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Sequentially Ordered Sobolev Inner Product and Laguerre–Sobolev Polynomials

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Abstract: We study the sequence of polynomials $\{S_n\}_{n \geq 0}$ that are orthogonal with respect to the general discrete Sobolev-type inner product $\langle f, g \rangle_s = \int f(x)g(x)d\mu(x) + \sum_{j=1}^{N} \sum_{k=0}^{d_j} \lambda_{j,k} f^{(k)}(c_j) g^{(k)}(c_j)$, where μ is a finite Borel measure whose support $\text{supp}(\mu)$ is an infinite set of the real line, $\lambda_{j,k} \geq 0$, and the mass points c_i, $i = 1, \ldots, N$ are real values outside the interior of the convex hull of $\text{supp}(\mu)$ ($c_i \in \mathbb{R} \setminus \mathbf{C}_h(\text{supp}(\mu))^\circ$). Under some restriction of order in the discrete part of $\langle \cdot, \cdot \rangle_s$, we prove that S_n has at least $n - d^*$ zeros on $\mathbf{C}_h(\text{supp}(\mu))^\circ$, being d^* the number of terms in the discrete part of $\langle \cdot, \cdot \rangle_s$. Finally, we obtain the outer relative asymptotic for $\{S_n\}$ in the case that the measure μ is the classical Laguerre measure, and for each mass point, only one order derivative appears in the discrete part of $\langle \cdot, \cdot \rangle_s$.

Keywords: orthogonal polynomials; Sobolev orthogonality; zeros location; asymptotic behavior

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1. Introduction

Let μ be a positive finite Borel measure with finite moments, whose support $\Delta \subset \mathbb{R}$ contains infinitely many points. We will denote by $\mathbf{C}_h(A)$ the convex hull of a set A and by A° its interior.

Let $\{P_n\}_{n \geq 0}$ be the monic orthogonal polynomial sequence with respect to the inner product

$$\langle f, g \rangle_\mu = \int_\Delta f(x)g(x)d\mu(x).$$

An inner product is called standard if the multiplication operator is symmetric with respect to the inner product. Obviously, $\langle xf, g \rangle_\mu = \langle f, xg \rangle_\mu$, i.e., $\langle \cdot, \cdot \rangle_\mu$ is standard. Significant parts of the applications of orthogonal polynomials in mathematics and particular sciences are based on the following three consequences of this fact.

1. The polynomial P_n has exactly n real simple zeros in $\mathbf{C}_h(\Delta)^\circ$. Moreover, there is a zero of P_{n-1} between any two consecutive zeros of P_n.
2. The three-term recurrence relation

    $$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x); \quad P_0(x) = 1, \ P_{-1}(x) = 0,$$

    where $\gamma_n = \|P_n\|_\mu / \|P_{n-1}\|_\mu$ for $n \geq 1$, $\beta_n = \langle P_n, xP_n \rangle_\mu / \|P_n\|_\mu^2$ and $\|\cdot\|_\mu = \sqrt{\langle \cdot, \cdot \rangle_\mu}$ denotes the norm induced by $\langle \cdot, \cdot \rangle_\mu$.
3. For the kernel polynomials

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\|P_k\|_\mu^2},\tag{1}$$

we have the Christoffel–Darboux identities

$$K_n(x,y) = \begin{cases} \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{\|P_n\|_\mu^2 \, (x-y)}, & \text{if } x \neq y, \\ \frac{P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)}{\|P_n\|_\mu^2}, & \text{if } x = y. \end{cases}\tag{2}$$

These identities play a fundamental role in the treatment of Fourier expansions with respect to a system of orthogonal polynomials (see [1], Section 2.2). For a review of the use of (1) and (2) in the spectral theory of orthogonal polynomials, we refer the reader to [2]. In addition, see the usual references [3–5], for a basic background on these and other properties of $\{P_n\}_{n\geq 0}$.

Let $(a,b) = \mathbf{C}_h(\operatorname{supp}(\mu))^\circ$, $N, d_j \in \mathbb{Z}_+$, $\lambda_{j,k} \geq 0$, for $j = 1, \ldots, N$, $k = 0, 1, \ldots, d_j$, $\{c_1, c_2, \ldots, c_N\} \subset \mathbb{R} \setminus (a,b)$, where $c_i \neq c_j$ if $i \neq j$ and $I_+ = \{(j,k) : \lambda_{j,k} > 0\}$. We consider the following *Sobolev-type (or discrete Sobolev) inner product*

$$\langle f, g \rangle_{\mathsf{s}} = \int f(x)g(x)d\mu(x) + \sum_{j=1}^{N} \sum_{k=0}^{d_j} \lambda_{j,k} f^{(k)}(c_j) g^{(k)}(c_j)$$

$$= \int f(x)g(x)d\mu(x) + \sum_{(j,k) \in I_+} \lambda_{j,k} f^{(k)}(c_j) g^{(k)}(c_j),\tag{3}$$

where $f^{(k)}$ denotes the k-th derivative of the function f. Without loss of generality, we also assume $\{(j, d_j)\}_{j=1}^{N} \subset I_+$ and $d_1 \leq d_2 \leq \cdots \leq d_N$. For $n \in \mathbb{Z}_+$, we shall denote by S_n the monic polynomial of the lowest degree satisfying

$$\langle x^k, S_n \rangle_{\mathsf{s}} = 0, \quad \text{for } k = 0, 1, \ldots, n-1.\tag{4}$$

It is easy to see that for all $n \geq 0$, there exists such a unique polynomial S_n of degree n. This is deduced by solving a homogeneous linear system with n equations and $n + 1$ unknowns. Uniqueness follows from the minimality of the degree for the polynomial solution. We refer the reader to [6,7] for a review of this type of non-standard orthogonality.

Clearly, (3) is not standard, i.e., $\langle xp, q \rangle_{\mathsf{s}} \neq \langle p, xq \rangle_{\mathsf{s}}$, for some $p, q \in \mathbb{P}$. It is well known that the properties of orthogonal polynomials with respect to standard inner products differ from those of the Sobolev-type polynomials. In particular, the zeros of the Sobolev-type polynomials can be complex, or if real, they can be located outside the convex hull of the support of the measure μ, as can be seen in the following example.

Example 1 (Zeros outside the convex hull of the measures supports). *Set*

$$\langle f, g \rangle_{\mathsf{s}} = \int_0^\infty f(x)g(x)e^{-x}dx + 2f'(-1)g'(-1),$$

then the corresponding second-degree monic Sobolev-type orthogonal polynomial is $S_2(z) = z^2 - 2$, whose zeros are $z_{1,2} = \pm\sqrt{2}$. Note that $-\sqrt{2} \notin [-1, \infty)$.

Let $\{Q_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle_{\mu_\rho} = \int f(x)\, g(x)\, d\mu_\rho(x), \quad \text{where } \rho(x) = \prod_{c_j \leq a}(x - c_j)^{d_j+1} \prod_{c_j \geq b}(c_j - x)^{d_j+1}$$

$$\text{and } d\mu_\rho(x) = \rho(x)d\mu(x).$$

Note that ρ is a polynomial of degree $d = \sum_{j=1}^{N}(d_j + 1)$, which is positive on (a, b). If $n > d$, from (4), $\{S_n\}$ satisfies the following quasi-orthogonality relations with respect to μ_ρ

$$\langle S_n, f \rangle_{\mu_\rho} = \langle S_n, \rho f \rangle_\mu = \int S_n(x) f(x) \rho(x) d\mu(x) = \langle S_n, \rho f \rangle_s = 0,$$

for $f \in \mathbb{P}_{n-d-1}$, where $\mathbb{P}_n$ is the linear space of polynomials with real coefficients and the degree at most $n \in \mathbb{Z}_+$. Hence, *the polynomial S_n is quasi-orthogonal of order d with respect to μ_ρ* and by this argument, we obtain that S_n has at least $(n - d)$ changes of sign in (a, b).

The results obtained for measures μ with bounded support (see [8], (1.10)) suggest that the number of zeros located in the interior of the support of the measure is closely related to $d^* = |I_+|$, the number of terms in the discrete part of $\langle \cdot, \cdot \rangle_s$ (i.e., $\lambda_{j,k} > 0$), instead of this greater quantity d.

Our first result, Theorem 1, goes in this direction for the case when the inner product is sequentially ordered. This kind of inner product is introduced in Section 2 (see Definition 1).

Theorem 1. *If the discrete Sobolev inner product* (3) *is sequentially ordered, then S_n has at least $n - d^*$ changes of sign on (a, b), where d^* is the number of positive coefficients $\lambda_{j,k}$ in* (3).

Previously, this result was obtained for more restricted cases in ([9], Th. 2.2) and ([10], Th. 1). In ([9], Th. 2.2), the authors proved this result for the case $N = 1$. In ([10], Th. 1), the notion of a sequentially ordered inner product is more restrictive than here, because it did not include the case when the Sobolev inner product has more than one derivative order at the same mass point.

In the second part of this paper, we focus our attention on the Laguerre–Sobolev-type polynomials (i.e., $d\mu = x^\alpha e^{-x} dx$, with $\alpha > -1$). In the case of the inner product, (3) takes the form

$$\langle f, g \rangle_s = \int_0^\infty f(x) g(x) x^\alpha e^{-x} dx + \sum_{j=1}^N \lambda_j f^{(d_j)}(c_j) g^{(d_j)}(c_j), \tag{5}$$

where $\lambda_j := \lambda_{j,d_j} > 0$, $c_j < 0$, for $j = 1, 2, \ldots, N$, we obtain the outer relative asymptotic of the Laguerre–Sobolev-type polynomials.

Theorem 2. *Let $\{L_n^\alpha\}_{n \geq 0}$ be the sequence of monic Laguerre polynomials and let $\{S_n\}_{n \geq 0}$ be the monic orthogonal polynomials with respect to the inner product* (5). *Then,*

$$\frac{S_n(x)}{L_n^\alpha(x)} \rightrightarrows \prod_{j=1}^N \left(\frac{\sqrt{-x} - \sqrt{|c_j|}}{\sqrt{-x} + \sqrt{|c_j|}} \right), \quad K \subset \overline{\mathbb{C}} \setminus \mathbb{R}_+. \tag{6}$$

Throughout this paper, we use the notation $f_n \rightrightarrows f$, $K \subset \mathbb{U}$ when the sequence of functions f_n converges to f uniformly on every compact subset K of the region $\mathbb{U}$.

Combining this result with Theorem 1, we obtain that the Sobolev polynomials S_n, orthogonal with respect to a sequentially ordered inner product in the form (5), have at least n–N zeros in $(0, \infty)$ and, for sufficiently large n, each one of the other N zeros are contained in a neighborhood of each mass point c_j ($j = 1, \ldots, N$). Then, we have located all zeros of S_n and we obtain that for a sufficiently large n, they are simple and real, as in the Krall case (see [11]) or the Krall–Laguerre-type orthogonal polynomial (see [12]). This is summarized in the following corollary.

Corollary 1. *Let $\mu = \mu_\alpha$ be the classical Laguerre measure ($d\mu_\alpha(x) = x^\alpha e^{-x} dx$) and* (5) *a sequentially ordered discrete Sobolev inner product. Then, the following statements hold:*

1. *Every point c_j attracts exactly one zero of S_n for sufficiently large n, while the remaining n–N zeros are contained in $(0, \infty)$. This means:*

For every $r > 0$, there exists a natural value $\mathcal{N}$ such that if $n \geq \mathcal{N}$, then the n zeros of S_n $\{\xi_i\}_{i=1}^n$ satisfy

$$\xi_j \in B(c_j, r) \text{ for } j = 1, \ldots, N \quad \text{and} \quad \xi_i \in (0, \infty) \text{ for } i = N+1, N+2, \ldots, n.$$

2. The zeros of S_n are real and simple for large-enough values of n.
3. The zeros of $\{S_n\}_{n=1}^\infty$ are at a finite distance from $(0, \infty)$. This means that there exists a positive constant M such that if ξ is a zero of S_n, then

$$d(\xi, (0, \infty)) := \inf_{x>0} \{|x - \xi|\} < M.$$

Section 2 is devoted to introducing the notion of a sequentially ordered Sobolev inner product and to prove Theorem 1. In Section 3, we summarize some auxiliary properties of Laguerre polynomials to be used in the proof of Theorem 2. Some results about the asymptotic behavior of the reproducing kernels are given. The aim of the last section is to prove Theorem 2 and some of its consequences stated in Corollary 2.

2. Sequentially Ordered Inner Product

Definition 1 (Sequentially ordered Sobolev inner product). *Consider a discrete Sobolev inner product in the general form* (3) *and assume* $d_1 \leq d_2 \leq \cdots \leq d_N$ *without loss of generality. We say that a discrete Sobolev inner product is* sequentially ordered *if the conditions*

$$\Delta_k \cap \mathbf{C}_h\left(\cup_{i=0}^{k-1}\Delta_i\right)^\circ = \varnothing, \qquad k = 1, 2, \ldots, d_N,$$

hold, where

$$\Delta_k = \begin{cases} \mathbf{C}_h(\operatorname{supp}(\mu) \cup \{c_j : \lambda_{j,0} > 0\}), & \text{if } k = 0, \\ \mathbf{C}_h\left(\{c_j : \lambda_{j,k} > 0\}\right), & \text{if } 1 \leq k \leq d_N. \end{cases} \tag{7}$$

Note that Δ_k is the convex hull of the support of the measure associated with the k-th order derivative in the Sobolev inner product (3). Let us see two examples.

Example 2 (Sequentially ordered inner product).
Set

$$\langle f, g \rangle_{\mathsf{s}} = \int_0^\infty f(x)g(x)e^{-x}dx + 10f(-1)g(-1) + 5f'(-3)g'(-3) \\ + 5f'(-9)g'(-9) + 20f'''(-10)g'''(-10),$$

then the corresponding fifth-degree Sobolev orthogonal polynomial has the following exact expression

$$S_5(x) = x^5 + \frac{380961336355365}{16894750106161}x^4 + \frac{1836311881214045}{16894750106161}x^3 - \frac{7830454972601355}{16894750106161}x^2 \\ - \frac{36972053870326650}{16894750106161}x - \frac{22386262325875230}{16894750106161},$$

whose zeros are approximately $\xi_1 \approx 4.46$, $\xi_2 \approx -0.74$, $\xi_3 \approx -2.8$, $\xi_4 \approx -11.74 + 2.51i$ *and* $\xi_5 \approx -11.74 - 2.51i$. *Note that four of them are outside of* $(0, \infty)$ *and two are even complex.*

Example 3 (Non-sequentially ordered inner product).
Set $\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}dx + f'(-15)g'(-15) + f''(-9)g''(-9)$, *then the corresponding fifth-degree Sobolev orthogonal polynomial has the following exact expression*

$$S_5(x) = x^5 + \frac{55079160}{21682477}x^4 - \frac{5053767275}{21682477}x^3 + \frac{40953207555}{21682477}x^2$$
$$- \frac{98030649090}{21682477}x + \frac{42523040550}{21682477},$$

whose zeros are approximately $\xi_1 \approx 0.55$, $\xi_2 \approx 3.36$, $\xi_3 \approx 6.66 + 3.02i$, $\xi_4 \approx 6.66 - 3.02i$ and $\xi_5 \approx -19.77$. Note that, in spite of Theorem 1, $d^* = 2$ and three of the zeros of S_5 are outside of $(0, \infty)$, with two of them as not even real.

In the sequentially ordered example (Example 2), S_5 has exactly $1 = 5 - 4 = n - d^*$ simple zeros on the interior of the convex hull of the support of the Laguerre measure $(0, \infty)$, and thus, the bound of Theorem 1 is sharp. In addition, this example shows that the remaining d^* zeros might even be complex, although Corollary 1 shows that this does not happen when n is sufficiently large.

On the other hand, in the non-sequentially ordered example (Example 3), this condition is not satisfied, since S_5 has only $2 < 3 = 5 - 2 = n - d^*$ zeros on $(0, \infty)$, showing that the sequential order plays a main role in the localization of the zeros of S_n, at least to obtain this property for every value of n.

Throughout the remainder of this section, we will consider inner products of the form (3) that are sequentially ordered. The next lemma is an extension of ([13], Lemma 2.1) and ([10], Lemma 3.1).

Lemma 1. *Let $\{I_i\}_{i=0}^m$ be a set of $m + 1$ intervals on the real line and let P be a polynomial with real coefficients of degree $\geq m$. If*

$$I_k \cap \mathbf{C}_h\left(\cup_{i=0}^{k-1} I_i\right)^{\circ} = \varnothing, \qquad k = 1, 2, \ldots, m, \tag{8}$$

then

$$\mathbf{N}_z(P; J) + \mathbf{N}_{\circ}(P; I_0 \setminus J) + \sum_{i=1}^m \mathbf{N}_{\circ}\left(P^{(i)}; I_i\right) \leq \mathbf{N}_z\left(P^{(m)}; J\right)$$

$$+ \mathbf{N}_{\circ}\left(P^{(m)}; \mathbf{C}_h(\cup_{i=0}^m I_i) \setminus J\right) + m, \tag{9}$$

for every closed subinterval J of $I_0{}^{\circ}$ (both empty set and unitary sets are assumed to be intervals). Here, given a real set A and a polynomial P, $\mathbf{N}_{\circ}(P; A)$ denotes the number of values where the polynomial P vanishes on A (i.e., zeros of P on A without counting multiplicities), and $\mathbf{N}_z(P; A)$ denotes the total number of zeros (counting multiplicities) of P on A.

Proof. First, we point out the following consequence of Rolle's Theorem. If I is a real interval and J is a closed subinterval of I°, then

$$\mathbf{N}_z(P; J) + \mathbf{N}_{\circ}(P; I \setminus J) \leq \mathbf{N}_z\left(P'; J\right) + \mathbf{N}_{\circ}\left(P'; I^{\circ} \setminus J\right) + 1. \tag{10}$$

It is easy to see that (9) holds for $m = 0$. We now proceed by induction on m. Suppose that we have $m + 1$ intervals $\{I_i\}_{i=0}^m$ satisfying (8); thus, the first m intervals $\{I_i\}_{i=0}^{m-1}$ also satisfy (8), and we obtain (9) by induction hypothesis (taking $m - 1$ instead of m). Then

$$\mathbf{N}_z(P; J) + \mathbf{N}_{\circ}(P; I_0 \setminus J) + \sum_{i=1}^m \mathbf{N}_{\circ}\left(P^{(i)}; I_i\right),$$

$$\leq \mathbf{N}_z\left(P^{(m-1)}; J\right) + \mathbf{N}_{\circ}\left(P^{(m-1)}; \mathbf{C}_h\left(\cup_{i=0}^{m-1} I_i\right) \setminus J\right) + m - 1 + \mathbf{N}_{\circ}\left(P^{(m)}; I_m\right),$$

$$\leq \mathbf{N}_z\left(P^{(m)}; J\right) + \mathbf{N}_{\circ}\left(P^{(m)}; \mathbf{C}_h\left(\cup_{i=0}^{m-1} I_i\right)^{\circ} \setminus J\right) + m + \mathbf{N}_{\circ}\left(P^{(m)}; I_m\right),$$

$$\leq \mathbf{N}_z\left(P^{(m)}; J\right) + \mathbf{N}_{\circ}\left(P^{(m)}; \mathbf{C}_h(\cup_{i=0}^m I_i) \setminus J\right) + m,$$

where in the second inequality, we have used (10). □

As an immediate consequence of Lemma 1, the following result is obtained.

Lemma 2. *Under the assumptions of Lemma 1, we have*

$$\mathbf{N}_z(P; J) + \mathbf{N}_\circ(P; I_0 \setminus J) + \sum_{i=1}^{m} \mathbf{N}_\circ\left(P^{(i)}; I_i\right) \leq \deg P \qquad (11)$$

for every J closed subinterval of $I_0{}^\circ$. In particular, for $J = \emptyset$, we obtain

$$\sum_{i=0}^{m} \mathbf{N}_\circ\left(P^{(i)}; I_i\right) \leq \deg P. \qquad (12)$$

Lemma 3. *Let $\{(r_i, v_i)\}_{i=1}^{M} \subset \mathbb{R} \times \mathbb{Z}_+$ be a set of M ordered pairs. Then, there exists a unique monic polynomial U_M of minimal degree (with $0 \leq \deg U_M \leq M$), such that*

$$U_M^{(v_i)}(r_i) = 0, \quad i = 1, 2, \ldots, M. \qquad (13)$$

Furthermore, if the intervals $I_k = \mathbf{C}_h(\{r_i : v_i = k\})$, $k = 0, 1, 2, \ldots, v_M$, satisfy (8), then U_M has degree $\mathfrak{u}_M = \min \mathfrak{I}_M - 1$, where

$$\mathfrak{I}_M = \{i : 1 \leq i \leq M \text{ and } v_i \geq i\} \cup \{M + 1\}.$$

Proof. The existence of a nonidentical zero polynomial with degree $\leq M$ satisfying (13) reduces to solving a homogeneous linear system with M equations and $M + 1$ unknowns (its coefficients). Thus, a non-trivial solution always exists. In addition, if we suppose that there exist two different minimal monic polynomials U_M and $\widetilde{U}_M$, then the polynomial $\widehat{U}_M = U_M - \widetilde{U}_M$ is not identically zero, it satisfies (13), and $\deg \widehat{U}_M < \deg U_M$. Thus, if we divide $\widehat{U}_M$ by its leading coefficient, we reach a contradiction.

The rest of the proof runs by induction on the number of points M. For $M = 1$, the result follows taking

$$U_1(x) = \begin{cases} x - r_1, & \text{if } v_1 = 0, \\ 1, & \text{if } v_1 \geq 1. \end{cases}$$

Suppose that, for each sequentially ordered sequence of M ordered pairs, the corresponding minimal polynomial U_M has degree $\mathfrak{u}_M$.

Let $\{(r_i, v_i)\}_{i=1}^{M}$ be a set of M ordered pairs satisfying (8). Obviously, $\{(r_i, v_i)\}_{i=1}^{M-1}$ also satisfies (8) and U_M satisfies (13) for $i = 1, 2, \ldots, M - 1$; thus, we obtain $\deg U_{M-1} = \mathfrak{u}_{M-1}$ and $\deg U_M \geq \deg U_{M-1}$. Now, we divide the proof into two cases:

1. If $\mathfrak{u}_M = M$, then for all $1 \leq i \leq M$ we have $v_i < i$, which yields

   $$\deg U_M \geq \deg U_{M-1} = \mathfrak{u}_{M-1} = M - 1 \geq v_M.$$

   Since $\{(r_i, v_i)\}_{i=1}^{M}$ satisfies (8), from (12) we obtain

   $$M \leq \sum_{i=0}^{v_M} \mathbf{N}_\circ\left(U_M^{(i)}; I_i\right) \leq \deg U_M,$$

   which implies that $\deg U_M = M = \mathfrak{u}_M$.

2. If $\mathfrak{u}_M \leq M - 1$, then there exists a minimal j ($1 \leq j \leq M$), such that $v_j \geq j$, and $v_i < i$ for all $1 \leq i \leq j - 1$. Therefore, $\mathfrak{u}_M = j - 1 = \mathfrak{u}_{M-1}$. From the induction hypothesis, we obtain

   $$\deg U_{M-1} = \mathfrak{u}_{M-1} = j - 1 \leq v_j - 1 \leq v_M - 1,$$

   which gives $U_{M-1}^{(v_M)} \equiv 0$. Hence, $U_M \equiv U_{M-1}$ and, consequently, we obtain

   $$\deg U_M = \deg U_{M-1} = \mathfrak{u}_{M-1} = \mathfrak{u}_M.$$

   □

Note that, in Lemma 3, condition (8) is necessary for asserting that the polynomial U_M has degree $\mathfrak{u}_M$. If we consider $\{(-1,0),(1,0),(0,1)\}$, whose corresponding convex hulls $I_0 = [-1,1]$ and $I_1 = \{0\}$ do not satisfy (8), we obtain $U_3(x) = x^2 - 1$ and $\mathfrak{u}_3 = 3 \neq \deg U_3$.

Now we are able to prove the zero localization theorem for sequentially ordered discrete Sobolev inner products.

Proof of Theorem 1. Let $\xi_1 < \xi_2 < \cdots < \xi_\eta$ be the points on $(a,b) = \mathbf{C}_h(\mathrm{supp}(\mu))^\circ$ where S_n changes sign and suppose that $\eta < n - d^*$. Consider the set of ordered pairs

$$\{(r_i, \nu_i)\}_{i=1}^{d^*+\eta} = \{(\xi_i, 0)\}_{i=1}^{\eta} \cup \{(c_j, k) : \eta_{j,k} > 0, \ j = 1, 2, \ldots, N, k = 1, \ldots, d_j\}.$$

Since $\langle \cdot, \cdot \rangle_{\mathbf{s}}$ is sequentially ordered, the intervals $I_k = \Delta_k$ for $k = 0, 1, \ldots, \nu_N$ (see (7)) satisfy (8) (we can assume without loss of generality that $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{d^*+\eta}$). Consequently, from Lemma 3, there exists a unique monic polynomial $U_{d^*+\eta}$ of minimal degree, such that

$$\begin{aligned} U_{d^*+\eta}(\xi_i) &= 0; && \text{for } i = 1, \ldots, \eta, \\ U_{d^*+\eta}^{(k)}(c_j) &= 0; && \text{for each } (j,k) : \eta_{j,k} > 0, \end{aligned} \tag{14}$$

and $\deg U_{d^*+\eta} = \min \mathfrak{I}_{d^*+\eta} - 1 \leq d^* + \eta$, where

$$\mathfrak{I}_{d^*+\eta} = \{i : 1 \leq i \leq d^* + \eta \text{ and } \nu_i \geq i\} \cup \{d^* + \eta + 1\}. \tag{15}$$

Now, we need to consider the following two cases.

1. If $\deg U_{d^*+\eta} = d^* + \eta$, from (15), we obtain $\deg U_{d^*+\eta} \geq \nu_{\eta+d^*} + 1$. Thus, taking the closed interval $J = [\xi_1, \xi_\eta] \subset (a,b)$ in (11), we obtain

$$\begin{aligned} d^* + \eta &\leq \sum_{k=0}^{\nu_{d^*+\eta}} \mathbf{N}_\circ \left(U_{d^*+\eta}^{(k)}; I_k \right) \leq \mathbf{N}_z \left(U_{d^*+\eta}; [\xi_1, \xi_\eta] \right) + \mathbf{N}_\circ \left(U_{d^*+\eta}; I_0 \setminus [\xi_1, \xi_\eta] \right) \\ &\quad + \sum_{k=1}^{\nu_{d^*+\eta}} \mathbf{N}_\circ \left(U_{d^*+\eta}^{(k)}; I_k \right) \leq \deg U_{d^*+\eta} = d^* + \eta. \end{aligned}$$

2. If $\deg U_{d^*+\eta} < d^* + \eta$, from (15), there exists $1 \leq j \leq d^* + \eta$ such that $\deg U_{d^*+\eta} = j - 1$, $\nu_j \geq j$ and $\nu_i \leq i - 1$ for $i = 1, 2, \ldots, j - 1$. Hence,

$$\nu_{j-1} + 1 \leq j - 1 = \deg U_{d^*+\eta}$$

and, again, from (11) we have

$$\begin{aligned} j - 1 &\leq \sum_{k=0}^{\nu_{j-1}} \mathbf{N}_\circ \left(U_{d^*+\eta}^{(k)}; I_k \right) \leq \mathbf{N}_z \left(U_{d^*+\eta}; [\xi_1, \xi_\eta] \right) + \mathbf{N}_\circ \left(U_{d^*+\eta}; I_0 \setminus [\xi_1, \xi_\eta] \right) \\ &\quad + \sum_{k=1}^{\nu_{j-1}} \mathbf{N}_\circ \left(U_{d^*+\eta}^{(k)}; I_k \right) \leq \deg U_{d^*+\eta} = j - 1. \end{aligned}$$

In both cases, we obtain that $U_{d^*+\eta}$ has no other zeros in I_0 than those given by construction, and from $\mathbf{N}_\circ \left(U_{d^*+\eta}; [\xi_1, \xi_\eta] \right) = \mathbf{N}_z \left(U_{d^*+\eta}; [\xi_1, \xi_\eta] \right)$, all the zeros of S_n on I° are simple. Thus, in addition to (14), we obtain that $S_n U_{d^*+\eta}$ does not change sign on I°. Now, since $\deg U_{d^*+\eta} \leq d^* + \eta < n$, we arrive at the contradiction

$$\begin{aligned} 0 = \langle S_n, U_{d^*+\eta} \rangle &= \int S_n(x) U_{d^*+\eta}(x) d\mu(x) + \sum_{j=1}^{N} \sum_{k=0}^{d_j} \lambda_{j,k} S_n^{(k)}(c_j) U_{d^*+\eta}^{(k)}(c_j) \\ &= \int_a^b S_n(x) U_{d^*+\eta}(x) d\mu(x) \neq 0. \end{aligned}$$

$\square$

3. Auxiliary Results

The family of Laguerre polynomials is one of the three very well-known classical orthogonal polynomials families (see [3–5]). It consists of the sequence of polynomials $\{L_n^{(\alpha)}\}$ that are orthogonal with respect to the measure $d\mu = x^\alpha e^{-x} dx$, $x \in (0, \infty)$, for $\alpha > -1$, and that are normalized by taking $\frac{(-1)^n}{n!}$ as the leading coefficient of the n-th degree polynomial of the sequence. Laguerre polynomials play a key role in applied mathematics and physics, where they are involved in the solutions of the wave equation of the hydrogen atom (c.f. [14]).

Some of the structural properties of this family are listed in the following proposition in order to be used later.

Proposition 1. *Let $\{L_n^{(\alpha)}\}_{n\geq 0}$ (note the brackets in parameter α) be the sequence of Laguerre polynomials and let $\{L_n^\alpha\}_{n\geq 0}$ be the monic sequence of Laguerre polynomials. Then, the following statements hold.*

1. *For every $n \in \mathbb{N}$,*

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} L_n^\alpha(x). \tag{16}$$

2. *Three-term recurrence relation. For every $n \geq 1$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n + \alpha + 1)L_n^\alpha(x) + n(n + \alpha)L_{n-1}^\alpha(x)$$
$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (2n + \alpha + 1)L_n^{(\alpha)}(x) - (n + \alpha)L_{n-1}^{(\alpha)}(x)$$

with $L_{-1}^{(\alpha)} \equiv L_{-1}^\alpha = 0$, and $L_0^{(\alpha)} \equiv L_0^\alpha \equiv 1$.

3. *Structure relation. For every $n \in \mathbb{N}$,*

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x).$$

4. *For every $n \in \mathbb{N}$,*

$$||L_n^{(\alpha)}||_\mu^2 = \Gamma(\alpha + 1)\binom{n + \alpha}{n} = \frac{\Gamma(\alpha + n + 1)}{n!}. \tag{17}$$

In addition, we have

$$||L_n^\alpha||_\mu^2 = n!\Gamma(n + \alpha + 1)$$

5. *Hahn condition. For every $n \in \mathbb{N}$,*

$$[L_n^{(\alpha)}]'(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{18}$$

6. *Outer strong asymptotics (Perron's asymptotics formula on $\mathbb{C} \setminus \mathbb{R}_+$). Let $\alpha \in \mathbb{R}$. Then*

$$L_n^{(\alpha)}(x) = \frac{e^{x/2}n^{\alpha/2-1/4}e^{2(-nx)^{1/2}}}{2\pi^{1/2}(-x)^{\alpha/2+1/4}}\left\{\sum_{k=0}^{p-1} C_k(x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\}. \tag{19}$$

Here, $\{C_k(x)\}_{k=0}^{p-1}$ are certain analytic functions of x independent of n, with $C_0 \equiv 1$. This relation holds for x in the complex plane with a cut along the positive part of the real axis. The bound for the remainder holds uniformly in every closed domain with no points in common with $x \geq 0$ (see [5], Theorem 8.22.3).

Now, we summarize some auxiliary lemmas to be used in the proof of Theorem 2 (see ([15], Lem. 1) and ([16], Prop. 6)).

Lemma 4. *For $z \in \mathbb{C} \setminus [0, \infty)$, $\alpha, \beta \in \mathbb{R}$ and $j, k \geq -n$ we have*

$$\frac{L_{n+j}^{(\alpha+\beta)}(z)}{L_{n+k}^{(\alpha)}(z)} = \begin{cases} 1 + \frac{(j-k)\sqrt{-z}}{\sqrt{n}} + \left(\frac{\alpha}{2} - \frac{1}{4} - z\frac{(j-k)}{2}\right)\frac{(j-k)}{n} + \mathcal{O}_z\left(n^{-\frac{3}{2}}\right) & \text{if } \beta = 0 \\ \left(\frac{\sqrt{n}}{\sqrt{-z}}\right)^\beta \left(1 + \mathcal{O}_z\left(n^{-1/2}\right)\right) & \text{if } \beta \neq 0. \end{cases} \tag{20}$$

where $\mathcal{O}_z(n^{-j})$ denotes some analytic function sequence $\{g_n(z)\}_{n=1}^\infty$ such that $\{n^j g_n\}$ is uniformly bounded on every compact subset of $\mathbb{C} \setminus [0, \infty)$.

To study the outer relative asymptotic between the standard Laguerre polynomials and the Laguerre–Sobolev orthogonal polynomials (see Formula (6)), we need to compute the behavior of the Laguerre kernel polynomials and their derivatives when n approaches infinity. To this end, we prove the following auxiliary result, which is an extension of ([17], Ch. 5, Th. 16).

Lemma 5. *Let G and G' be two open subsets of the complex plane and $f_n : G \times G' \longrightarrow \mathbb{C}$ be a sequence of functions that are analytic with respect to each variable separately. If $\{f_n\}_{n=1}^\infty$ is a uniformly bounded sequence on each set in the form $K \times K'$, where $K \subset G$ and $K' \subset G'$ are compact sets, then any of its partial derivative sequences are also uniformly bounded on each set in the form $K \times K'$.*

Proof. Note that it is sufficient to prove this for the first derivative order with respect to any of the variables and then proceed by induction. Let $K \subset G$ and $K' \subset G'$ be two compact sets. Denote $G^c = \mathbb{C} \setminus G$, $d(K, G^c) = \inf_{z \in K, w \in G^c} |z - w|$, $r = d(K, G^c)/2 > 0$ and $B(z, r) = \{\zeta \in \mathbb{C} : |z - \zeta| < r\}$. Take K^* as the closure of $\bigcup_{z \in K} B(z, r)$; thus, K^* is a compact subset of G. Thus, there exists a positive constant $M > 0$ such that $|f_n(z, w)| \leq M$ for all $z \in K^*$, $w \in K'$ and $n \in \mathbb{N}$. Hence, for all $z \in K$, $w \in K'$ and $n \in \mathbb{N}$, we obtain

$$\left|\frac{\partial f_n}{\partial z}(z, w)\right| = \left|\frac{1}{2\pi i} \int_{c(z,r)} \frac{f_n(\xi, w)}{(\xi - z)^2} d\xi\right| \leq \frac{V(c(z,r))}{2\pi} \max_{\xi \in c(z,r)} \left\{\frac{|f_n(\xi, w)|}{|\xi - z|^2}\right\}$$

$$= \frac{2\pi r}{2\pi r^2} \max_{\xi \in c(z,r)} \{|f_n(\xi, w)|\} \leq \frac{M}{r},$$

where $c(z, r)$ denotes the circle with center at z, radius r and length $V(c(z, r))$. $\square$

From the Fourier expansion of S_n in terms of the basis $\{L_n^\alpha\}_{n \geqslant 0}$ we obtain

$$S_n(x) = \sum_{i=0}^n \langle S_n, L_i^\alpha \rangle_\mu \frac{L_i^\alpha(x)}{\|L_i^\alpha\|_\mu^2} = L_n^\alpha(x) + \sum_{i=0}^{n-1} \langle S_n, L_i^\alpha \rangle_\mu \frac{L_i^\alpha(x)}{\|L_i^\alpha\|_\mu^2}$$

$$= L_n^\alpha(x) + \sum_{i=0}^{n-1} \left(\langle S_n, L_i^\alpha \rangle_s - \sum_{(j,k) \in I_+} \lambda_{j,k} S_n^{(k)}(c_j)(L_i^\alpha)^{(k)}(c_j)\right) \frac{L_i^\alpha(x)}{\|L_i^\alpha\|_\mu^2}$$

$$= L_n^\alpha(x) - \sum_{(j,k) \in I_+} \lambda_{j,k} S_n^{(k)}(c_j) \sum_{i=0}^{n-1} \frac{L_i^\alpha(x)(L_i^\alpha)^{(k)}(c_j)}{\|L_i^\alpha\|_\mu^2}$$

$$= L_n^\alpha(x) - \sum_{(j,k) \in I_+} \lambda_{j,k} S_n^{(k)}(c_j) K_{n-1}^{(0,k)}(x, c_j), \tag{21}$$

where we use the notation $K_n^{(j,k)}(x, y) = \dfrac{\partial^{j+k} K_n(x, y)}{\partial^j x \partial^k y}$ to denote the partial derivatives of the kernel polynomials defined in (1). Differentiating Equation (21) ℓ-times and evaluating then at $x = c_i$ for each ordered pair $(i, \ell) \in I_+$, we obtain the following system of d^* linear equations and d^* unknowns $S_n^{(k)}(c_j)$.

$$(L_n^\alpha)^{(\ell)}(c_i) = \left(1 + \lambda_{i,\ell} K_{n-1}^{(\ell,\ell)}(c_i, c_i)\right) S_n^{(\ell)}(c_i) + \sum_{\substack{(j,k)\in I_+ \\ (j,k)\neq(i,\ell)}}^{d_j} \lambda_{j,k} K_{n-1}^{(\ell,k)}(c_i, c_j) S_n^{(k)}(c_j). \quad (22)$$

Lemma 6. *The Laguerre kernel polynomials and their derivatives satisfy the following behavior when n approaches infinity for $x, y \in \mathbb{C} \setminus [0, \infty)$*

$$K_{n-1}^{(i,j)}(x,y) = \frac{\partial^{i+j} K_{n-1}}{\partial^i x \partial^j y}(x,y) = \frac{L_n^{(\alpha+i)}(x) L_n^{(\alpha+j)}(y)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x} + \sqrt{-y})}\left((-1)^{i+j} + \mathcal{O}_{x,y}(n^{-1/2})\right), \quad i, j \geq 0,$$

where $\mathcal{O}_{x,y}(n^{-k})$ denotes some sequence of functions $\{g_n(x,y)\}_{n=1}^\infty$ that are holomorphic with respect to each variable and whose sequence $\{n^k g_n\}$ is uniformly bounded on every set $K \times K'$, such that K and K' are compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$.

Proof. The proof is by induction on $k = i + j$. First, suppose $k = 0$ (i.e., $i = j = 0$) and split the proof into two cases according to whether $x = y$ or not. If $x = y$, from (2), (16), (18) and (20), we obtain

$$\begin{aligned}
\frac{\|L_{n-1}^{(\alpha)}\|_\mu^2}{n} K_{n-1}(x,x) &= L_n^{(\alpha)}(x)(L_{n-1}^{(\alpha)})'(x) - (L_n^{(\alpha)})'(x) L_{n-1}^{(\alpha)}(x) \\
&= L_{n-1}^{(\alpha+1)}(x) L_{n-1}^{(\alpha)}(x) - L_{n-2}^{(\alpha+1)}(x) L_n^{(\alpha)}(x) \\
&= L_{n-2}^{(\alpha+1)}(x) L_{n-1}^{(\alpha)}(x)\left(\frac{L_{n-1}^{(\alpha+1)}(x)}{L_{n-2}^{(\alpha+1)}(x)} - \frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}\right) \\
&= L_{n-2}^{(\alpha+1)}(x) L_{n-1}^{(\alpha)}(x)\left[1 + \frac{\sqrt{-x}}{\sqrt{n}} + \left[\frac{\alpha+1}{2} - \frac{1}{4} - \frac{x}{2}\right]\frac{1}{n} + \mathcal{O}_x(n^{-3/2})\right. \\
&\quad - \left.\left(1 + \frac{\sqrt{-x}}{\sqrt{n}} + \left[\frac{\alpha}{2} - \frac{1}{4} - \frac{x}{2}\right]\frac{1}{n} + \mathcal{O}_x(n^{-3/2})\right)\right] \\
&= L_{n-2}^{(\alpha+1)}(x) L_{n-1}^{(\alpha)}(x)\left(\frac{1}{2n} + \mathcal{O}_x(n^{-3/2})\right) \\
&= \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)}(x)}{2n}\left(\frac{\sqrt{n}}{\sqrt{-x}}\right)\left(1 + \mathcal{O}_x(n^{-1/2})\right) \\
&= \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)}(x)}{2\sqrt{n}\sqrt{-x}}\left(1 + \mathcal{O}_x(n^{-1/2})\right).
\end{aligned}$$

On the other hand, if $x \neq y$, from (2) and (20) we obtain

$$\begin{aligned}
\frac{\|L_{n-1}^{(\alpha)}\|_\mu^2}{n} K_{n-1}(x,y) &= \frac{L_{n-1}^{(\alpha)}(x) L_n^{(\alpha)}(y) - L_n^{(\alpha)}(x) L_{n-1}^{(\alpha)}(y)}{x - y} \\
&= \frac{L_{n-1}^{(\alpha)}(x) L_{n-1}^{(\alpha)}(y)}{x - y}\left(\frac{L_n^{(\alpha)}(y)}{L_{n-1}^{(\alpha)}(y)} - \frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}\right) \\
&= \frac{L_{n-1}^{(\alpha)}(x) L_{n-1}^{(\alpha)}(y)}{x - y}\left(\frac{\sqrt{-y} - \sqrt{-x}}{\sqrt{n}} + \mathcal{O}_{x,y}(n^{-1})\right) \\
&= \frac{L_{n-1}^{(\alpha)}(x) L_{n-1}^{(\alpha)}(y)}{\sqrt{-x} + \sqrt{-y}}\left(\frac{1}{\sqrt{n}} + \mathcal{O}_{x,y}(n^{-1})\right) \\
&= \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)}{\sqrt{n}(\sqrt{-x} + \sqrt{-y})}\left(1 + \mathcal{O}_{x,y}(n^{-1/2})\right).
\end{aligned}$$

From (17) and ([18], Appendix, (1.14))

$$\|L_{n-1}^{(\alpha)}\|_{\mu}^2 = \frac{\Gamma(n+\alpha)}{\Gamma(n)} = n^{\alpha}(1 + \mathcal{O}(n^{-1})),$$

which proves the case $k = 0$. Now, we assume that the theorem is true for $i + j = k$ and we will prove it for $i + j = k + 1$. By the symmetry of the formula, the proof is analogous when any of the variables increase its derivative order; thus, we only will prove it when the variable y does.

$$\frac{\partial^{k+1} K_{n-1}}{\partial x^i \partial^{j+1} y}(x,y) = \frac{\partial}{\partial y} \left(\frac{L_n^{(\alpha+i)}(x) L_n^{(\alpha+j)}(y)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x}+\sqrt{-y})} \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) \right)$$

$$= \frac{L_n^{(\alpha+i)}(x)}{n^{\alpha-\frac{1}{2}}} \left[\frac{\partial}{\partial y} \left(\frac{L_n^{(\alpha+j)}(y)}{\sqrt{-x}+\sqrt{-y}} \right) \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) \right.$$

$$\left. + \frac{L_n^{(\alpha+j)}(y)}{\sqrt{-x}+\sqrt{-y}} \frac{\partial}{\partial y} \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) \right]$$

$$= \frac{L_n^{(\alpha+i)}(x)}{n^{\alpha-\frac{1}{2}}} \left[\frac{-(\sqrt{-x}+\sqrt{-y}) L_{n-1}^{(\alpha+j+1)}(y) + \frac{1}{2} L_n^{(\alpha+j)}(y)(-y)^{-1/2}}{(\sqrt{-x}+\sqrt{-y})^2} \right.$$

$$\left. \cdot \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) + \frac{L_n^{(\alpha+j)}(y)}{\sqrt{-x}+\sqrt{-y}} \mathcal{O}_{x,y}(n^{-1/2}) \right]$$

$$= \frac{L_n^{(\alpha+i)}(x) L_{n-1}^{(\alpha+j+1)}(y)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x}+\sqrt{-y})} \left[\left(-1 + \frac{\frac{\sqrt{-y}}{\sqrt{n}} + \mathcal{O}_{x,y}(n^{-1})}{2\sqrt{-y}(\sqrt{-x}+\sqrt{-y})} \right) \right.$$

$$\left. \cdot \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) + \left(\frac{\sqrt{-y}}{\sqrt{n}} + \mathcal{O}_{x,y}(n^{-1}) \right) \mathcal{O}_{x,y}(n^{-1}) \right]$$

$$= \frac{L_n^{(\alpha+i)}(x) L_{n-1}^{(\alpha+j+1)}(y)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x}+\sqrt{-y})} \left[\left(-1 + \mathcal{O}_{x,y}(n^{-1/2}) \right) \left((-1)^k + \mathcal{O}_{x,y}(n^{-1/2}) \right) \right.$$

$$\left. + \mathcal{O}_{x,y}(n^{-3/2}) \right]$$

$$= \frac{L_n^{(\alpha+i)}(x) L_n^{(\alpha+j+1)}(y)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x}+\sqrt{-y})} \left[(-1)^{k+1} + \mathcal{O}_{x,y}(n^{-1/2}) \right],$$

where in the third equality we use Lemma 5 to guarantee that $\frac{\partial}{\partial y} \mathcal{O}_{x,y}(n^{-1}) = \mathcal{O}_{x,y}(n^{-1})$, and in the fourth equality, we use (20). □

4. Proof of Theorem 2 and Consequences

Proof of Theorem 2. Without loss of generality, we will consider the polynomials $L_n^{(\alpha)} = (-1)^n/n! \, L_n^{\alpha}$ and $\widehat{S}_n = (-1)^n/n! \, S_n$, instead of the monic polynomials L_n^{α} and S_n.

Multiplying both sides of (21) by $(-1)^n/n!$, we obtain

$$\widehat{S}_n(x) = L_n^{(\alpha)}(x) - \sum_{j=1}^{N} \lambda_j \widehat{S}_n^{(d_j)}(c_j) K_{n-1}^{(0,d_j)}(x, c_j), \tag{23}$$

Dividing by $L_n^{(\alpha)}(x)$ on both sides of (23), we obtain

$$\frac{\widehat{S}_n(x)}{L_n^{(\alpha)}(x)} = 1 - \sum_{j=1}^{N} \lambda_j \widehat{S}_n^{(d_j)}(c_j) \frac{K_{n-1}^{(0,d_j)}(x, c_j)}{L_n^{(\alpha)}(x)}. \tag{24}$$

Recall that we are considering the Laguerre–Sobolev polynomials $\{\widehat{S}_n\}$ that are orthogonal with respect to (5). In this case, the consistent linear system (22) becomes

$$\left(L_n^{(\alpha)}\right)^{(d_k)}(c_k) = \left(1 + \lambda_k K_{n-1}^{(d_k,d_k)}(c_k,c_k)\right)\widehat{S}_n^{(d_k)}(c_k) + \sum_{\substack{j=1 \\ j \neq k}}^{N} \lambda_j K_{n-1}^{(d_k,d_j)}(c_k,c_j)\widehat{S}_n^{(d_j)}(c_j),$$

(25)

for $k = 1, 2, \ldots, N$. Let us define

$$P_{n,j}^\alpha(x) := -\lambda_j \widehat{S}_n^{(d_j)}(c_j)\frac{K_{n-1}^{(0,d_j)}(x,c_j)}{L_n^{(\alpha)}(x)} \quad \text{and} \quad P_j^\alpha(x) := \lim_{n\to\infty} P_{n,j}^\alpha(x).$$

From (24), in order to prove the existence of the limit (6), we need to figure out the values of $P_j^\alpha(x)$. Note that

$$\widehat{S}_n^{(d_j)}(c_j) = -\frac{L_n^{(\alpha)}(x)P_{n,j}^{(\alpha)}(x)}{\lambda_j K_{n-1}^{(0,d_j)}(x,c_j)}.$$

If we replace these expressions in (25), then we obtain the following linear system in the unknowns $P_{n,j}(x)$

$$\begin{pmatrix} a_{1,1}(n,x) & a_{1,2}(n,x) & \cdots & a_{1,N}(n,x) \\ a_{2,1}(n,x) & a_{2,2}(n,x) & \cdots & a_{1,N}(n,x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1}(n,x) & a_{N,2}(n,x) & \cdots & a_{N,N}(n,x) \end{pmatrix}\begin{pmatrix} P_{n,1}^\alpha(x) \\ P_{n,2}^\alpha(x) \\ \vdots \\ P_{n,N}^\alpha(x) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix},$$

(26)

where

$$a_{k,j}(n,x) = \begin{cases} \dfrac{L_n^{(\alpha)}(x)K_{n-1}^{(d_k,d_j)}(c_k,c_j)}{\left(L_n^{(\alpha)}\right)^{(d_k)}(c_k)K_{n-1}^{(0,d_j)}(x,c_j)}, & j \neq k, \\[4mm] \dfrac{L_n^{(\alpha)}(x)\left(\frac{1}{\lambda_k} + K_{n-1}^{(d_k,d_k)}(c_k,c_k)\right)}{\left(L_n^{(\alpha)}\right)^{(d_k)}(c_k)K_{n-1}^{(0,d_k)}(x,c_k)}, & j = k. \end{cases}$$

Now, we will find the behavior of the coefficients $a_{k,j}(n,x)$ when n approaches infinity. If $k = j$, we have

$$a_{k,k}(n,x) = \frac{L_n^{(\alpha)}(x)\left(\frac{1}{\lambda_k} + K_{n-1}^{(d_k,d_k)}(c_k,c_k)\right)}{\left(L_n^{(\alpha)}\right)^{(d_k)}(c_k)K_{n-1}^{(0,d_k)}(x,c_k)}$$

$$= \frac{L_n^{(\alpha)}(x)\left(\frac{1}{\lambda_k} + \frac{L_n^{(\alpha+d_k)}(c_k)L_n^{(\alpha+d_k)}(c_k)}{n^{\alpha-\frac{1}{2}}\sqrt{-c_k}+\sqrt{-c_k}}\left((-1)^{d_k+d_k} + \mathcal{O}(n^{-1/2})\right)\right)}{(-1)^{d_k}L_{n-d_k}^{(\alpha+d_k)}(c_k)\frac{L_n^{(\alpha+d_k)}(c_k)L_n^{(\alpha)}(x)}{n^{\alpha-\frac{1}{2}}(\sqrt{-x}+\sqrt{-c_k})}\left((-1)^{d_k} + \mathcal{O}_x(n^{-1/2})\right)}$$

$$= \frac{\sqrt{-x}+\sqrt{-c_k}}{2\sqrt{-c_k}}\frac{\left(\frac{n^{\alpha-\frac{1}{2}}}{\lambda_k L_n^{(\alpha+d_k)}(c_k)} + L_n^{(\alpha+d_k)}(c_k)\left(1 + \mathcal{O}(n^{-1/2})\right)\right)}{L_{n-d_k}^{(\alpha+d_k)}(c_k)\left(1 + \mathcal{O}_x(n^{-1/2})\right)}$$

$$= \frac{\sqrt{-x}+\sqrt{-c_k}}{2\sqrt{-c_k}}\frac{\left(\frac{n^{\alpha-\frac{1}{2}}}{\lambda_k\left(L_n^{(\alpha+d_k)}(c_k)\right)^2} + 1 + \mathcal{O}(n^{-1/2})\right)}{1 + \mathcal{O}_x(n^{-1/2})}$$

$$= \frac{\sqrt{-x}+\sqrt{-c_k}}{2\sqrt{-c_k}}\frac{1 + \mathcal{O}(n^{-1/2})}{1 + \mathcal{O}_x(n^{-1/2})},$$

where in the last equality we use Perron's Asymptotic Formula (19) to obtain

$$\frac{n^{\alpha-\frac{1}{2}}}{(L_n^{(\alpha+d_k)}(c_k))^2} = \frac{4\pi n^{\alpha-\frac{1}{2}}}{e^{c_k+4\sqrt{-c_k}\sqrt{n}}} \frac{(-c_k)^{\alpha+d_k+\frac{1}{2}}}{n^{\alpha+d_k-\frac{1}{2}}}\mathcal{O}(1) = \frac{1}{n^{d_k}e^{4\sqrt{-c_k}\sqrt{n}}}\mathcal{O}(1),$$

which has exponential decay ($c_k < 0$). On the other hand, if $k \neq j$, we obtain

$$
\begin{aligned}
a_{k,j}(n,x) &= \frac{L_n^{(\alpha)}(x)K_{n-1}^{(d_k,d_j)}(c_k,c_j)}{\left(L_n^{(\alpha)}\right)^{(d_k)}(c_k)K_{n-1}^{(0,d_j)}(x,c_j)} \\
&= \frac{\frac{L_n^{(\alpha+d_k)}(c_k)}{\sqrt{-c_k}+\sqrt{-c_j}}\left((-1)^{d_k+d_j}+\mathcal{O}(n^{-1/2})\right)}{(-1)^{d_k}\frac{L_{n-d_k}^{(\alpha+d_k)}(c_k)}{\sqrt{-x}+\sqrt{-c_j}}\left((-1)^{d_j}+\mathcal{O}(n^{-1/2})\right)} \\
&= \frac{\sqrt{-x}+\sqrt{-c_j}}{\sqrt{-c_k}+\sqrt{-c_j}}\frac{\left(1+\mathcal{O}(n^{-1/2})\right)}{\left(1+\mathcal{O}(n^{-1/2})\right)}.
\end{aligned}
$$

Hence,

$$\lim_{n\to\infty} a_{k,j}(n,x) = \begin{cases} \frac{\sqrt{-x}+\sqrt{-c_j}}{\sqrt{-c_k}+\sqrt{-c_j}}, & \text{if } j \neq k \\ \frac{\sqrt{-x}+\sqrt{-c_k}}{2\sqrt{-c_k}}, & \text{if } j = k \end{cases} = \frac{\sqrt{-x}+\sqrt{|c_j|}}{\sqrt{|c_k|}+\sqrt{|c_j|}}.$$

Next, taking limits on both sides of (26) when n approaches ∞, we obtain

$$\begin{pmatrix} \frac{\sqrt{-x}+\sqrt{|c_1|}}{\sqrt{|c_1|}+\sqrt{|c_1|}} & \frac{\sqrt{-x}+\sqrt{|c_2|}}{\sqrt{|c_1|}+\sqrt{|c_2|}} & \cdots & \frac{\sqrt{-x}+\sqrt{|c_N|}}{\sqrt{|c_1|}+\sqrt{|c_N|}} \\ \frac{\sqrt{-x}+\sqrt{|c_1|}}{\sqrt{|c_2|}+\sqrt{|c_1|}} & \frac{\sqrt{-x}+\sqrt{|c_2|}}{\sqrt{|c_2|}+\sqrt{|c_2|}} & \cdots & \frac{\sqrt{-x}+\sqrt{|c_N|}}{\sqrt{|c_2|}+\sqrt{|c_N|}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{-x}+\sqrt{|c_1|}}{\sqrt{|c_N|}+\sqrt{|c_1|}} & \frac{\sqrt{-x}+\sqrt{|c_2|}}{\sqrt{|c_N|}+\sqrt{|c_2|}} & \cdots & \frac{\sqrt{-x}+\sqrt{|c_N|}}{\sqrt{|c_N|}+\sqrt{|c_N|}} \end{pmatrix}\begin{pmatrix} P_1^\alpha(x) \\ P_2^\alpha(x) \\ \vdots \\ P_N^\alpha(x) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

Using Cauchy determinants, it is not difficult to prove that the N solutions of the above linear system are

$$P_j^\alpha(x) = \frac{-2\sqrt{|c_j|}}{\sqrt{-x}+\sqrt{|c_j|}}\prod_{\substack{l=1 \\ l\neq j}}^N\left(\frac{\sqrt{|c_j|}+\sqrt{|c_l|}}{\sqrt{|c_j|}-\sqrt{|c_l|}}\right).$$

Now, from (24), we obtain

$$\lim_{n\to\infty}\frac{\widehat{S}_n(x)}{L_n^{(\alpha)}(x)} = 1 + \sum_{j=1}^N\frac{2\sqrt{|c_j|}}{\sqrt{-x}+\sqrt{|c_j|}}\prod_{\substack{l=1 \\ l\neq j}}^N\left(\frac{\sqrt{|c_j|}+\sqrt{|c_l|}}{\sqrt{|c_j|}-\sqrt{|c_l|}}\right).$$

If we consider the change of variable $z = \sqrt{-x}$ and for simplicity we also consider the notation $t_j = \sqrt{|c_j|}$, then we obtain the following partial fraction decomposition

$$1 + \sum_{j=1}^N\frac{2t_j}{z+t_j}\prod_{\substack{l=1 \\ l\neq j}}^N\left(\frac{t_j+t_l}{t_j-t_l}\right).$$

Thus, we only have to prove that this is the partial fraction decomposition of

$$\prod_{j=1}^{N}\left(\frac{z-t_j}{z+t_j}\right).$$

Let $P_N(z) = \prod_{j=1}^{N}(z-t_j)$ and $Q_N(z) = \prod_{j=1}^{N}(z+t_j)$, then

$$\prod_{j=1}^{N}\left(\frac{z-t_j}{z+t_j}\right) = \frac{P_N(z)}{Q_N(z)} = 1 + \frac{P_N(z)-Q_N(z)}{Q_N(z)} = 1 + \sum_{j=1}^{N}\frac{A_j}{z+t_j},$$

where

$$A_j = \lim_{z\to -t_j}(z+t_j)\frac{P_N(z)-Q_N(z)}{Q_N(z)} = \frac{P_N(-t_j)-Q_N(-t_j)}{Q'_N(-t_j)}$$

$$= \frac{\displaystyle\prod_{l=1}^{N}(-t_j-t_l)-\prod_{l=1}^{N}(-t_j+t_l)}{\displaystyle\prod_{\substack{l=1\\l\neq j}}^{N}(-t_j+t_l)} = \frac{(-1)^N 2t_j}{(-1)^N}\prod_{\substack{l=1\\l\neq j}}^{N}\left(\frac{t_j+t_l}{t_j-t_l}\right) = 2t_j\prod_{\substack{l=1\\l\neq j}}^{N}\left(\frac{t_j+t_l}{t_j-t_l}\right),$$

which completes the proof. □

Obviously, the inner product (5) and the monic polynomial S_n depend on the parameter $\alpha > -1$, so that in what follows, we will denote $S_n^\alpha = S_n$. Formula (6) allows us to obtain other asymptotic formulas for the polynomials S_n^α. Three of them are included in the following corollary.

Corollary 2. *Let $\alpha, \beta > -1$, $n \in \mathbb{Z}_+$ and $k \geq -n$. Under the hypotheses of Theorem 2, we obtain*

$$(1) \quad \frac{S_{n+k}^{\alpha+\beta}(z)}{n^{k+\beta/2}\,L_n^\alpha(z)} \rightrightarrows (-1)^k\left(\sqrt{-z}\right)^{-\beta}\prod_{j=1}^{N}\left(\frac{\sqrt{-x}-\sqrt{|c_j|}}{\sqrt{-x}+\sqrt{|c_j|}}\right), \qquad K \subset \overline{\mathbb{C}}\setminus\mathbb{R}_+. \quad (27)$$

$$(2) \quad \frac{S_{n+k}^{\alpha+\beta}(z)}{n^{k+\beta/2}\,S_n^\alpha(z)} \rightrightarrows (-1)^k\left(\sqrt{-z}\right)^{-\beta}, \qquad K \subset \overline{\mathbb{C}}\setminus\mathbb{R}_+. \quad (28)$$

$$(3) \quad \frac{\left(S_n^\alpha(z)\right)^{(\nu)}}{\left(L_n^\alpha(z)\right)^{(\nu)}} \rightrightarrows \prod_{j=1}^{N}\left(\frac{\sqrt{-x}-\sqrt{|c_j|}}{\sqrt{-x}+\sqrt{|c_j|}}\right), \qquad K \subset \overline{\mathbb{C}}\setminus\mathbb{R}_+. \quad (29)$$

Proof. Formulas (27) and (28) are direct consequences of Theorem 2 and Lemma 4.

The proof of (29) is by induction on ν. Of course, (6) is (29) for $\nu = 0$. Assume that (29) is true for $\nu = \kappa \geq 0$. Note that

$$\frac{\left(S_n^\alpha(z)\right)^{(\kappa+1)}}{\left(L_n^\alpha(z)\right)^{(\kappa+1)}} = \frac{\left(L_n^\alpha(z)\right)^{(\kappa)}}{\left(L_n^\alpha(z)\right)^{(\kappa+1)}}\left(\frac{\left(S_n^\alpha(z)\right)^{(\kappa)}}{\left(L_n^\alpha(z)\right)^{(\kappa)}}\right)' + \frac{\left(S_n^\alpha(z)\right)^{(\kappa)}}{\left(L_n^\alpha(z)\right)^{(\kappa)}}$$

From (16), (18) and Lemma 4

$$\frac{\left(L_n^\alpha(z)\right)^{(\kappa)}}{\left(L_n^\alpha(z)\right)^{(\kappa+1)}} = \frac{L_{n-\kappa}^{(\alpha+\kappa)}}{L_{n-\kappa-1}^{(\alpha+\kappa+1)}} \rightrightarrows 0, \qquad K \subset \overline{\mathbb{C}}\setminus\mathbb{R}_+.$$

Hence, from Theorem 2, we obtain (29) for $\nu = \kappa + 1$. □

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