russ. Math. Surv.* **1998**, *53*, 57–108. [CrossRef]
11. Movsisyan, Y.M. *Hyperidentities and Hypervarieties in Algebras*; University in Yerevan: Yerevan, Armenia, 1990.
12. Artamonov, V.A; Pilz, G.F. *The Concise Handbook of Algebra*; Springer Science+Business Media: Dordrecht, The Netherlands, 2002.
13. Plotkin, B. *Universal Algebra, Algebraic Logic, and Databases*; Kluwer Academic: New York, NY, USA, 1994.
14. Cohn, P.M. *Universal Algebra*; Harper and Row Publishers: New York, NY, USA, 1965.
15. Ludkowski, S.V. Skew continuous morphisms of ordered lattice ringoids. *Mathematics* **2016**, *4*, 17. . [CrossRef]
16. Akivis, M.A. The local algebras of a multidimensional three-web. *Sib. Math. J.* **1976**, *17*, 3–8. [CrossRef]
17. Shestakov, I.P. Linear representability of Akivis algebras. *Dokl. Akad. Nauk.* **1999**, *368*, 21–23.
18. Shestakov, I.P. Every Akivis algebra is linear. *Geom. Dedicata* **1999**, *77*, 215–223. [CrossRef]
19. Bremner, M.R.; Hentzel, I.R.; Peresi, L.A. Dimension formulas for free Akivis algebras and primitive elements in free nonassociative algebras. *Comm. Algebra* **2005**, *33*, 4063–4081. [CrossRef]
20. Artamonov, V.A.; Klimakov, A.V.; Mikhalev, A.A.; Mikhalev, A.V. Primitive and almost primitive elements of Schreier varieties. *J. Math. Sci. N. Y.* **2019**, *237*, 157–179. [CrossRef]
21. Shestakov, I.P.; Umirbaev, U.U. Free Akivis algebras, primitive elements, and hyperalgebras. *J. Algebra* **2002**, *250*, 533–548. [CrossRef]
22. Perez-Izquierdo, J.M. Algebras, hyperalgebras, nonassociative bialgebras and loops. *Adv. Math.* **2007**, *208*, 834–876. [CrossRef]
23. Goto, M.; Grosshans, F.D. *Semisimple Lie Algebras*; Marcel Dekker, Inc.: New York, NY, USA, 1978.

24. Pérez-Izquierdo, J.M. An envelope for Bol algebras. *J. Algebra* **2005**, *284*, 480–493. [CrossRef]
25. Pérez-Izquierdo, J.M.; Shestakov, I.P. An envelope for Malcev algebras. *J. Algebra* **2004**, *272*, 379–393. [CrossRef]
26. Shestakov, I.P.; Zhukavets, N.M. The free Malcev superalgerbra on one odd generator and related superalgebras. *J. Math. Sci. N. Y.* **2007**, *140*, 243–249. [CrossRef]
27. Sabinin, L.V.; Miheev, P.O. On local analytic loops and their corresponding hyperalgebras. In *Proceedings of 9th Conference of Young Researchers*; VINITI Press: Moscow, Russia, 1986; pp. 34–54.
28. Sabinin, L.V.; Miheev, P.O. Infinitesimal theory of local analytic loops. *Soviet Math. Dokl.* **1988**, *36*, 545–548.
29. Kurosh, A. Radicals in rings and algebras. *Math. Sb.* **1953**, *33*, 13–26.
30. Hoffman, A.E.; Leavitt, W.G. Properties inherited by the lower radical. *Port. Math.* **1968**, *27*, 63–66.
31. Belov, A.; Bokut, L.; Rowen, L.; Yu, J.-T. The Jacobian conjecture, together with Specht and Burnside-type problems. *Math. Stat.* **2014**, *79*, 249–285.
32. Huynh, D.V. (Ed.) *Advances in Ring Theory*; Birkhäuser: Basel, Switzerland, 2010.
33. Tuganbaev, A.A. *Semidistributive Modules and Rings*; Springer: Berlin/Heidelberg, Germany, 1998.
34. Amitsur, S.A. Radicals in rings and bicategories. *Am. J. Math.* **1954**, *76*, 100–125. [CrossRef]
35. Tangeman, R.; Kreiling, D. Lower radicals in nonassociative rings. *J. Aust. Math. Soc.* **1972**, *14*, 419–423. [CrossRef]
36. Ore, O. Theory of noncommutative polynomials. *Ann. Math.* **1933**, *34*, 480–508. [CrossRef]
37. Pumplün, S. Finite nonassociative algebras obtained from skew polynomials and possible applications to (f, σ, δ)-codes. *Adv. Math. Comm.* **2017**, *11*, 615–634. [CrossRef]
38. Brown, C.; Pumplün, S. How a nonassociative algebra reflects the properties of a skew polynomial. *Glasg. Math. J.* **2021**, *63*, 6–26. [CrossRef]
39. Petit, J.-C. Sur les quasi-corps distributifes à base momogène. *C. R. Acad. Sci. Paris Sér. A* **1968**, *266*, 402–404.
40. Pumplün, S. Nonassociative differential extensions of characteristic p. *Results Math.* **2017**, *72*, 245–262. [CrossRef]
41. Lam, T.Y.; Leroy, A. Algebraic conjugacy classes and skew-polynomial rings. In *Perspectives in Ring Theory*; Springer: Antwerp, Belgium, 1988; pp. 153–203.
42. Lam, T.Y.; Leung, K.H.; Leroy, A.; Matczuk, J. Invariant and semi-invariant polynomials in skew polynomial rings. In *Proceedings of the Ring Theory 1989, Ramat Gan and Jerusalem, 1988/1989*; Weizmann Science Press of Israel: Jerusalem, Israel, 1989; pp. 247–261.
43. Jacobson, N. Pseudo-linear transformations. *Ann. Math.* **1937**, *38*, 484–507. [CrossRef]
44. Tkachev, V.G. The universality of one half in commutative nonassociative algebras with identities. *J. Algebra* **2021**, *569*, 466–510. [CrossRef]
45. Koecher, M. On commutative nonassociative algebras. *J. Algebra* **1980**, *62*, 479–493. [CrossRef]
46. Reed, M.L. Algebraic structure of genetic inheritance. *Bull. Am. Math. Soc.* **1997**, *34*, 107–130. [CrossRef]
47. Fernández, J.C.G. Principal and plenary train algebras. *Commun. Algebra* **2000**, *28*, 653–667. [CrossRef]
48. Lyubich, Y.I. *Mathematical Structures in Population Genetics*; Springer: Berlin/Heidelberg, Germany, 1992.
49. Meyberg, K.; Osborn, J.M. Pseudo-composition algebras. *Math. Z.* **1993**, *214*, 67–77. [CrossRef]
50. Nourigat, M.; Varro, R. Etude des ω-PI algèbres commutatives de degré 4, III: Algèbres barycentriques invariantes par gamétisation. *Commun. Algebra* **2013**, *41*, 2825–2851. [CrossRef]
51. Bordemann, M. Nondegenerate invariant bilinear forms on nonassociative algebras. *Acta Math. Univ. Comen.* **1997**, *66*, 151–201.
52. Koecher, M. *The Minnesota Notes on Jordan Algebras and Their Applications*; Springer: Berlin/Heidelberg, Germany, 1999.
53. Norton, S. The Monster algebra: Some new formulae. *Contemp. Math. Am. Math. Soc.* **1996**, *193*, 297–306.
54. Ivanov, A.A. Majorana representation of the Monster group. *Contemp. Math. Am. Math. Soc.* **2017**, *694*, 11–17.
55. Medts, D.T.; Couwenberghe, V.M. Modules over axial algebras. *Algebras Represent. Theory* **2020**, *23*, 209–227. [CrossRef]
56. Sandler, R. Autotopism groups of some finite nonassociative algebras. *Am. J. Math.* **1962**, *84*, 239–264. [CrossRef]
57. Astier, V.; Pumplün, S. Nonassociative quaternion algebras over rings. *Isr. J. Math.* **2006**, *155*, 125–147.
58. Waterhouse, W.C. Nonassociative quaternion algebras. *Algebra Groups Geom.* **1987**, *4*, 365–378.
59. Pumplün, S.; Steele, A. Fast-decodable MIDO codes from nonassociative algebras. *Int. J. Inf. Coding Theory* **2015**, *3*, 15–38.
60. Pumplün, S. Tensor products of nonassociative cyclic algebras. *J. Algebra* **2016**, *451*, 145–165. [CrossRef]
61. Brown, C.; Pumplün, S. Nonassociative cyclic extensions of fields and central simple algebras. *J. Pure Appl. Algebra* **2019**, *223*, 2401–2412. [CrossRef]
62. Petit, J.-C. Sur certains quasi-corps généralisant un type d'anneau-quotient. *Séminaire Dubriel Algèbre et Théorie des Nombres* **1967**, *20*, 1–18.
63. Hartwig, J.T.; Larsson, D.; Silvestrov, S.D. Deformations of Lie algebras using σ-derivations. *J. Algebra* **2006**, *295*, 314–361. [CrossRef]
64. Makhlouf, A.; Silvestrov, S. Hom-algebras and Hom-coalgebras. *J. Algebra Appl.* **2010**, *9*, 553–589. [CrossRef]
65. Makhlouf, A.; Silvestrov, S. Hom-Lie admissible hom-coalgebras and hom-Hopf algebras. In *Generalized Lie Theory in Mathematics, Physics and Beyond*; Silvestrov, S., Paal, E., Abramov, V., Stolin, A., Eds.; Springer: Berlin/Heidelberg, Germany, 2009.
66. Nystedt, P.; Öinert, J.; Richter, J. Nonassociative Ore extensions. *Isr. J. Math.* **2018**, *224*, 263–292. [CrossRef]
67. Bäck, P.; Richter, J.; Silvestrov, S. Hom-associative Ore extensions and weak unitalizations. *Int. Electron. J. Algebra* **2018**, *24*, 174–194. [CrossRef]

68. Bäck, P.; Richter, J. Hilbert's basis theorem for non-associative and hom-associative Ore extensions. *Algebras Represent. Theory* **2022**. [CrossRef]
69. Knus, M.-A.; Merkurjev, A.; Rost, M.; Tignol, J.-P. *The Book of Involutions*; American Mathematical Society: Providence, RI, USA, 1998.
70. Dickson, L.E. *The Collected Mathematical Papers*; Chelsea Publishing Co.: New York, NY, USA, 1975.
71. Zhevlakov, K.A.; Slin'ko, A.M.; Shestakov, I.P.; Shirshov, A.I. *Rings That Are Nearly Associative*; Academic Press: New York, NY, USA, 1982.
72. Albert, A.A. Quadratic forms permitting composition. *Ann. Math.* **1942**, *43*, 161–177. [CrossRef]
73. Lam, T.Y. *Introduction to Quadratic Forms over Real Fields*; American Mathematical Society: Providence, RI, USA, 2005.
74. Darpö, E.; Nystedt, P. Von-Neumann finiteness and reversibility in some classes of non-associative algebras. *Algebras Represent. Theory* **2021**, *24*, 1245–1258. [CrossRef]
75. Fell, J.M.G.; Doran, R.S. *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles*; Academic Press: Boston, MA, USA, 1988.
76. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis*; Springer: Berlin/Heidelberg, Germany, 1994; Volume 2.
77. Naimark, M.A. *Normed Rings*; Nauka: Moscow, Russia, 1968.
78. Ghahramani, F.; Runde, V.; Willis, G. Derivations on group algebras. *Proc. Lond. Math. Soc.* **2000**, *80*, 360–390. [CrossRef]
79. Johnson, B.E. The derivation problem for group algebras of connected locally compact groups. *J. Lond. Math. Soc.* **2001**, *63*, 441–452. [CrossRef]
80. Losert, V. The derivation problem for group algebras. *Ann. Math.* **2008**, *168*, 221–246. [CrossRef]
81. Belopolskaya, Y.I.; Dalecky, Y.L. *Stochastic Equations and Differential Geometry*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1989.
82. Dalecky, Y.L.; Fomin, S.V. *Measures and Differential Equations in Infinite-Dimensional Spaces*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1991.
83. Dalecky, Y.L.; Shnaiderman, Y.L. Diffusion and quasi-invariant measures on infinite-dimensional Lie groups. *Funct. Anal. Its Appl.* **1969**, *3*, 156–158.
84. Ludkovsky, S.V. Topological transformation groups of manifolds over non-Archimedean fields, representations and quasi-invariant measures, I. *J. Math. Sci. N. Y.* **2008**, *147*, 6703–6846. [CrossRef]
85. Ludkovsky, S.V. Topological transformation groups of manifolds over non-Archimedean fields, representations and quasi-invariant measures, II. *J. Math. Sci. N. Y.* **2008**, *150*, 2123–2223. [CrossRef]
86. Ludkovsky, S.V. Quasi-invariant measures on non-Archimedean groups and semigroups of loops and paths, their representations. II. *Ann. Math. B. Pascal.* **2000**, *7*, 55–80. [CrossRef]
87. Ludkovsky, S.V. Stochastic processes on geometric loop groups, diffeomorphism groups of connected manifolds, associated unitary representations. *J. Math. Sci. N. Y.* **2007**, *141*, 1331–1384. [CrossRef]
88. Ludkovsky, S.V. Quasi-invariant measures on a group of diffeomorphisms of an infinite-dimensional real manifold and induced irreducible unitary representations. *Rend. dell'Istituto di Matem. dell'Università di Trieste. Nuova Serie.* **1999**, *30*, 101–134.
89. Ludkovsky, S.V. Semidirect products of loops and groups of diffeomorphisms of real, complex and quaternion manifolds, and their representations. In *Focus on Groups Theory Research*; Ying, L.M., Ed.; Nova Science Publishers, Inc.: New York, NY, USA, 2006; pp. 59–136.
90. Ludkovsky, S.V. Operators on a non locally compact group algebra. *Bull. Sci. Math.* **2013**, *137*, 557–573. [CrossRef]
91. Ludkovsky, S.V. Meta-centralizers of non-locally compact group algebras. *Glasg. Math. J.* **2015**, *57*, 349–364. [CrossRef]
92. Ludkovsky, S.V. Properties of quasi-invariant measures on topological groups and associated algebras. *Ann. Math. B. Pascal.* **1999**, *6*, 33–45. [CrossRef]
93. Ludkowski, S.V. Integral operators for nonlocally compact group modules. *Quaest. Math.* **2022**, *45*, 1125–1144. [CrossRef]
94. Ludkowski, S.V. Spectra and structures of non locally compact group algebras. *Southeast Asian Bull. Math.* **2020**, *44*, 377–415.
95. Hilbert, D. *The Theory of Algebraic Number Fields*; Springer: Berlin/Heidelberg, Germany, 1998.
96. Lorenz, F. Ein Scholion zum Satz 90 von Hilbert. *Abh. Math. Semin. Univ. Hamb.* **1998**, *68*, 347–362. [CrossRef]
97. Lang, S. *Algebra*; Addison-Wesley: Boston, MA, USA, 1993.
98. Lundström, P. Hilbert 90 for algebras with conjugation. *Algebras Represent. Theory* **2012**, *15*, 119–135. [CrossRef]
99. Zaicev, M.V. Integrality of exponents of growth of identities of finite-dimensional Lie algebras. *Izv. Math.* **2002**, *66*, 463–487.
100. Gordienko, A.S. Codimensions of polynomial identities of representations of Lie algebras. *Proc. Am. Math. Soc.* **2013**, *141*, 3369–3382. [CrossRef]
101. Giambruno, A.; Zelmanov, E. On growth of codimensions of Jordan algebras. *Contemp. Math.* **2011**, *537*, 205–210.
102. Belov-Kanel, A.; Giambruno, A.; Rowen, L.H.; Vishne, U. Zariski closed algebras in varieties of universal algebra. *Algebras Represent. Theory* **2014**, *17*, 1771–1783. [CrossRef]
103. Belov, A.; Rowen, L.H.; Vishne, U. Structure of Zariski closed algebras. *Trans. Am. Math. Soc.* **2010**, *362*, 4695–4734. [CrossRef]
104. Zaicev, M.V.; Mishchenko, S.P. Codimension sequences and their asymptotic behavior. *J. Math. Sci. N. Y.* **2021**, *257*, 825–833. [CrossRef]
105. Nichita, F.F. Yang-Baxter systems, algebra factorizations and braided categories. *Axioms* **2013**, *2*, 437–442. [CrossRef]

106. Bokut, L.A.; Chen, Y.; Li, Y. Gröbner-Shirshov bases for Vinberg-Koszul-Gerstenhaber right-symmetric algebras. *J. Math. Sci. N. Y.* **2010**, *166*, 603–612. [CrossRef]
107. Tuganbaev, A. On rings of weak global dimension at most one. *Mathematics* **2021**, *9*, 2643. [CrossRef]
108. Huang, J.; Chen, Y. Gröbner-Shirshov bases theory for trialgebras. *Mathematics* **2021**, *9*, 1207. [CrossRef]
109. Allcock, D. Reflection groups and octave hyperbolic plane. *J. Algebra* **1998**, *213*, 467–498. [CrossRef]
110. Baez, J.C. The octonions. *Bull. Am. Math. Soc.* **2002**, *39*, 145–205. [CrossRef]
111. Kantor, I.L.; Solodovnikov, A.S. *Hypercomplex Numbers*; Springer: Berlin/Heidelberg, Germany, 1989.
112. Ludkowski, S.V. Automorphisms and derivations of nonassociative C^* algebras. *Linear Multil. Algebra* **2019**, *67*, 1531–1538. [CrossRef]
113. Ludkowski, S.V. Cohomology theory of nonassociative algebras with metagroup relations. *Axioms* **2019**, *8*, 78. [CrossRef]
114. Ludkowski, S.V. Homotopism of homological complexes over nonassociative algebras with metagroup relations. *Mathematics* **2021**, *9*, 734. [CrossRef]
115. Ludkowski, S.V. Torsion for homological complexes of nonassociative algebras with metagroup relations. *Axioms* **2021**, *10*, 319. [CrossRef]
116. Ludkowski, S.V. Splitting extensions of nonassociative algebras and modules with metagroup relations. *Axioms* **2022**, *11*, 131. [CrossRef]
117. Ludkowski, S.V. Satellites of functors for nonassociative algebras with metagroup relations. *Mathematic* **2022**, *10*, 1169. [CrossRef]
118. Ludkowski, S.V. Smashed and twisted wreath products of metagroups. *Axioms* **2019**, *8*, 127. [CrossRef]
119. Ludkowski, S.V. Separability of nonassociative algebras with metagroup relations. *Axioms* **2019**, *8*, 139. [CrossRef]
120. Mikhalev, A.V. Alternative rings with single-valued addition. *J. Math. Sci. N. Y.* **1994**, *69*, 1092–1097. [CrossRef]
121. Makhlouf, A. Hom-alternative algebras and hom-Jordan algebras. *Int. Electron. J. Algebra* **2010**, *8*, 177–190.
122. Sun, Q. On hom-prealternative bialgebras. *Algebras Represent. Theory* **2016**, *19*, 657–677. [CrossRef]
123. Golubkov, A.Y. Construction of special radicals of algebras. *J. Math. Sci., N. Y.* **2017**, *223*, 530–580. [CrossRef]
124. Dzhumadil'daev, A.M. Zinbiel algberas under q-commutators. *J. Math. Sci. N. Y.* **2007**, *144*, 3909–3925. [CrossRef]
125. Brown, R.B. On generalized Cayley-Dickson algebras. *Pacific J. Math.* **1967**, *20*, 415–422. [CrossRef]
126. Culbert, C. Cayley-Dickson algebras and loops. *J. Gener. Lie Theory Appl.* **2007**, *1*, 1–17. [CrossRef]
127. Elduque, A; Myung, H.C. Colour algebras and Cayley-Dickson algebras. *Proc. R. Soc. Edinb.* **1995**, *125A*, 1287–1303. [CrossRef]
128. Pengelley, D.J.; Williams, F. The global structure of odd-primary Dickson algebras as algebras over the Steenrod algebra. *Math. Proc. Camb. Phil. Soc.* **2004**, *136*, 67–73. [CrossRef]
129. Ludkowski, S.V. Octonion orthocomplemantable modules. *Adv. Appl. Clifford Algebra* **2017**, *27*, 1551–1566. [CrossRef]
130. Schafer, R.D. Inner derivations of nonassociative algebras. *Bull. Am. Math. Soc.* **1949**, *55*, 769–776. [CrossRef]
131. Jacobson, N. Abstract derivation and Lie algebras. *Trans. Am. Math. Soc.* **1937**, *42*, 206–224. [CrossRef]
132. Akemann, C.A.; Pedersen, G.K. Central sequences and inner derivations of separable C^*-algebras. *Am. J. Math.* **1979**, *101*, 1047–1061. [CrossRef]
133. Ludkowski, S.V. Completions and homomorphisms of infinite dimensional Cayley-Dickson algebras. *Linear Multil. Algebra* **2021**, *69*, 2040–2049. [CrossRef]
134. Ludkovsky, S.V. Differentiable functions of Cayley-Dickson numbers and line integration. *J. Math. Sci. N. Y.* **2007**, *141*, 1231–1298. [CrossRef]
135. Ludkowski, S.V. On a class of right linearly differentiable functions of Cayley-Dickson variables. *Adv. Appl. Clifford Algebra* **2014**, *24*, 781–803. [CrossRef]
136. Ludkovsky, S.V. Functions of several Cayley-Dickson variables and manifolds over them. *J. Math. Sci. N. Y.* **2007**, *141*, 1299–1330. [CrossRef]
137. Ludkovsky, S.V. Normal families of functions and groups of pseudoconformal diffeomorphisms of quaternion and octonion variables. *J. Math. Sci. N. Y.* **2008**, *150*, 2224–2287. [CrossRef]
138. Ludkovsky, S.V. Studies of Diophantine equations with the help of Cayley-Dickson algebras. *J. Algebra Number Theory Acad.* **2011**, *2*, 63–79.
139. Ludkovsky, S.V.J. Noncommutative quasi-conformal integral transforms over quaternions and octonions. *J. Math. Sci. N. Y.* **2009**, *157*, 199–251. [CrossRef]
140. Kac, V.G. *Infinite-Dimensional Lie Algebras*; Cambridge Univ. Press: Cambridge, UK, 1990.
141. Ludkovsky, S.V. Affine and wrap algebras over octonions. *J. Math. Sci. N. Y.* **2010**, *167*, 767–809. [CrossRef]
142. Ludkovsky, S.V. Wrap groups of connected fiber bundles: Their structure and cohomologies. *Int. J. Math., Game Theory, Algebra.* **2009**, *19*, 53–128.
143. Ludkovsky, S.V.; Sprössig, W. Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces. *Adv. Appl. Clifford Algebra* **2010**, *20*, 321–342.
144. Ludkovsky, S.V.; Sprössig, W. Spectral theory of super-differential operators of quaternion and octonion variables. *Adv. Appl. Clifford Algebra* **2011**, *21*, 165–191.
145. Ludkowski, S.V. Quasi-permutable normal operators in octonion Hilbert spaces and spectra. *Adv. Appl. Clifford Algebra* **2014**, *24*, 163–178. [CrossRef]

146. Ludkovsky, S.V.; Sprössig, W. Spectral representations of operators in Hilbert spaces over quaternions and octonions. *Complex Var. Elliptic Equat.* **2012**, *57*, 1301–1324. [CrossRef]
147. Ludkovsky, S.V. Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra. *J. Math. Sci. N. Y.* **2007**, *144*, 4301–4366. [CrossRef]
148. Ludkovsky, S.V. C^*-algebras of meta-invariant operators in modules over Cayley-Dickson algebras. *Southeast Asian Bull. Math.* **2015**, *39*, 625–684.
149. Bourbaki, N. *Algèbre*; Springer: Berlin/Heidelberg, Germany, 2007.
150. Pierce, R.S. *Associative Algebras*; Springer: New York, NY, USA, 1982.
151. Markov, V.N.; Mikhalev, A.V.; Nechaev, A.A. Nonassociative algebraic tsructures in cryptography and coding. *J. Math. Sci. N. Y.* **2020**, *245*, 178–196. [CrossRef]
152. Markov, V.T.; Mikhalev, A.V.; Gribov, A.V.; Zolotykh, P.A.; Skazhenik, S.S. Quasigroups and rings in the coding and construction of cryptocircuits. *Prikl. Diskr. Mat.* **2012**, *4*, 31–52.
153. Gribov, A.V. Some homomorphic cryptosystems based on nonassociative structures. *J. Math. Sci. N. Y.* **2017**, *223*, 581–586. [CrossRef]
154. Gribov, A.V. The prime radical of alternative rings and loops. *J. Math. Sci. N. Y.* **2017**, *223*, 587–601. [CrossRef]
155. Arakelov, G.G.; Gribov, A.V.; Mikhalev, A.V. Applied homomorphic cryptography: Examples. *J. Math. Sci.* **2019**, *237*, 353–361. [CrossRef]
156. Gonzalez, S.; Kouselo, E.; Markov, V.; Nechaev, A. Group codes and their nonassociative generalizations. *Diskret. Mat.* **2004**, *14*, 146–156.
157. Blahut, R.E. *Algebraic Codes for Data Transmission*; Cambridge University Press: Cambridge, UK, 2003.
158. Mikhalev, A.V.; Shirshova, E.E. Interpolation pseudo-ordered rings. *Fundam. Prikl. Mat.* **2022**, *24*, 177–191. [CrossRef]
159. Liu, X.; Wang, M.; Guo, X. Shen-Larsson's modules for simple generalized Witt algebras. *Algebras Represent. Theory* **2022**, 1–23. [CrossRef]
160. Barnes, G.E.; Schenkel, A.; Szabo, R.J. Nonassociative geometry in quasi-Hopf representation categories I: Bimodules and their internal homomorphisms. *J. Geom. Phys.* **2015**, *89*, 111–152. [CrossRef]
161. Dzhunushaliev, V. Non-associative slave-boson decomposition. *J. Gener. Lie Theory Appl.* **2007**, *1*, 129–134. [CrossRef]
162. Dzhunushaliev, V. Toy models of a nonassociative quantum mechanics. *Adv. High Energy Phys.* **2007**, *12387*, 1–10. [CrossRef]
163. Dirac, P.A.M. *Die Prinzipen der Quantenmechanik*; Hirzel: Leipzig, Germany, 1930.
164. Frenod, E.; Ludkowski, S.V. Integral operator approach over octonions to solution of nonlinear PDE. *Far East J. Math. Sci.* **2018**, *103*, 831–876. [CrossRef]
165. Ludkovsky, S.V. Integration of vector hydrodynamical partial differential equations over octonions. *Complex Var. Elliptic Equat.* **2013**, *58*, 579–609. [CrossRef]
166. Ludkovsky, S.V. Integration of vector Sobolev type PDE over octonions. *Complex Var. Elliptic Equat.* **2016**, *61*, 1014–1035. [CrossRef]
167. Ludkowski, S.V. Decompositions of PDE over Cayley-Dickson algebras. *Rendic. dell'Ist. di Math. dell'Università di Trieste. Nuova Serie.* **2014**, *46*, 1–23.
168. Ludkovsky, S.V. Line integration of Dirac operators over octonions and Cayley-Dickson algebras. *Computat. Meth. Funct. Theory* **2012**, *12*, 279–306. [CrossRef]
169. Ludkovsky, S.V. Multidimensional Laplace transforms over quaternions; octonions and Cayley-Dickson algebras; their applications to PDE. *Adv. Pure Math.* **2012**, *2*, 63–103. [CrossRef]
170. Gürsey, F.; Tze, C.-H. *On the Role of Division, Jordan and Related Algebras in Particle Physics*; World Scientific Publishing Co.: Singapore, 1996.
171. Nichita, F.F. Unification theories: New results and examples. *Axioms* **2021**, *8*, 60. [CrossRef]
172. Iantovics, L.B.; Nichita, F.F. On the colored and the set-theoretical Yang-Baxter equations. *Axioms* **2021**, *10*, 146. [CrossRef]
173. Majid, S. Gauge theory on nonassociative spaces. *J. Math. Phys.* **2005**, *46*, 103519. [CrossRef]
174. Bruck, R.H. *A Survey of Binary Systems*; Springer: Berlin/Heidelberg, Germany, 1971.
175. Pickert, G. *Projektive Ebenen*; Springer: Berlin/Heidelberg, Germany, 1955.
176. Pickert, G. Doppelebenen und loops. *J. Geom.* **1991**, *41*, 133–144. [CrossRef]
177. Markl, M.; Shnider, S.; Stasheff, J. *Operads in Algebra, Topology and Physics*; American Mathematical Society: Providence, RI, USA, 2002.
178. Hasiewicz, Z.; Defever, F.; Troost, W. Nonassociative superconformal algebras. *J. Math. Phys.* **1991**, *32*, 2285. [CrossRef]
179. Mylonas, D.; Schupp, P.; Szabo, R.J. Non-geometric fluxes, quasi-Hopf twist deformations, and nonassociative quantum mechanics. *J. Math. Phys.* **2014**, *55*, 122301. [CrossRef]
180. Kerner, E.H. Nonassociative structure of quantum mechanics in curved space-time. *J. Math. Phys.* **1999**, *40*, 4664. [CrossRef]
181. Castro, C. On the noncommutative and nonassociative geometry of octonionic space time, modified dispersion relations and grand unification. *J. Math. Phys.* **2007**, *48*, 073517. [CrossRef]
182. Pommaret, J.F. *Systems of Partial Differential Equations and Lie Pseudogroups*; Gordon and Breach Science Publishers: New York, NY, USA, 1978.
183. Shang, Y. Lie algebraic discussion for affinity based information diffusion in social networks. *Open Phys.* **2017**, *15*, 705–711. [CrossRef]
184. Cartan, H.; Eilenberg, S. *Homological Algebra*; Princeton University Press: Princeton, NJ, USA, 1956.

185. Bourbaki, N. Algèbre Homologique. In *Algèbre*; Springer: Berlin/Heidelberg, Germany, 2007.
186. Bredon, G.E. *Sheaf Theory*; McGarw-Hill: New York, NY, USA, 2012.
187. Hochschild, G. On the cohomology theory for associative algebras. *Ann. Math.* **1946**, *47*, 568–579. [CrossRef]
188. Georgantas, G.T. Derivations in central separable algebras. *Glasgow Math. J.* **1978**, *19*, 75–77. [CrossRef]
189. Lee, D.-W. On the digital cohomology modules. *Mathematics* **2020**, *8*, 1451. [CrossRef]
190. Van Oystaeyen, F. Separable algebras. In *Handbook of Algebra*; Hazewinkel, M., Ed.; Elsevier: Amsterdam, The Netherlands, 2000; Volume 2, pp. 463–505.
191. Rumynin, D.A. Cohomomorphisms of separable algebras. *Algebra Log.* **1994**, *33*, 233–237. [CrossRef]
192. Sproston, J.P. Derivations on some (possibly non-separable) C^*-algebras. *Glasgow Math. J.* **1981**, *22*, 43–56. [CrossRef]

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