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Spectrum of Zariski Topology in Multiplication Krasner Hypermodules

Ergül Türkmen *, Burcu Nişancı Türkmen and Öznur Kulak

Department of Mathematics, Faculty of Arts and Sciences, Amasya University, Amasya 05100, Turkey; oznur.kulak@amasya.edu.tr (Ö.K.)
* Correspondence: ergul.turkmen@amasya.edu.tr

Abstract: In this paper, we define the concept of pseudo-prime subhypermodules of hypermodules as a generalization of the prime hyperideal of commutative hyperrings. In particular, we examine the spectrum of the Zariski topology, which we built on the element of the pseudo-prime subhypermodules of a class of hypermodules. Moreover, we provide some relevant properties of the hypermodule in this topological hyperspace.

Keywords: pseudo-prime spectrum; Zariski topology; spectral hyperspace

MSC: 46H99; 20N20

Citation: Türkmen, E.; Nişancı Türkmen, B.; Kulak, Ö. Spectrum of Zariski Topology in Multiplication Krasner Hypermodules. *Mathematics* **2023**, *11*, 1754. https://doi.org/10.3390/math11071754

Academic Editors: Lakshmi Kanta Dey and Pratulananda Das

Received: 21 January 2023
Revised: 24 March 2023
Accepted: 4 April 2023
Published: 6 April 2023

1. Introduction

Hypergroup theory, which was defined in [1] as a more comprehensive algebraic structure of group theory, has been investigated by different authors in modern algebra. It has been developed using hyperring and hypermodule theory studies by many authors in a series of papers [1–15]. Following these papers, let us start by giving the basic information necessary for the algebraic structure that we will study as Krasner S-hypermodule in studying the S-hypermodule class on a fixed Krasner hyperring class S. Let N be a non-empty set; $(N, \cdot)$ is called a *hypergroupoid* if for the map defined as $\cdot : N \times N \longrightarrow P^*(N)$ is a function. Here "$\cdot$" is called a *hyperoperation* on N. Let X and Y be subsets of N. The hyperproduct $X \cdot Y$ is defined as

$$X \cdot Y = \bigcup_{(x,y) \in X \times Y} x \cdot y.$$

If $x, y \in N$, then $\{x\} \cdot Y$ and $X \cdot \{y\}$ are simply represented as $x \cdot Y$ and $X \cdot y$, respectively. A hypergroupoid $(N, \cdot)$ is called a *semihypergroup* if for each $x, y, z \in N$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. A semihypergroup $(N, \cdot)$ is called a *hypergroup* if for each $x \in N$, $x \cdot N = N \cdot x = N$. A hypergroup $(N, \cdot)$ is called *commutative* provided $x \cdot y = y \cdot x$ for all $x, y \in N$.

A commutative hypergroup $(N, +)$ is said to be *canonical* if

(1) There exists a unique $0 \in N$ such that for each $x \in N$ there exists a unique element x' in N, denoted by $-x$, such that $0 \in x + (-x)$;
(2) $z \in x + y$ implies $y \in z - x := z + (-x)$ for each $x, y, z \in N$.

As it is proved in [13], if $(N, +)$ is a canonical hypergroup, then $x + 0 = x$ for all $x \in N$.

Let $(S, +, \cdot)$ be a hyperstructure. $(S, +, \cdot)$ is called a *Krasner hyperring* if

(1) $(S, +)$ is a canonical hypergroup;
(2) $(S, \cdot)$ is a semigroup with a bilaterally absorbing element 0, i.e.,
   (a) $a \cdot b \in R$ for all $a, b \in S$;
   (b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in S$;

(c) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$;

(3) The multiplication distributes over the addition on both sides.

A Krasner hyperring $(S, +, \cdot)$ is called *commutative* if it is commutative with respect to the multiplication. If $a = a \cdot 1_S = 1_S \cdot a$ for every $a \in S$, then element 1_S is called an *identity element* of the Krasner hyperring $(S, +, \cdot)$. From now on, when we say hyperring, we mean commutative Krasner hyperrings with identity.

Let $(S, +, \cdot)$ be a hyperring and J be a non-empty subset of S. Then, J is called a *hyperideal* of S provided $(J, +)$ is a subhypergroup and $r \cdot a,\ a \cdot r \in J$ for all $a \in J$ and $r \in S$. Let S be a hyperring and I, J be hyperideals of S. Then the product $IJ = \{x \mid x \in \sum_\Lambda a_\lambda b_\lambda,\ a_\lambda \in I \ and \ b_\lambda \in J\}$ is a hyperideal of S.

Let S be a hyperring. A canonical hypergroup $(N, +)$ together with a left external map $S \times N \longrightarrow N$ defined by

$$(x,\, a) \mapsto x \cdot a \in N$$

such that for every $x, y \in S$ and $a, b \in N$, we have

(1) $x \cdot (a + b) = x \cdot a + x \cdot b$,
(2) $(x + y) \cdot a = x \cdot a + y \cdot a$,
(3) $(x \cdot y) \cdot a = x \cdot (y \cdot a)$,
(4) $a = 1_S \cdot a$,
(5) $x \cdot 0_N = 0_N$,

which is called a *Krasner left hypermodule* over S.

Throughout this paper, for a simple explanation, when we say hypermodule, we mean the left Krasner hypermodule. Note that a non-zero hypermodule always has two different subhypermodules, which are *trivial subhypermodules*. It is known that a non-empty subset K of an S-hypermodule N is a subhypermodule of N if and only if $a - b \subseteq K$ and $ra \in K$ for all a, $b \in K$ and $r \in S$.

Let N be a S-hypermodule and $x \in N$. Then $S \cdot x = \{s \cdot x : s \in S\}$ is a subhypermodule of a hypermodule N.

Let K and T be subhypermodules of N. Then $K + T = \{x \in k + t : k \in K, t \in T\}$ is a subhypermodule of N. Let N and K be S-hypermodules and let $f : N \to K$ be a function. If $f(x + y) \subseteq f(x) + f(y)$ and $f(s.x) = s.f(x)$ for every $x, y \in K$ and $s \in S$, f is called a hypermodule S-homomorphism from N to K. Instead of this statement, if the inclusion satisfies $f(x + y) = f(x) + f(y)$, then f is said to be a *strong S-homomorphism* from N to K. The class of every strong S-homomorphism from N to K is denoted by $Hom_S(N, K)$; sets are defined as $ker(f) := \{x \in N : f(x) = 0_K\}$ and $Im(f) := \{y \in K : \exists x \in N, y \in f(x)\}$. The homomorphism $f \in Hom_S(N, K)$ is called *strongly injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for every $x_1, x_2 \in N$, and f is called *strongly surjective* if $K = f(N)$. To simplify denoting annihilator of an S-hypermodule N for a subhypermodule K, we use the symbol $K :_S N$, and the set is a hyperideal, which is defined as $\{s \in S : s.N \subseteq K\}$. Another representation of $D :_S N$ is $Ann_S(N)$.

As a generalization of a prime spectrum of the ring of commutative topology defined on S with Zariski topology [16] inspired by the interaction between the theoretical properties of the hyperring S of the text, over a commututive hyperring S on a several hypermodule N, we examine a Zariski topology on these spectrum X_N of pseudo-prime subhyper-modules, and we give the interaction between topological hyperspace.

We give topological conditions such as connectedness, Noetherianness, and irreducibility in the pseudo-prime spectrum of hypermodules and obtain more information about the algebraic hyperstructure of these hypermodules. Further, we prove this topological hyperspace in terms of spectral hyperspace, which is a topological hyperspace and homeomorphic to $Spec(S)$ for any hyperring S.

2. Condition of Pseudo-Prime for Krasner Hypermodules

In this section, we present pseudo-prime subhypermodules as a new concept of hypermodules theory. Then we investigate the connection between spectral hyperspace and Zariski topology. Recall from [17] that a proper hyperideal J of a hyperring S is called *prime* if for hyperideals X, Y of S the relation $XY \subseteq J$ implies $X \subseteq J$ or $Y \subseteq J$.

Definition 1. *Let N be an S-hypermodule and K be a subhypermodule of N.*

(1) *K is called pseudo-prime if $(K :_S N)$ is a prime hyperideal of S.*
(2) *We call a pseudo-prime spectrum of N as the set of all pseudo-prime submodules of N expressed by X_N. For any prime hyperideal $J \in X_S = Spec(S)$, the collection N of whole pseudo-prime subhypermodules of N with $(K : N) = J$*
(3) *We define the set $V(K) = \{Y \in X_N : K \leq Y\}$.*
(4) *If $X_N \neq \emptyset$, the function $\eta : X_N \to Spec(S/Ann(N))$ via*

$$\eta(Y) = (Y : N)/Ann(N)$$

is called natural map of X_N. If either $N = \{0\}$ or $N \neq \{0\}$ and the natural map of X_N is strongly surjective, then we call N pseudo-primeful.
(5) *If the natural map of X_N is strongly injective, then we call N a pseudo-injective.*

According to our above definition, prime hyperideals of a hyperring S and the pseudo-prime S-hypermodule of the hypermodule S are the same. It is obtained that the concept of prime hyperideal to hypermodules is a strong notion of the strongly pseudo-prime subhypermodule S. Let N be an S-hypermodule. Following [18], a proper subhypermodule K of N is called *prime* if, for a hyperideal J of S and a subhypermodule X of N, the ralation

$$[J.X] = \bigcup \left\{ \sum_{j=1}^{p} u_j.x_j : l \in \mathbb{N}, u_j \in J \text{ and } x_j \in X, \text{ for all } j \right\}$$

implies $J \subseteq (K : N)$ or $X \subseteq K$.

Therefore, a proper subhypermodule K of N is prime if N/K is a torsion-free $S/(K : N)$-hypermodule, i.e., N/K is a hypermodule on S such that the only element destroyed by a non-zero divisor of hyperring $S/(K : N)$ is zero. Using Definition 1, every prime subhypermodule K is a pseudo-prime subhypermodule because $(K : N) \in Spec(S)$.

Recall from [11] that a hypermodule N is multiplication S-hypermodule if, for each subhypermodule K of N, there exists a hyperideal J of S with $K = [J.N]$. A proper subhypermodule K of N is called maximal if for each subhypermodule L of N with $K \leq L \leq N$, then $K = L$ or $L = N$.

Lemma 1. *The following assertions are equivalent for a finitely generated S-hypermodule N.*

(1) *N is a multiplication hypermodule.*
(2) *N is a pseudo-injective hypermodule.*
(3) *$\left| X_{N,J} \right| \leq 1$ for each maximal hyperideal J of S.*
(4) *$N/J.N$ is simple for each maximal hyperideal J of S.*

Proof. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are clear.

$(3) \Rightarrow (4)$ It can be proven clearly that $J.N = N$ for a maximal hyperideal J of S. Hence, suppose that $J.N \neq N$ and $K/J.N \subset N/J.N$. Then K is a proper subhypermodule containing the subhypermodule $J.N$ of N. Thus we have $J = (J.N : N) = (K : N)$. Since K and $J.N$ belong to $X_{N:J}$, then $K = J.N$ by the assumption. Therefore, $N/J.N$ is a simple S-hypermodule. By [11], N is a multiplication hypermodule. □

Further, we use the concept of pseudo-prime subhypermodules to describe another new hypermodule class, namely, the topological hypermodule. We explore some algebraic properties of this hypermodule class. Then, in the next section, we connect a topology to the set of all pseudo-prime subhypermodules of topological hypermodules, called the Zariski topology. Let L be a subset of X_N for an S-hypermodule N. We show as notation the intersection of all elements in L by $\Im(L)$.

Definition 2. *Let N be an S-hypermodule.*

(1) *A subhypermodule K of N is called* pseudo-semiprime *if it is an intersection of pseudo-prime subhypermodules of N.*
(2) *A pseudo-prime subhypermodule K of N is called* extraordinary *if, whenever T and L are pseudo-semiprime subhypermodules of N, $T \cap L \leq K$ implies that $L \subseteq K$ or $T \subseteq K$.*
(3) *The pseudo-prime radical of K is shown as notation $Prad(K)$ is the intersection of each pseudo-prime subhypermodules of N containing K, i.e.,* $Prad(K) = \Im(V(K)) = \bigcap_{P \in V(K)} P$ *. If $V(K) = \emptyset$, then we get $Prad(K) = N$ for a subhypermodule K* of *N.*
(4) *If $K = Prad(K)$, then the subhypermodule K of N is said to be a pseudo-prime radical subhypermodule.*
(5) *If $X_N = \emptyset$ or each pseudo-prime subhypermodule of N is extraordinary, then N is said to be topological.*

Using Definition 2 we prove that every prime hyperideal of S is an extraordinary pseudo-prime subhypermodule for the S-hypermodule S.

Theorem 1. *Let N be a topological S-hypermodule. Then the following statements hold.*

(1) *Every strong homomorphic image of N is a topological S-hypermodule.*
(2) *N_J is a topological S_J-hypermodule for every prime hyperideal J of S.*

Proof. (1) Let K be a subhypermodule of N. We have a factor S-hypermodule N/K, say L. Let U/K be a pseudo-prime subhypermodule of L. Since $(U/K:L) = (U:N)$, we obtain that U is a pseudo-prime subhypermodule of N. Let V/K and W/K be pseudo-semiprime subhypermodule of L so that $V/K \cap W/K \subseteq U/K$. Therefore, V and W are pseudo-semiprime subhypermodules of N such that $V \cap W \subseteq U$. By the hypothesis, $V \subseteq U$ or $W \subseteq U$. Therefore, $V/K \subseteq U/K$ or $W/K \subseteq U/K$. Consequently, L is a topological S-hypermodule.

(2) Let L be a pseudo-prime subhypermodule of the S_J-hypermodule N_J and let Ψ: $N \to N_J$ be the canonical strong homomorphism. First we shall prove that $L \cap N$ is a pseudo-prime subhypermodule of N. Let I and I' be hyperideals of S so that $II' \subseteq (L \cap N :_S N)$. Using the canonical strong homomorphic image of N by Ψ, we have $\left(I_J I'_J\right);\ N_J \subseteq L = (L \cap N)_J$. Since L is a pseudo-prime subhypermodule of the S_J-hypermodule N_J, either $I_J \subseteq \left(L : N_J\right)$ or $I'_J \subseteq \left(L : N_J\right)$. Therefore, we have $I.N \subseteq (I.N)_J \cap N \subseteq L \cap K$ or $I : N \subseteq L \cap K$. It follows that $L \cap K$ is a pseudo-prime subhypermodule of N. Take pseudo-semiprime subhypermodules K_1 and K_2 of N_J with $K_1 \cap K_2 \subseteq L$. We have that $K_1 \cap N$ and $K_2 \cap N$ are pseudo-semiprime subhypermodules of N with $(K_1 \cap N) \cap (K_2 \cap N) = (K_1 \cap K_2) \cap N \subseteq L \cap N$ that $K_1 = (K_1 \cap N)_J \subseteq (L \cap N)_J = H$ or $K_2 = (K_2 \cap N)_J \subseteq (L \cap N)_J = H$. Therefore, H is extraordinary and N_J is a topological S_J-hypermodule. □

Recall that the pseudo-prime subhypermodules of S as on S-hypermodule are the pseudo-prime hyperideals for any hyperring S. In the following theorem, we extend the fact in Theorem 1 to multiplication hypermodules.

Theorem 2. *Let N be a finitely generated S-hypermodule. Then the following assertions are equivalent.*

(1) *N is a multiplication hypermodule.*
(2) *There exists a hyperideal J of S such that $V(K) = V(J.K)$ for every subhypermodule K of N.*
(3) *N is a topological hypermodule.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Let L be a pseudo-prime subhypermodule of N. Assume that K and U are pseudo-semiprime subhypermodules of N with $K \cap U \subseteq L$. Then we have $V(K) = V(J.N)$ and $V(U) = V\left(J' : N\right)$ for hyperideals J and J' of S. Take some collection of pseudo-prime

subhypermodules $\{K'_\alpha\}_{\alpha\in\Omega}$ such that $K = \bigcap_{\alpha\in\Omega} K'_\alpha$. Therefore, we get $(J \cap J').N \subseteq K'_\alpha$ for every $\alpha \in \Omega$ using the conclusion

$$K'_\alpha \in V(K) \subseteq V(K) \cup V(U) = V(J.N) \cup V(J'.N = V((J \cap J').N)).$$

Hence $(J \cap J').N \subseteq \bigcap_{\alpha\in\Omega} K'_\alpha = K$. By a similar way, we have the conclusion $(J \cap J').N \subseteq U$. Thus $(J \cap J').N \subseteq K \cap U \subseteq L$. It follows from $J \cap J' \subseteq (L : N)$ that $L \in V(J.N) = V(K)$ or $L \in V(J{:}N) = V(U)$, that is $K \subseteq L$ or $U \subseteq L$.

(3) $\Rightarrow$ (1) Clear by Lemma 2. □

Definition 3. *Let N be an S-hypermodule. Then N is called content if $b \in c(b)N$, where $c(b) = \bigcap\{J : J$ is a hyperideal of S and $b \in J.N\}$ for every $b \in N$.*

It is can be seen that N is a content S-hypermodule if and only if $\left(\bigcap_{\alpha\in\Omega} J_\alpha\right).N = \bigcap_{\alpha\in\Omega} (J_\alpha.N)$ for every family of $\{J_\alpha : \alpha \in \Omega\}$ *of* S.

Theorem 3. *Let N be an S-hypermodule. Consider the following conditions:*

(1) N is a content and pseudo-injective S-hypermodule.

(2) $Prad(K) = \sqrt{(K{:}N)}N$ for every subhypermodule K of N.

Then if N satisfies one of these above conditions, it is topological.

Proof. If $Prad(L) = N$, then we have $V(K) = V(S.N)$. Suppose that $Prad(L) \neq N$. Therefore, $Prad(L)$ is a pseudo-semiprime subhypermodule of N. There exist pseudo-prime subhypermodules L_α for every $\alpha \in \Omega$ with $Prad(L) = \bigcap_{\alpha\in\Omega} L_\alpha$ and $(L_\alpha : N) = p_\alpha \in Spec(S)$. Therefore $p_\alpha N = (p_\alpha : N).N = ((p_\alpha : N).N : N)$ and N is pseudo-injective for every $\alpha \in \Omega$ with $L_\alpha = p_\alpha N$. Since N is a content hypermodule,

$$\begin{aligned} Prad(L) = \bigcap_{\alpha\in\Omega} L_\alpha = \bigcap_{\alpha\in\Omega} (p_\alpha N) = \left(\bigcap_{\alpha\in\Omega} p_\alpha\right) N \\ = \bigcap_{\alpha\in\Omega} (L_\alpha : N)N = \left(\bigcap_{\alpha\in\Omega} L_\alpha : N\right) N \\ = (Prad(L) : N)N. \end{aligned}$$

Then we obtain $V(L) = V(Prad(L)) = V((Prad(L) : N)N)$. It follows from Theorem 2 that N is a topological hypermodule.

Suppose that, for all subhypermodule K of N, $Prad(K) = \sqrt{(K : N)}N$. Then $V(K) = V(Prad(K)) = V\left(\sqrt{(K : N)}N\right)$. It follows from Theorem 2 that N is a topological hypermodule. □

3. Pseudo-Prime Spectrum over Topological Hypermodules

We denote N as a topological S-hypermodule in the rest of this text. In [11], we investigated the Zariski topology over multiplication hypermodules. Zariski topology is built on topological modules in [16]. In this section, inspired by this source, this class will be examined in hypermodules by looking at it from a different spectrum. Briefly, J and $\overline{J}$ will be used instead of $S/Ann(N)$ and $J/Ann(N)$ for every hyperideal $J \in V^s(Ann(N))$.

Theorem 4. *If X_N is connected for a pseudo-primeful S-hypermodule N, then $X_{\overline{S}}$ is connected.*

Proof. Let $\varphi : X_N \to Spec(S/Ann(N))$ be a natural map. As φ is surjective, we must show that φ is continuous. Take a hyperideal J of S containing $Ann(N)$. Let $K \in \varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right)$. There is a hyperideal $\overline{J'} \in V^{\overline{S}}(\overline{J})$ such that $\varphi(K) = \overline{J'}$. Thus $J \subseteq (K{:}N) = J$. It follows from

$J.N \subseteq K$ that $K \in V^N(J.N)$. Let $L \in V^N(J.N)$. Then we obtain $J \subseteq (J.N:N) \subseteq (L:N)$. Therefore $L \in \varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right)$. φ is continuous as $\varphi^{-1}\left(V^{\overline{S}}(\overline{J})\right) = V^N(J.N)$. □

In the following proposition, we obtain basic properties of the subhypermodules of N, taking the topological hyperspace X_N as a T_1-hyperspace.

Proposition 1. *Let $Y \subseteq X_N$ and $K \in X_{N,J}$ for any $J \in Spec(S)$. Then the following statements hold.*

(1) $Cl(Y) = V(\Im(Y))$. Thus $Y = V(\Im(Y))$ if and only if Y is closed.
(2) $\langle 0 \rangle \in Y$ provided that Y is dense in X_N.
(3) X_N is a T_0-hyperspace.
(4) Every pseudo-prime subhypermodule of N is a maximal element in the set of whole pseudo-prime subhypermodules of N if and only if X_N is a T_1-hyperspace.
(5) $Spec(S)$ is a T_1-hyperspace provided that X_N is a T_1-hyperspace.

Proof. (1) The inclusion $V(\Im(Y)) \supseteq Y$ is clear. Let $V(K)$ be any closed subset of X_N containing Y. Then, $V(\Im(Y)) \subseteq V(\Im(V(K))) = V(Prad(K)) = V(K)$ since $\Im(V(K)) \subseteq \Im(Y)$. It follows that $V(\Im(Y))$ is the smallest closed subset of X_N containing Y. Therefore, the equality is obtained.

(2) It can be seen clearly due to condition (1).

(3) To show X_N is a T_0-hyperspace, we have to prove that all closures of distinct points in X_N are distinct. Let H and K be any distinct point of X_N. According to condition (1), we have $Cl(\{H\}) = V(H) \neq V(K) = Cl(\{K\})$; this is also desired.

(4) Topologically, we know that for X_N to be a T_1-hyperspace, it must be that each singleton subset is closed. Let L be a maximal element in the set of all pseudo-prime subhypermodules of N; using condition (1) we get that $Cl(\{L\}) = V(L) = \{L\}$. Therefore, $\{L\}$ is closed. We obtain that X_N is a T_1-hyperspace. Conversely, let $\{L\}$ be closed as X_N is a T_1-hyperspace. Therefore, we can write the following equality:

$$\{L\} = Cl(\{L\}) = V(\Im(\{L\})) = V(L).$$

Therefore L is a maximal element in the set of whole pseudo-prime subhypermodules of N.

(5) Let L be a pseudo-prime subhypermodule of N. We have $Cl(\{L\}) = V(L)$ using condition (1). Let $H \in V(L)$. By the hypothesis, we have $(L:N) = (H:N) \in Max(S)$. Thus, L and H are prime subhypermodules of N. By Theorem 2, we can write $H = L$. It follows from $Cl(\{L\}) = L$ that X_N is a T_1-hyperspace. □

Definition 4. *A topological hyperspace N is called irreducible if for every decomposition $N = N_1 \cup N_2$ as closed subsets N_1 and N_2 of N provided that $N_1 = NN_2 = N$. In addition, a maximal irreducible subset of N is said to be an irreducible component of N.*

The next theorem reveals the relation between pseudo-prime subhypermodules of the S-hypermodule N and an irreducible subset of the topological hyperspace X_N. It is clear that for a hyperring S, a subset K of $Spec(S)$ is irreducible if and only if $\Im(K)$ is a prime hyperideal of S.

Theorem 5. *Let N be an S-hypermodule and K be a subset of X_N. Then $\Im(K)$ is a pseudo-prime subhypermodule of N if and only if K is an irreducible hyperspace.*

Proof. ($\Rightarrow$) Let us take a pseudo-prime subhypermodule $\Im(K)$ of N with $K \subseteq K_1 \cup K_2$ where K_1 and K_2 are closed subsets of X_N. Thus there exist subhypermodules L and T of N such that $V(L) = K_1$ and $V(T) = K_2$. Therefore, $\Im(V(L) \cup V(T)) = \Im(V(L)) \cap \Im(V(T)) = Prad(L) \cap Prad(T) \subseteq \Im(K)$. Then we have that $\Im(K)$ is an extraordinary subhypermodule because N is a topological hypermodule. It is obtained that $Prad(L) \subseteq$

$\Im(L)$ or $Prad(T) \subseteq \Im(K)$ and so $K \subseteq V(\Im(K)) \subseteq V(Prad(L)) = V(L) = K_1$ or $K \subseteq K_2$. It means that K is irreducible.

($\Leftarrow$) Let K be an irreducible hyperspace, T and U be hyperideals of S with $TU \subseteq (\Im(K):N)$. Then we have $K \subseteq V(\Im(K)) \subseteq V((TU).N) = V(T.N) = \bigcup V(U.N)$. By the assumption, we obtain that $K \subseteq V(T.N)$ or $K \subseteq V(U.N)$. Therefore, $T.N \subseteq Prad(T.N) = \Im(V(U.N)) \subseteq \Im(N)$ or $U.N \subseteq \Im(K)$. Since $T \subseteq (\Im(K):N)$ or $U \subseteq (\Im(K):N)$, then $\Im(K)$ is a pseudo-prime subhypermodule of N. □

Corollary 1. *Let N be an S-hypermodule and let K be a subhypermodule of N.*

(1) $Prad(K)$ is a pseudo-prime subhypermodule of N if and only if $V(K)$ is an irreducible hyperspace.
(2) $Prad(0)$ is a pseudo-prime subhypermodule of N if and only if N is an irreducible hyperspace.
(3) If $X_{N,U} \neq \emptyset$ for any $u \in Spec(S)$, then $X_{N,U}$ is an irreducible hyperspace.

Proof. (1) It follows from $Prad(K) = \Im(V(K))$ that the proof is obtained directly using Theorem 3.

(2) Clear from (1) by taking $K = (0)$.

(3) Since $\Im(X_{N,U} : N) = \bigcap_{Q \in X_{N,U}} (Q:N) = U \in Spec(S)$, the claim holds due to Theorem 3. □

Definition 5. *Let N be an S-hypermodule and U be a hyperideal of S. Then U is said to be a radical hyperideal of S if $U = \bigcap_i u_i$ where u_i runs through $V^S(U)$.*

Lemma 2. *Let N be a non-zero pseudo-primeful S-hypermodule and U be a radical hyperideal of S. Then $Ann(N) \subseteq U$ if and only if $(U.N : N) = U$.*

Proof. ($\Rightarrow$) By the hypothesis, $Ann(N) \subseteq U = \bigcap_i u_i$, where u_i runs through $V^S(U)$. Then there is a pseudo-prime subhypermodule K_i of N with $(K_i:N) = u_i$ for a pseudo-primeful S-hypermodule N and $u_i \in V^S(Ann(N))$. We have $U \subseteq (U.N:N) = \left(\left(\bigcap_i u_i\right).N:N\right) \subseteq \bigcap_i (u_i.N:N) \subseteq \bigcap_i (K_i:N) = \bigcap_i u_i = U$. Hence $(U.N:N) = U$.

($\Leftarrow$) It is clear. □

Let N be an S-hypermodule and L be a subhypermodule of N. In [19], L is called *small* if $N \neq T + L$ for every proper subhypermodule T of N. Following [19], we denote by $Rad(N)$ the sum of all small subhypermodules of N.

Now let us adapt Nakayama's Lemma to hypermodules in the next proposition.

Proposition 2. *Let N be a pseudo-primeful S-hypermodule and U be a hyperideal of S that is contained in $Rad(S)$ such that $U.N = N$. Then $N = (0)$.*

Definition 6. *Let T be closed subset of a topological hyperspace N and a be an element of T. If $T = Cl(\{a\})$, then a is said to be the* generic point *of T.*

By Proposition 1 (1), we obtain that each element K of X_N is a *generic point* of the irreducible closed subset $V(K)$. Note that if the topological hyperspace is a T_0-hyperspace, the general point T of a closed subset of the topological hyperspace is unique by Proposition 1. The following theorem is an excellent implementation of Zariski topology on hypermodules. Indeed, the following theorem shows that there is a relationship between the irreducible closed subsets of X_N and the pseudo-prime subhypermodules of the S-hypermodule N.

Theorem 6. *Let N be a S-hypermodule and $U \subseteq X_N$. Then U is an irreducible closed subset of X_N if and only if $U = V(W)$ for each $W \in X_N$.*

Proof. ($\Rightarrow$) It follows from Proposition 1 that $U = V(W)$ is an irreducible closed subset of X_N for each pseudo-prime subhypermodule W of N.

($\Leftarrow$) Let $U = V(W)$ be an irreducible closed subset of X_N. Therefore, we have $U = V(T)$ for some subhypermodule T of N and $\Im(U) = \Im(V(T)) = Prad(T) \in X_N$ by using Theorem 5. Then we get $U = V(T) = V(Prad(T))$. This completes the proof. □

Recall from [17] that a hyperring S is said to be *Noetherian* if it satisfies the ascending chain condition on hyperideals of S, i.e., for each ascending chain of hyperideals

$$J_1 \subseteq J_2 \subseteq \dots,$$

there is an element $k \in \mathbb{N}$ such that $J_k = J_t$ for every $k \geq t$.

Definition 7. *A topological hyperspace X is said to be Noetherian hyperspace if the open subset of the hyperspace possesses the ascending chain condition.*

We use the notion of Noetherian S-hypermodules for pseudo-prime spectrum of hypermodules and radical hyperideals of S satisfying the ascending chain condition ACC.

Theorem 7. *Let N be an S-hypermodule. Then N possesses Noetherian pseudo-prime spectrum if and only if the ACC is provided pseudo-prime radical subhypermodules of N.*

Proof. ($\Rightarrow$) Let N have a Noetherian pseudo-prime spectrum and

$$U_1 \subseteq U_2 \subseteq \dots$$

be an ascending chain of pseudo-prime radical subhypermodules of N. Hence $U_j = \Im\big(\big(V\big(U_j\big)\big)\big) = Prad\big(U_j\big)$ for $j \in \mathbb{N}$. It follows that $V(U_1) \supseteq V(U_2) \supseteq \dots$ is a descending chain of closed subsets of X_N. By the hypothesis there exists an element $l \in \mathbb{N}$ such that $V(U_l) = V(U_{l+n})$ for each $n \in \mathbb{N}$. Thus

$$N_1 = Prad(U_l) = \Im(V(U_l)) = \Im(V(U_{l+n})) = Prad(U_{l+n}) = U_{l+n}.$$

($\Leftarrow$) Suppose that the ACC is provided for pseudo-prime radical subhypermodules of N. Let

$$V(U_1) \supseteq V(U_2) \supseteq \dots$$

be a descending chain of closed subsets of X_N for $U_j \leq N$. Then $\Im(V(U_1)) \subseteq \Im(V(U_2)) \subseteq \dots$ is an ascending chain of psudo-prime radical subhypermodules $\Im\big(V\big(U_j\big)\big) = Prad\big(U_j\big)$ of the hypermodule N. By the hypothesis, there is an element $l \in \mathbb{N}$ such that $\Im(V(U_l)) = \Im\Big(V\Big(U_{l+j}\Big)\Big)$ for each $j \in \mathbb{N}$. It follows from Proposition 1 that $V(U_l) = V(\Im(V(U_l))) = V\Big(\Im\Big(V\Big(U_{l+j}\Big)\Big)\Big) = V\Big(U_{l+j}\Big)$. Therefore, X_N is a Noetherian hyperspace. □

Definition 8. *Let S be a hyperring according to the Zariski topology and N be an S-hypermodule with topological hyperspace. A topological hyperspace N is called a* spectral hyperspace *if it is homeomorphic to $Spec(S)$.*

Theorem 8. *Let N be an S-hypermodule. Then X_N is a spectral hyperspace if each of the following conditions are met.*

(1) There exists a hyperideal J of S so that $V(U) = V(J.N)$ for a Noetherian hyperring S and for every subhypermodule U of N.

(2) Let N be an content pseudo-injective S-hypermodule and $Spec(S)$ be a Noetherian topological hyperspace.

Proof. (1) If it is shown that every subset of X_N is quasi-compact, the desired result is obtained. Let K be an open subset of X_N and $\{A_i\}_{i\in\mathbb{N}}$ be an open cover of K. Then there exist subhypermodules L and L_i so that $K = X_N \backslash V(L)$, $A_i = X_N \backslash V(L_i)$ for every $i \in I$ and $K \subseteq \bigcup\limits_{i\in I} A_i = X_N \backslash \bigcap\limits_{i\in I} V(L_i)$. By assumption, there is a hyperideal I_i in S so that $V(L_i) = V(I_i.N)$ for every $i \in I$. Then we have $L \subseteq X_N \backslash V\left(\sum\limits_{i\in I} I_i.N\right) = X_N \backslash V\left(\left(\sum\limits_{i\in I} I_i\right).N\right)$. As S is a Noetherian hyperring, there is a finite subset I' of I so that $L \subseteq \bigcup\limits_{j\in I'} A_j$. Hence X_N is a both of Noetherian hyperspace and spectral hyperspace.

(2) Let us show that X_N is Noetherian. Let $V(L_1) \supseteq V(L_2) \supseteq \ldots$ be a descending chain of closed subsets of X_N. Therefore, $Prad(L_1) \subseteq Prad(L_2) \subseteq \ldots$ As $Spec(S)$ is Noetherian, the ACC $(Prad(L_1) : N) \subseteq (Prad(L_2) : N) \subseteq \ldots$ of radial hyperideals shall be stationary by Theorem 8. Therefore there exists an element $l \in \mathbb{N}$ so that $(Prad(L_l) : N) = \left(Prad\left(L_{l+j}\right) : N\right) = \ldots$, for every $j = 1, 2, \ldots$ If the proof technique in Theorem 3 is applied, it is seen that $Prad\left(L_j\right) = \left(Prad\left(L_j\right) : N\right).N$. Thus, we get $Prad(L_l) = Prad\left(L_{l+j}\right) = \ldots$ for every $j = 1, 2, \ldots$ It follows that $V(L_l) = V(Prad(L_l)) = V\left(Prad\left(L_{l+j}\right)\right) = V\left(L_{l+j}\right) = \ldots$ Therefore, X_N is Noetherian, and the desired result is achieved. □

Author Contributions: Conceptualization, B.N.T.; methodology, E.T., B.N.T. and Ö.K. ; investigation, E.T., B.N.T. and Ö.K.; writing—original draft preparation, E.T. and Ö.K.; writing—review and editing, Ö.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees who contributed to the development of the article.

Conflicts of Interest: The authors declare that they have no competing interests.

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