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The Heat Equation on Submanifolds in Lie Groups and Random Motions on Spheres

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Abstract: We studied the random variable $V_t = \mathrm{vol}_{S^2}(g_t B \cap B)$, where B is a disc on the sphere S^2 centered at the north pole and $(g_t)_{t\geq 0}$ is the Brownian motion on the special orthogonal group $SO(3)$ starting at the identity. We applied the results of the theory of compact Lie groups to evaluate the expectation of V_t for $0 \leq t \leq \tau$, where τ is the first time when V_t vanishes. We obtained an integral formula using the heat equation on some Riemannian submanifold Γ_B seen as the support of the function $f(g) = \mathrm{vol}_{S^2}(gB \cap B)$ immersed in $SO(3)$. The integral formula depends on the mean curvature of Γ_B and the diameter of B.

Keywords: Brownian motion; Lie group; heat kernel; Riemannian manifold

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1. Introduction

We studied the behavior of the shape of a body under random transformations.The random motion of a particle on the unit sphere S^2 in $\mathbb{R}^3$ can be used to model the tracking of animals equipped with a transmitter, which has a range given by a disc B of a certain radius depending on the power of the signal issued by a radar. The most-common way to model an erratic motion at least for a sufficiently small body is the Brownian motion on the sphere S^2. The reason for that is the property that such processes have no memory, which means that the motion in the future only depends on the present, not on the past. There are at least two ways to simulate a Brownian motion on the sphere [1]. The most-natural one is to use the Brownian motion of the sphere S^2; its exact density is well-known and has been computed explicitly by Yosida [2]. Another way to simulate a Brownian motion on the sphere is by using the group action point of view. Indeed, we fix a point, say the north pole N, then choose a Brownian motion valued in the group of direct isometries of the sphere S^2, namely the group $SO(3)$. The required Markov process $X_t = r_t(N)$ will give rise to a random motion on the sphere, which differs from the Brownian motion on the sphere, which starts from N. The second point of view requires the exact density of the Brownian motion of $SO(3)$. Fortunately, this theory is well-developed now and can be recast in the Fourier theory of compact Lie groups using unitary representations and Peter–Weyl decomposition. This point of view has been used by M. Liao in order to deduce the stochastic property of the random motion of a rigid body subject to white noise perturbation [3]. It is possible to use Levy processes instead of the Brownian motion, but those have points of discontinuity, while we are considering continuous motions. This was recently performed by S. Albeverio and M. Gordina for matrix Lie groups such asthe special linear group and the Heisenberg group [4]. In our case, we deal with the compact Lie groups for which the complete picture is completely understood using unitary representations and their characters.

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Set Up and Main Result

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\omega \in \Omega \mapsto (r_t(\omega))_{t\geq 0}$ be a continuous Brownian motion starting at identity and valued in the group $SO(3)$ of rotations in $\mathbb{R}^3$. Here, $SO(3)$ is seen as a compact Lie group acting on S^2, the unit sphere in $\mathbb{R}^3$, and the action is transitive. We equip S^2 with a volume area measure denoted as $volS^2$, which has density $f(\theta, \varphi) = \sin\phi d\varphi d\theta$ with respect to the Lebesgue measure on $[0, 2\pi) \times [-\pi/2, \pi/2]$. This measure is invariant under the action of $SO(3)$. We studied the following real-valued continuous stochastic process:

$$V_t(\omega) := vol_{S^2}(r_t(\omega)B \cap B)$$

where B is a Borel subset of S^2 with $vol_{S^2}(B) > 0$. In particular, $V_0(\omega) = vol_{S^2}(B)$ since $r_0(\omega) = I_3$.

For each $g \in SO(3)$, let us consider the function on $SO(3)$ corresponding to $(V_t)_{t\geq 0}$ given by

$$f(g) = vol_{S^2}(gB \cap B).$$

Thus, the random process $(V_t)_{t\geq 0}$ is just the image of the Brownian process $(g_t)_{t\geq 0}$ under the map $f : SO(3) \to \mathbb{R}_{\geq 0}$. We are more particularly interested in the Brownian motion (g_t) valued in $SO(3)$, but stopped at the boundary of the support of f. Namely, if $\tau = \inf\{t > 0 : vol_{S^2}(g_t B \cap B) = 0\}$ is the corresponding stopping time, then $(g_t)_{t\geq 0}$ will be the Brownian motion valued in $SO(3)$, which starts at identity and stops at time τ. The unit sphere S^2 can be equipped with the spherical distance given by

$$d_{S^2}(x, y) = \arccos(\langle x, y\rangle)$$

where $\langle x, y\rangle$ is just the Euclidean inner product in $\mathbb{R}^3$. Until the end, we assume that B is the spherical disc with the north pole $N = (0, 0, 1)$ as its center and with diameter $\mathrm{diam}(B)$. Using the property of $(g_{t\wedge\tau})_{t\geq 0}$, we are able to prove a closed formula for the expectation of $(V_t)_{t\geq 0}$.

Theorem 1. *Let B be the spherical disc with the north pole $N = (0, 0, 1)$ as its center in S^2, and let $(g_t)_{t\geq 0}$ be a Brownian motion on $SO(3)$, which starts at the identity and stops at $\tau = \inf\{\mathrm{t} > 0 : \mathrm{vol}_{S^2}(g_t B \cap B) = 0\}$. Then, the expectation of $V_{t\wedge\tau} = \mathrm{vol}_{S^2}(g_{t\wedge\tau} B \cap B)$ is given by*

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_0^\pi \mathcal{J}(t, \theta) e^{L_t(\theta)} \sin^2(\theta/2) d\theta$$

where, for each $0 \leq t \leq \tau$, $\mathcal{J}(t, \theta) = J_0 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta) J_n$ with

$$J_n = \int_0^{\mathrm{diam}(B)} f(\beta) \chi_n(\beta) \sin^2(\beta/2)\, d\beta$$

and $\chi_n(u) = \dfrac{\sin((2n+1)u/2)}{\sin(u/2)}$ for all $n \geq 0$ and where L_t is a function that depends on the mean curvature of the support of f.

2. Motivation and Literature Review

The Brownian motion is the most-natural way to encode a random motion. It has all the properties that make it the most-unpredictable behavior possible, and it is the most-suitable candidate to model molecular rotations in fluids (FPL model). The probability density function of a Brownian motion satisfies the heat equation. We are interested in the rotational Brownian motion, that is the Brownian motion on the sphere. This kind of random process has been well-studied in the past, and it is still an active area of research. For instance, let us mention the work of Furry [5], Favro [6], Ivanov [7], and Hubbard [8].

For nice surveys on rotational Brownian motions, we invite the reader to read the survey of Valiev and Ivanov [9] and McClung [10] for the rotational Fokker–Planck equation. The problem we are interested in is geometrical. Given a subset B on the two-dimensional sphere, say a cloud, one can use Brownian rotation to move the cloud B. The question is to give the expectation of the volume of the intersection of the cloud with its translation. To treat this question, we need to introduce the Brownian motion on the Lie group $SO(3)$ corresponding to the group of positively oriented rotations on the sphere; this is the aim of Section 3. The point of view taken is to treat a Lie group as a Riemannian manifold; for such a class of spaces, the Brownian motion was studied for instance by Graham [11], van Kampen [12], and Risken [13]. For the general theory of the Brownian motion on manifolds, we refer to the classical book of Elworthy ([14]). The study of such stochastic models has many applications in physics. Let us mention the work of Castro-Villarreal et al. [15,16], Novikov et al. [17], Gómez et al. [18] and Yang-Li [19].

3. The Heat Kernel in $SO(3)$

In this section, we review the spectral theory of the Laplace operator of $SO(3)$ within the theory of compact Lie groups (see, e.g., [20–23]).

3.1. The Lie Group $SO(3)$

The group of isometries of the sphere S^2 is the group of all the space transformations g such that $\langle gx, gy\rangle_{S^2} = \langle x, y\rangle_{S^2}$ for any $x, y \in S^2$. Using duality, such isometries have to satisfy the relation $g^t g = 1$. The group of all such transformations is denoted $O(3)$ and is called the orthogonal group in three dimensions. The orthogonality relation $g^t g = 1$ implies that $\det g = \pm 1$. The elements of $O(3)$ such that $\det g = 1$ preserve the orientation (i.e., act with the positive Jacobian) and form what we call the special orthogonal group given by

$$SO(3) = \{g \in \mathrm{SL}_3(\mathbb{R}) \mid g^t g = I_3\}.$$

The group $SO(3)$ is a maximal Lie compact subgroup of $\mathrm{SL}_3(\mathbb{R})$; in particular, it has a Lie group structure. The Lie algebra of $SO(3)$, namely the tangent space at $g = I_3$, is given by

$$\mathfrak{so}(3) = \{X \in \mathcal{M}_3(\mathbb{R}) \mid X^t = -X\}$$

which consists of skew-symmetric matrices. A basis of $\mathfrak{so}(3)$ is given by the following three matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be seen that $\mathfrak{so}(3)$ is closed under the Lie bracket by noting the following commutation relations:

$$[X_1, X_2] = X_3 \quad [X_2, X_3] = X_1 \quad [X_3, X_1] = X_2.$$

A Lie group closely related to $SO(3)$ is the group $SU(2)$ of unitary matrices of size two:

$$SU(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \;:\; z_1, z_2 \in \mathbb{C},\ |z_1|^2 + |z_2|^2 = 1 \right\}.$$

The Lie algebra of $SU(2)$ is denoted $\mathfrak{su}(2)$, and it is generated by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfy the commutation relations $[\sigma_1, \sigma_2] = 2\sigma_3$, $[\sigma_2, \sigma_3] = 2\sigma_3$, and $[\sigma_3, \sigma_1] = 2\sigma_2$. The $SU(2)$ group is homeomorphic to the unit sphere in $\mathbb{C}^2$. As a consequence, $SU(2)$ is simply connectedand compact. The group $SO(3)$ is not simply connected; its universal covering is given by $SU(2)$. More precisely, $SO(2)$ has a two-sheet universal covering realized by the adjoint representation of $SU(2)$:

$$\mathrm{Ad} : SU(2) \to SO(3)$$

given by $Ad(X)g = X^{-1}gX$. The kernel of this map is given by the center of $SU(2)$, which is $\{\pm I_2\}$. It is less trivial to find the image of this map, and it can be proven that it is surjective. The latter fact can be checked by working on the Lie algebra level. Indeed, the differential of the adjoint map at the identity is given by $ad : \mathfrak{su}(2) \to \mathfrak{so}(3)$, $X \mapsto adX = [X, .]$. In particular,

$$SO(3) \simeq SU(2)/\{\pm I_2\}.$$

3.2. Euler Parametrization and Haar Measure on $SO(3)$

For our purposes, we need a precise description of the group $SO(3)$ in terms of the Euler angles. This will give a well-suited parametrization of the elements of the group in order to perform the analysis. The group $SO(3)$ is a compact Lie group, which is given by

$$SO(3) = \{g \in \mathrm{SL}n(3) \mid g^t g = I_3\}.$$

The tangent space of G at some $g \in G$ is just the set of matrices of the form gX, where X is some element in $\mathfrak{so}(3)$. The exponential map $\exp : \mathfrak{so}(3) \to SO(3)$ is surjective. We use the polar coordinate for an element X of the Lie algebra $\mathfrak{so}(3)$; indeed, such an X can be written as $X = \theta T(u, v, w)$ with $(u, v, w) \in S^2$ and where

$$T(u, v, w) = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}.$$

Any element of $g \in SO(3)$ can be written in the form:

$$g = e^{\psi X_3} e^{\theta X_1} e^{\phi X_3} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The normalized Haar measure is, therefore, given by (see [20])

$$\mu_{SO(3)}(d\theta, d\psi, d\phi) = \frac{2}{\pi^2} \sin^2(\frac{\theta}{2}) \sin\psi d\psi d\phi. \quad (2)$$

3.3. Brownian Motion on a Riemannian Manifold

For an introduction to Brownian motion on manifolds, we refer to the book of Elworthy [14]. There are several ways to construct the Brownian motion on a Lie group G. An elegant one consists of defining the density function of the Brownian motion as the solution of the heat equation on G. In fact, only the underlying structure of the Riemannian manifold on G is needed. Let us assume more generally that we are given an n-dimensional Riemannian (M, h), where the metric h is a symmetric bilinear form on the tangent bundle h. Given local coordinates $(x_1, \ldots, x_n)$ of a point $x \in M$ with a local frame $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, which forms a basis of $T_x(M)$, the metric is then locally determined by its coefficients $h_{ij} = h(\partial/\partial x_i, \partial\, \partial x_j)$ giving the length element:

$$ds^2 = \sum_{i,j} h_{ij} dx_i \otimes dx_j.$$

Let $\mathcal{C}(TM)$ denote the space of smooth sections of the tangent bundle, then one can define a covariant derivative using a connection $\nabla : \mathcal{C}(TM) \times \mathcal{C}(TM) \to \mathcal{C}(TM)$ depending on the metric h. This connection assigns to a pair of vectors fields $X, Y \in \mathcal{C}(TM)$ the vector field

$\nabla_X Y$, which may be seen as the derivative of Y. The torsion associated with the connection is the quantity defined by

$$T'(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{3}$$

In local coordinates, the connection is essentially characterized by its values on the basis $(\partial_{x_1}, \ldots, \partial_{x_n})$ of TM:

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{1 \leqslant k \leqslant n} \Gamma_{ij}^k \partial_{x_k}$$

where the coefficients Γ_{ij}^k of the connection, known as the Christoffel symbols, can be computed explicitly by using the first derivatives of the metric components h_{ij}. The gradient of a smooth function f associated with the metric h is defined by the relation $h(X, \mathrm{grad} f) = df(X)$ for any $X \in \mathcal{C}(TM)$. The central role in the theory of heat diffusion is played by the Laplace–Beltrami operator on (M,h), which is defined in local coordinates by

$$\Delta_{M,h} = \frac{1}{\sqrt{h}} \sum_{i,j} \frac{\partial}{\partial x_i} \left[\sqrt{h} h^{ij} \frac{\partial}{\partial x_j} \right]$$

where h and h^{ij} are, respectively, the determinant and the inverse of the coordinates of the metric tensor (h_{ij}) in the local chart. With the Laplace–Beltrami operator one can associate the heat equation on M with an initial condition f:

$$\begin{cases} \frac{1}{2}\Delta_M u(t,x) + \partial_t u(t,x) = 0 \\ \\ u(0,x) = f(x) \qquad \text{on} \quad M. \end{cases} \tag{4}$$

In the compact case, which is our main concern, this equation always has a smooth solution denoted by $p_t(x)$. The Brownian motion $(X_t)_{t \geq 0}$ on M is just a Feller process, which has a transition operator of the form:

$$P_t f(x) = \mathbb{E}[f(X_t) | X_0 = x]$$

for any f continuous with compact support on M. The kernel associated with this operator is given by $p_t(x,y)$; this quantity is the probability that the Brownian is at y at time t conditioned on the fact that it started at x. It satisfies the relation:

$$P_t f(x) = \mathbb{E}[f(X_t) | X_0 = x] = \int_M p_t(x,y) f(y) dy.$$

3.4. The Density Probability of the Brownian Motion in $SO(3)$

The notion of Brownian motion on a compact Lie group will be directly derived from the setting of the previous section, in that a Lie group has a structure of the Riemannian manifold. The aim is to find an explicit formulation of the solution of the heat equation in $SO(3)$. Before, we need to find the expression of the Laplace operator. The fact that the Brownian motion in a compact Lie group can be constructed from a solution of the heat kernel was developed by K. Ito [24]. In this case, one can do much better than proving the existence; indeed, using Fourier analysis on $SO(3)$, it is possible to give an explicit formula for the density $(p_t)_{t>0}$. Let us first recall that the Lie algebra of $SO(3)$ is generated by three matrices X_1, X_2, and X_3, which give rise to the three corresponding differential operators $\widetilde{X_i}$ $(i = 1,2,3)$, which act on the set of functions on $SO(3)$ via the rule:

$$(\widetilde{X_i}.f)(x) = \frac{d}{ds} f(e^{sX_i} x)_{|s=0} \quad i = 1,2,3.$$

The kth iteration of an operator X is just written $\widetilde{X}^k$. The Levi-Civita connection on $SO(3)$ is given by

$$\nabla_X Y = \frac{1}{4}[X, Y]$$

for any two vector fields X, Y [25]. This defines a Riemannian metric on $SO(3)$, which is just the identity. This gives the simple expression for the Laplace operator:

$$\Delta_{SO(3)} = \widetilde{X_1}^2 + \widetilde{X_2}^2 + \widetilde{X_3}^2.$$

The density of the Brownian motion on G starting at identity is given by the solution in $L^2(G) \cap \mathcal{C}_c^2(G)$ of the heat equation with initial data in L^2:

$$\begin{cases} \frac{1}{2}\Delta_{SO(3)} u(t, x) = -\partial_t u(t, x) \\ \\ u(id_G, x) = f(x) \quad \text{ on } \ G. \end{cases} \tag{5}$$

3.5. Root Decomposition of the Lie Algebra of a Compact Lie Group

The explicit description of a solution to the problem (5) is well understood in the setting of compact Lie groups, which consists of a vast generalization of the L^2-theory of n-dimensional tori $\mathbb{R}^n/2\pi\mathbb{Z}^n$. Let us be given a compact Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G over the complex numbers and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. In particular, since $\mathfrak{h}$ is abelian, the operators $ad(H) = [H, .]$ commute with each other for all $h \in \mathfrak{h}$. A general fact from linear algebra implies that all the operators $ad(H)$ $(H \in \mathfrak{h})$ are diagonalizable over the same basis. Thus, for any $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, there exists a not necessarily real eigenvalue $\alpha(H)$ such that

$$ad(H)X = [H, X] = \alpha(H)X.$$

This gives rises to a well-defined map $\alpha : \mathfrak{h} \to \mathbb{C}$, which is linear and, thus, can be seen as an element of the dual of $\mathfrak{h}$. The set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the set of all $\alpha \in \mathfrak{h}^*$ coming this way. We denote by $\mathcal{R}$ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. One can define a definite negative bilinear on $\mathfrak{g} \times \mathfrak{g}$, by the following rule $B(X, Y) = \mathrm{tr}(ad_X \circ ad_X)$. For each root $\alpha \in \mathcal{R}$ and any $H \in \mathfrak{h}$, there exists a unique $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = B(H, H_\alpha)$. Let us set $\mathfrak{h}_0 = \oplus_{\alpha \in \mathcal{R}} \mathbb{Q} H_\alpha$ as the $\mathbb{Q}$-span of H_α $(\alpha \in \mathfrak{h})$. One can define a positive definite inner product on $\mathfrak{h}_0^*$, by the rule:

$$(\alpha, \beta) = B(H_\alpha, H_\beta) \text{ for every } \alpha, \beta \in \mathcal{R}.$$

If we fix a set $H_1, \ldots, H_s$ that spans $\mathfrak{h}_0$, we say that an element α of $\mathfrak{h}_0$ is positive if there exists an integer $1 \leqslant j \leqslant s$ such that $\alpha(H_1) = \ldots, \alpha(H_{j-1}) = 0$ and $\alpha(H_j) > 0$. We denote $\alpha > 0$, and we denote by $\mathcal{R}^+$ the set of positive roots. If $\alpha \in \mathcal{R}$, then $-\alpha \in \mathcal{R}$. We have the following decomposition into eigenspaces:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-.$$

where $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) is the direct sum $\oplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}\alpha$ (respectively, $\oplus\alpha \in \mathcal{R}^+ \mathfrak{g}\alpha$). For any given irreducible representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, there exists a nonzero vector $v \in V$ and $\Lambda \in \mathfrak{h}_0^*$ such that $\rho(H)v = \Lambda(H)v$ and $\dfrac{2(\Lambda, \lambda)}{(\lambda, \lambda)}$ is a nonnegative integer for each $\lambda \in \mathcal{R}^+$. The vector v is called the highest weight vector, and Λ is the highest weight of the representation ρ. Actually, the highest weights of an irreducible representation characterize completely the equivalence class of an irreducible representation of $\mathfrak{g}$. For connected compact Lie groups, Abelian subgroups are just tori in the usual sense, and Cartan subgroups are replaced by the notion of maximal tori. In particular, this opens the way to the the generalization of Fourier analysis to compact Lie groups. In tori, the key role is played by irreducible characters, which are traces of the irreducible representations rather than the highest weights. The reason behind this is that maximal tori are conjugate in G, i.e., they form a unique orbit under the conjugation action. Thus, the trace of a representation restricted to any maximal torus is constant on the conjugacy class.

Since any element of G is contained in a maximal torus, central functions on G are completely characterized by their restriction to a maximal torus. In particular, this applies to the characters of irreducible representations. A character is characterized by the sets of highest weight of a maximal torus, that is for each highest weight λ, one has a corresponding irreducible character χ_λ of G.

3.6. Computation of the Characters

Since all maximal tori are conjugate under G, the action of G on the set of all maximal tori is transitive. If we fix a representative torus T for this action, the Weyl group is by definition $W(T) = N(T)/T$, where $N(T) = \{g \in G : g^{-1}Tg = T\}$ is the normalizer of T. Concretely, the elements of the Weyl group are generated by a finite set of reflections with respect to the hyperplanes $F_\alpha = \{\beta \in \mathcal{R} : (\alpha, \beta) = 0\}$. The set $C_\alpha = \{\beta \in \mathcal{R} : (\alpha, \beta) > 0\}$ is called the Weyl chamber associated with the root α. An important fact is that the Weyl group permutes Weyl chambers. For each $w \in W$, let us denote by N_w the number of reflections in the decomposition of w. The irreducible character corresponding to the highest weight λ evaluated for $H \in \mathrm{Lie}(T)$ is as follows:

$$\chi_\lambda(e^H) = \frac{\sum_{w \in W} (-1)^{N_w} e^{iw(\lambda+\rho)H}}{\sum_{w \in W} (-1)^{N_w} e^{iw\rho H}}$$

where ρ is the half sum of the positive roots. The dimension of the corresponding irreducible representation is given by

$$d_\lambda = \frac{\prod_{\alpha \in \mathcal{R}^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \mathcal{R}^+} (\rho, \alpha)}.$$

For each highest root λ and $g \in G$, one has

$$\Delta_G \chi_\lambda(g) = c(\lambda) \chi_\lambda(g) \tag{6}$$

with the corresponding eigenvalues being

$$c(\lambda) = (\lambda + \rho, \lambda + \rho) - (\lambda, \lambda). \tag{7}$$

3.7. Solution of the Heat Equation for Compact Lie Groups

We solve Equation (5) for an initial data f, which is a trace class function in $L^2(G)$, that is $f(hgh^{-1})$ for any $g, h \in G$. Under this assumption, the Peter–Weyl theorem gives us the Fourier expansion of f, which takes the following nice form

$$f(g) = \sum_{\lambda \in \Lambda^+} \sqrt{d_\lambda} \chi_\lambda(g)$$

where the equality is to be considered in the L^2 sense. Now, we set the following map $p_t : G \times G \to \mathbb{R}$ for each $t > 0$.

$$p_t^G(k, g) = \sum_{\lambda \in \Lambda_+} \frac{1}{\sqrt{d_\lambda}} \chi_\lambda(k) \chi_\lambda(g) e^{-tc(\lambda)} \quad (k, g \in G). \tag{8}$$

We claim that this function is a solution of (5). Indeed,

$$\Delta_G p_t^G(k, g) = \sum_{\lambda \in \Lambda_+} \frac{1}{\sqrt{d_\lambda}} \chi_\lambda(k) \Delta_G(\chi_\lambda(g)) e^{-tc(\lambda)}.$$

Using (6), one can deduce that

$$\Delta_G p_t^G(k,g) = \sum_{\lambda\in\Lambda_+} \frac{1}{\sqrt{d_\lambda}} \chi_\lambda(k)\chi_\lambda(g)c(\lambda)e^{-tc(\lambda)} = -\frac{\partial}{\partial t}[p_t^G(k,g)].$$

The solution is called the *heat kernel* of the compact Lie group G. We also call the heat kernel the one variable function $p_t(g) = p_t(I_3, g)$, the only ambiguity being that kernels in operators theory are defined on the product of the space with itself; indeed, the kernel defines an operator of $L^2(G)$ called the heat operator:

$$P_t f(k) = \int_G p_t(k,g) f(g) dg.$$

A notation of common use for $p_t(I_3, g)$ is $p_t(g)$, which is also called the heat kernel, and we employ both terms with no risk of confusion.

3.8. Solution of the Heat Equation for SO(3)

Now, we are able to give the explicit form of the heat kernel for $G = SO(3)$. In most of the presentations in the literature, it is always derived from the case $G = SU(2)$, for which the situation is much clearer due to the fact that it is simply connected. Here, we follow the presentation of M. Liao (Liao gives the formula for Levy processes, and it is easy to deduce the Brownian case, which corresponds to continuous Levy trajectories, which have a null Levy measure and the infinitesimal generator L, being the half of the Laplacian of G.) (Example 4.20 [23]). We recall that this construction is only valid if f is a conjugate invariant, which is the case for us. The equivalence classes of irreducible unitary representations of $SO(3)$ are indexed by the set of nonnegative integers $\{n = 0, 1, 2, 3, \ldots\}$, and the corresponding characters are trace class functions depending only the conjugacy class of a rotation depending only on an angle θ and given by

$$\chi_n(g) = \chi_n(\theta) = \frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}.$$

The expanded form of the heat kernel of $SO(3)$ is given by

$$p_t^{\mathrm{SO}(3)}(g) = p_t^{\mathrm{SO}(3)}(\theta) = 1 + \sum_{n\geq 1}(2n+1)e^{-atn(n+1)}\frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}. \quad (9)$$

with the corresponding kernel given by

$$p_t^{\mathrm{SO}(3)}(h,g) = p_t^{\mathrm{SO}(3)}(\beta,\theta) = 1 + \sum_{n\geq 1}(2n+1)e^{-n(n+1)t/2}\chi_n(\theta)\chi_n(\beta). \quad (10)$$

For a such that the infinitesimal generator is $L = a\Delta_G$, for the Brownian motion, we took $a = 1/2$. Thus, the density distribution of the Brownian motion on G is

$$p_t^{\mathrm{SO}(3)}(\theta) = 1 + \sum_{n\geq 1}(2n+1)e^{-n(n+1)t/2}\frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)}. \quad (11)$$

The action of the heat operator relative to $L = \frac{1}{2}\Delta$ on the space of the L^2-integrable function of G are conjugate invariant. Thus, using (2), it takes the following form:

$$P_t^G f(I_3) = \int_{\mathrm{SO}(3)} p_t^{\mathrm{SO}(3)}(g) f(g)\mu_{\mathrm{SO}(3)}(dg) = \frac{2}{\pi}\int_0^\pi p_t^{\mathrm{SO}(3)}(\theta) f(\theta)\sin^2(\theta/2)\, d\theta. \quad (12)$$

In other terms, this means that if $(g_t)_{t\geq 0}$ is a Brownian motion on SO(3) starting at the identity, which is conjugate invariant, we have

$$\mathbb{E}[f(g_t)] = \frac{2}{\pi}\int_0^{\pi} p_t^{\mathrm{SO}(3)}(\theta) f(\theta)\sin^2(\theta/2)\, d\theta. \tag{13}$$

4. The Brownian Motion on the Support of f

4.1. The Support of f

We introduce the support of f, which for us will be the following set:

$$\Gamma_B = \mathrm{supp} f = \{g \in SO(3) | f(g) = vol_{S^2}(gB \cap B) \geq 0\}.$$

The function f vanishes as soon as $d_{S^2}(gN, N) \geqslant \mathrm{diam}(B)$, which amounts to saying that f is supported by those g such that $\arccos\langle gN, N\rangle < \mathrm{diam}(B)$. Noting that cos is decreasing in the interval $[0, \pi]$, the latter condition is equivalent to $\langle gN, N\rangle > \cos\mathrm{diam}(B)$. The support of f is

$$\Gamma_B = \{g \in SO(3) | \langle gN, N\rangle \geqslant \cos\mathrm{diam}(B)\}.$$

The support of f, namely Γ_B, is then a closed subset of $SO(3)$, but not a Lie subgroup. The boundary of Γ_B is denoted Σ_B and is simply given by

$$\Sigma_B = \{g \in SO(3) | \langle gN, N\rangle = \cos\mathrm{diam}(B)\}.$$

The subset Σ_B can be seen as a smooth hypersurface of $SO(3)$ of equation $\theta(g) = \langle gN, N\rangle = \cos\mathrm{diam}(B)$. Reminding that N is the north pole, we readily obtain that $\theta(g) = g_{33}$. Thus,

$$\Sigma_B = \{g \in SO(3) | g_{33} = \cos\mathrm{diam}(B)\}.$$

Using the Euler parametrization of the rotations $(\theta(g), \varphi(g), \psi(g))$, we know that

$$g_{33} = \cos\theta(g).$$

This shows that the boundary of the support of f is then given by

$$\Sigma_B = \{g \in SO(3) | \theta(g) = \mathrm{diam}(B)\}.$$

4.2. The Support Γ Seen as Submanifold Embedded in $SO(3)$

There are several ways to construct a Brownian process on a Lie group viewed as a Riemannian manifold. The more suitable way in our case is to introduce the density of such a process, which is given by the solution of the heat equation on the support of f viewed as a Riemannian manifold. Indeed, the support Γ_B can be endowed with a structure of the Riemannian submanifold embedded in G with the induced metric of $SO(3)$. In particular, from this induced metric, one is able to extract the Laplace–Beltrami operator Δ_{Γ_B}. The reason we are interested in this operator is that it encodes the property of the Brownian motion killed outside Γ_B, in that the density of a Brownian process on such a submanifold is the solution of the heat operator associated with Γ_B. In other words, one can say that the infinitesimal generator of $(g_t)_{t\geq 0}$ stopped outside the support of f is just $\frac{1}{2}\Delta\Gamma_B$.

Tangent Space of the Submanifold Γ_B.

The set Γ_B is the set of all $g \in G$ such that

$$\theta(g) = \langle gN, N\rangle \leqslant \cos\mathrm{diam}(B).$$

Since we are going to work locally, it is more suitable to look at Γ_B as a union of level sets of ϕ, namely

$$\Gamma_B = \bigcup_{0 \leqslant \gamma \leqslant \cos \operatorname{diam}(B)} \{g \in G | \theta(g) = \gamma\}.$$

For each $\gamma \in [0, \cos \operatorname{diam}(B)]$, we first need to show that the level set:

$$\Gamma_B[\gamma] := \theta^{-1}(\gamma) = \{g \in G | \theta(g) = \gamma\}$$

is a smooth submanifold in G. A sufficient condition is that θ is a submersion, i.e., the differential is surjective at any point (see [26]). We check this fact in the following lemma.

Lemma 1. *The map $\theta : G \to \mathbb{R}$ is submersion, in particular the level sets of θ are smooth immersed submanifolds of G.*

Proof. Let us compute the differential of θ at a point $g \in \Gamma_f$ in the direction given by a vector field $X \in T_g(G)$. This is given by

$$(d\theta)_g X = (\widetilde{X}.\theta)(g) = \frac{d}{dt}\theta(e^{tX}g)|t = 0.$$

Thus,

$$(d\theta)_g X = \lim t \to 0 \frac{\theta(e^{tX}g) - \theta(g)}{t} = \lim_{t \to 0} \langle \frac{(e^{tX} - I_3)}{t} gN, N\rangle.$$

One has,

$$\lim_{t \to 0} \frac{e^{tX} - I_3}{t} = \lim_{t \to 0} \frac{1}{t}\left(tX + \frac{(tX)^2}{2} + \frac{(tX)^3}{3!} + \ldots\right) = X.$$

Therefore, we obtain

$$(d\theta)_g X = \langle XgN, N\rangle = \theta(Xg).g_{33}.$$

The kernel is given by

$$\operatorname{Ker}(d\theta)_g = \{X \in Tg(G)) \mid \theta(Xg) = 0\}$$

$$= \{X \in T_g(G)) \mid \langle XgN, N\rangle = 0\}$$

$$= \{X \in T_g(G)) \mid (Xg)_{33} = 0\}.$$

Thus,

$$\operatorname{Ker}(d\phi)_g = \{X \in Tg(G) \ \mid X_{31}g_{13} + X_{32}g_{23} + X_{33}g_{33} = 0\}.$$

This $\operatorname{Ker}(d\theta)_g$ is a hyperplane of $Tg(G)$ being of codimension one as the kernel of a linear form on $T_g(G)$. In particular,

$$\operatorname{rank}(d\theta)_g = \dim T_g(G) - \dim \operatorname{Ker}(d\theta)_g = 1.$$

Hence, for every $g \in G$, $(d\theta)_g$ is surjective since it is real-valued. □

Proposition 1. *There exists a vector field Z such that, for every $g \in G$,*

$$T_g(G) = T_g(\Gamma) \oplus \mathbb{R}Z_3.$$

In particular, Γ_B is a smooth hypersurface in G with the normal direction given by Z_3.

Proof. Since

$$\theta(g) = \langle gN, N\rangle = \langle gN, N\rangle$$

the tangent space of Γ_B is given by

$$T_g(\Gamma_B) = \{X \in T_g(G) \mid (Xg)_{33} = 0\}.$$

A vector field $X \in T_g(G$ is then in $T_g(\Gamma_B)$ if $X = Yg$ for some $Y \in \mathfrak{g}$ and, thus, if $(Xg)_{33} = (g^2 Y)_{33} = 0$. The later condition gives

$$(g^2)_{31} Y_{13} + (g^2)_{32} Y_{23} + (g^2)_{33} Y_{33} = 0.$$

Since $Y \in \mathfrak{so}(3)$, it is skew-symmetric, hence of the form:

$$Y = \begin{pmatrix} 0 & -Y_{21} & -Y_{31} \\ Y_{21} & 0 & -Y_{32} \\ Y_{31} & Y_{32} & 0 \end{pmatrix}$$

In particular, $Y_{33} = 0$, and therefore, one has the equation

$$(g^2)_{31} Y_{31} + (g^2)_{32} Y_{32} = 0.$$

This gives the relation $Y_{31} = -\rho(g) Y_{32}$, where

$$\rho(g) = \frac{(g^2)_{32}}{(g^2)_{31}} = \frac{g_{31}g_{12} + g_{32}g_{22} + g_{33}g_{32}}{g_{31}g_{11} + g_{32}g_{21} + g_{33}g_{31}}.$$

By recasting in Y, one obtains

$$Y = \begin{pmatrix} 0 & -Y_{21} & \rho(g)Y_{32} \\ Y_{21} & 0 & -Y_{32} \\ -\rho(g)Y_{32} & Y_{32} & 0 \end{pmatrix} = Y_{21} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + Y_{32} \begin{pmatrix} 0 & 0 & \rho(g) \\ 0 & 0 & -1 \\ -\rho(g) & 1 & 0 \end{pmatrix}.$$

Thus, we obtain that the elements X of the tangent space $T_g(\Gamma_B)$ are of the form:

$$X = \mathbb{R}Z_1 + \mathbb{R}Z_2$$

where Z_1 and Z_2 are two vector fields given by $Z_1(g) = gX_3$ and $Z_2(g) = gX_1 + \rho(g) gX_2$; here, $(X_i)_{1 \leqslant i \leqslant 3}$ is the basis of $\mathfrak{g} = \mathfrak{so}(3)$ given in §2.1. Hence, the tangent space at the point g of the submanifold ΓB is given by

$$T_g(\Gamma_B) = \operatorname{span}\{Z_1(g), Z_2(g)\}.$$

The normal bundle $\mathcal{N}(\Gamma_B)$ is the orthogonal complement of the tangent bundle in $T(G)$ with respect to the Killing inner product given by $\beta(X, Y) = \operatorname{Tr}(XY)$

$$T(G) = T(\Gamma_B) \oplus \mathcal{N}(\Gamma_B).$$

We already know that, for every $g \in \Gamma_B$, $\dim \mathcal{N} g(\Gamma B) = 1$. Its generator Z_3 satisfies the two conditions:

$$\operatorname{Tr}(Z_3 Z_1) = 0 \text{ and } \operatorname{Tr}(Z_3 Z_2) = 0.$$

□

4.3. Laplace–Beltrami Operator on Γ_B.

As remarked earlier, the support of f, Γ_B, is not a Lie group, but only a submanifold of G. For this reason, we cannot write the Laplace operator of Γ_B as squares of differential operators afforded to the basis of $\mathfrak{so}(3)$. The structure of the Riemannian manifold on Γ_B allows overcoming this issue. Indeed, there is a canonical way to obtain an expression of the Laplace–Beltrami operator of a submanifold as a function of the Laplace operator of the underlying manifold and the coefficients of the second fundamental form of Γ_B embedded in G.

There is a useful formula that allows expressing the Laplace operator of a submanifold as function of the following.

Lemma 2. *Let us assume that we have an n-dimensional Riemannian manifold M and a k-dimensional Riemannian submanifold N immersed in M. Let us denote by ∇_M, Δ_M (respectively, ∇_N, Δ_N) the connections and the Laplace operator on M (respectively, N). Suppose $(X_{k+1}, \ldots, X_n)$ is an orthonormal basis of the normal bundle of N, and H denotes the mean curvature vector of N in M.*
Then, for any $f \in \mathcal{C}^\infty(M)$, one has

$$\Delta_N f_{|N} = (\Delta_M f)_{|N} + Hf - \sum_{i=k+1}^{n} \nabla_M^2 f(X_i, X_i). \tag{14}$$

Proof. See, e.g., Lemma 2 in [27]. □

We applied the lemma to the case when $M = SO(3)$ and $N = \Gamma_B$ and with Z_3 as the generator of the normal bundle of $N = \Gamma_B$ (here $n = 3$ and $k = 2$), then for any $f \in \mathcal{C}^\infty(SO(3))$, we have

$$\Delta_{\Gamma_B} f_{|\Gamma_B} = (\Delta_{SO(3)} f)_{|\Gamma_B} + H_B f - \nabla^{(2)}_{SO(3)} f(Z_3, Z_3) \tag{15}$$

where H_B denotes the mean curvature of Γ_B in $SO(3)$. The last term can be simplified; indeed, the second covariant derivative is by definition equal to

$$\nabla^{(2)}_{SO(3)} f(Z_3, Z_3) = \nabla_{Z_3}\nabla_{Z_3} f - \nabla_{\nabla_{Z_3} Z_3} f.$$

The Levi-Civita connection on $SO(3)$ is just $\nabla_X Y = \dfrac{1}{2}[X, Y]$ for any vector fields X, Y. Thus,

$$\nabla^{(2)}_{SO(3)} f(Z_3, Z_3) = Z_3^2 f - \nabla_{\frac{1}{2}[Z_3, Z_3]} f = Z_3^2 f.$$

Let us denote by C_G the Casimir operator of G, the unique generator of the center of the enveloping algebra $U(\mathfrak{g})$, which is nothing but the Laplace operator on G. To sum up, we obtained the following.

Proposition 2. *The Laplace operator of the submanifold Γ_B takes the following form:*

$$\Delta_\Gamma = C_G + H_\Gamma - Z_3^2.$$

4.4. The Heat Kernel on Γ_B.

The density probability distribution of the Brownian motion in Γ_B is determined by the heat kernel of the Markov semi-group operator $P^\Gamma t = e^{-t\Delta\Gamma}$ acting on $L^2(\Gamma_B)$. The Casimir operator C_G lies in the center of the enveloping algebra $U(\mathfrak{g})$; in particular, it commutes with Z_3^2, i.e., $[C_G, Z_3^2] = 0$. Thus, the commutation relation and Proposition 2 give the identity:

$$P_t^\Gamma = e^{-tC_G} e^{tZ_3^2} e^{-th} \tag{16}$$

where h is the mean curvature scalar of Γ_B seen as the embedded Riemannian submanifold in G. The diffusion operator $P^\Gamma t$ has a heat kernel function $p_t^\Gamma : G \times G \to \mathbb{R}$ characterized by the following relation:

$$P_t^\Gamma f(g) = \int_\Gamma p_t^\Gamma(k, g) f(k) dk$$

for every $g \in \Gamma$ and $f \in L^2(\Gamma)$. The value $p_t^\Gamma(k, g)$ gives exactly the probability of the Brownian motion in Γ to be at k at time t provided it started at g. We need to make this probability transition explicit as a function ofthe one in G and in the normal direction given by the operator Z_3^2. Recall that the normal direction is given by the vector field Z_3, and thus, the normal direction at the point $g \in G$ is just given by $Z_3(g)$ and $\pi(g)$ is the normal component of $g \in G$ onto the geodesic $Z = \{e^{tZ_3} : t \in \mathbb{R}\}$. The element $\alpha(g)$ is a unique element of Lie (Z) such that $e^{\alpha(g)} = \pi(g)$. Since Z is a one-dimensional one, the element $\alpha(g) = \log(\pi(g))$ can be seen as an element of $\mathbb{R}$. To compute $\alpha(g)$, it suffices to consider the vector fields Z_1, Z_2, and Z_3, which form the basis of $\mathfrak{g}$. Indeed, let $g \in G$ be given; since the exponential is surjective onto G, there exists a $X \in \mathfrak{g}$ such that $g = e^X$. Now, writing the decomposition of X with respect to the basis $\{Z_i, i = 1, 2, 3\}$, we get that $X = z_1 Z_1 + z_2 Z_2 + z_3 Z_3$ for some real numbers z_i $(i = 1, 2, 3)$. Thus, we have

$$\alpha(g) = \alpha(e^X) = z_3.$$

The relation (16) gives

$$P^\Gamma t f(\gamma) = \int_\Gamma p_t^\Gamma(k, \gamma) f(k) dk = \int_G p_t^G(\gamma, g) e^{-th(g)} \left(\int_{\mathbb{R}} p_t^Z(s, \alpha(g)) f(e^{sZ_3}) ds \right) dg.$$

Finally, taking $f = \delta_{I_3}$, we obtain the solution of the heat equation in Δ with the initial condition $u(0^+, x) = \delta_{I_3}(x)$. Thus, the heat kernel of Δ_Γ is given by

$$p_t^\Gamma(\gamma) = p_t^\Gamma(I_3, \gamma) = \int_G p_t^G(\gamma, g) e^{-th(g)} p_t^Z(0, \alpha(g)) dg. \tag{17}$$

Now, Z is a the trajectory of a both-sided geodesic with initial velocity Z_3 in $SO(3)$. In particular, it is a totally geodesic submanifold and, therefore, minimal in $SO(3)$. Hence, the heat kernel on Z is just the one-dimensional heat kernel:

$$p^Z(z) = \frac{1}{\sqrt{\pi t}} e^{z^2/2t}.$$

The heat kernel of Δ_Γ takes the following form:

$$p_t^\Gamma(\gamma) = \frac{1}{\sqrt{\pi t}} \int_G p_t^G(\gamma, g) e^{-th(g)} e^{\alpha(g)^2/2t} dg. \tag{18}$$

Proof of Theorem 1. Now, we come to our initial problem, namely the study of the random process $V_t = vol_{S^2}(g_t B \cap B)$ for $t \geq 0$, where $(g_t)_{t \geqslant 0}$ is the Brownian motion, which is stopped when it hits the boundary of the support of f. More precisely, we define the stopping time:

$$\tau = \inf\{t > 0 \ : \ g_t \in \partial \Gamma_B\}.$$

Thus, the Brownian motion starting at identity and killed outside $\Gamma = \mathrm{supp} f$ has its density given by

$$p_t^\Gamma(k) = \frac{1}{\sqrt{\pi t}} \int_G p_t^G(k, g) e^{-th(g)} e^{\alpha(g)^2/2t} dg. \tag{19}$$

The expectation of $(V_{t\wedge\tau})_{t \geq 0} = (f(g_{t\wedge\tau}))$ is

$$\mathbb{E}[V_{t\wedge\tau}] = \mathbb{E}[V_t \mid t < \tau] = \int_\Gamma p_t^\Gamma(k) f(k) dk.$$

Using (19), we obtain

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{1}{\sqrt{\pi t}} \int_\Gamma \int_G p_t^G(k, g) f(k) e^{L_t(g)} dg\, dk$$

where $L_t(g) = -th(g) + \alpha(g)^2/2t$ for every $g \in G$. The function L_t defines a map that is $SO(3)$ invariant. Furthermore, the function f is conjugate invariant; indeed, for any $g, h \in G$, we have

$$f(hgh^{-1}) = \mathrm{vol}_{S^2}(hgh^{-1}B \cap B) = \mathrm{vol}_{S^2}(gh^{-1}B \cap h^{-1}B) = \mathrm{vol}_{S^2}(gB \cap B) = f(g).$$

The last equality is justified by vol_{S^2} being $SO(3)$ invariant. Thus, f is entirely determined by its values at a rotation of a given axis, thus depending only on the angle β,

$$f(\beta) = f(R_\beta).$$

The support of f is, thus, given by the interval $0 \leq \beta \leq \mathrm{diam}(B)$.

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_{\beta=0}^{\mathrm{diam}(B)} \int_{\alpha=0}^{\pi} p_t^G(\beta,\theta) f(\beta) e^{L_t(\theta)} \sin^2(\theta/2) \sin^2(\beta/2) d\theta\, d\beta.$$

Let us set

$$\mathcal{J}(t,\theta) = \int_{\beta=0}^{\mathrm{diam}(B)} p_t(\beta,\theta) f(\beta) \sin^2(\beta) d\beta.$$

Then, using the heat kernel expansion (9):

$$\mathcal{J}(t,\theta) = \int_{\beta=0}^{\mathrm{diam}(B)} \left(1 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta) \chi_n(\beta)\right) f(\beta) \sin^2(\beta/2) d\beta.$$

Let us denote $J_n = \displaystyle\int_0^{\mathrm{diam}(B)} f(\beta)\chi_n(\beta) \sin^2(\beta/2)\, d\beta$ for $n \geq 0$; therefore,

$$\mathcal{J}(t,\theta) = J_0 + \sum_{n\geq 1} (2n+1) e^{-n(n+1)t/2} \chi_n(\theta) J_n.$$

Finally, using Fubini's theorem, one has

$$\mathbb{E}[V_{t\wedge\tau}] = \frac{4}{\pi^2\sqrt{\pi t}} \int_0^{\pi} \mathcal{J}(t,\theta) e^{L_t(\theta)} \sin^2(\theta/2) d\theta.$$

This proves Theorem 1. □

5. Conclusions

Using all the variety of mathematical tools coming from the theory of the Brownian motions on manifolds, we were able to derive an integral expression for the expectation of the volume intersection of a subset of the sphere S^2 with its translation. Such results could be applied to concrete problems in physics and dynamical 3D image processing. A natural generalization of our result would be to try to find an analog of Theorem 1 by replacing Brownian motions on Lie groups by Levy processes, which are stochastic processes, which can have jump discontinuities using the recent results of Albeverio and Gordina [4].

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