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Fuzzy Mittag–Leffler–Hyers–Ulam–Rassias Stability of Stochastic Differential Equations

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Abstract: Stability is the most relevant property of dynamical systems. The stability of stochastic differential equations is a challenging and still open problem. In this article, using a fuzzy Mittag–Leffler function, we introduce a new fuzzy controller function to stabilize the stochastic differential equation (SDE) $\nu'(\gamma,\mu) = F(\gamma,\mu,\nu(\gamma,\mu))$. By adopting the fixed point technique, we are able to prove the fuzzy Mittag–Leffler–Hyers–Ulam–Rassias stability of the SDE.

Keywords: fuzzy Mittag–Leffler–Hyers–Ulam–Rassiass stability; fuzzy controller function; fuzzy normed space; random operator; stochastic differential equation

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1. Introduction and Mathematical Preliminaries

Morsi [1] used the concepts of Minkowski functionals of L-fuzzy sets and fuzzy metric space to introduce the notion of fuzzy (pseudo) normed spaces. Subsequently, Jäger and Shi [2], using random normed spaces, introduced the fuzzy normed spaces. In the last years, the fuzzy functional analysis and its applications, especially the Hyers–Ulam–Rassias stability [3–5] in fuzzy normed spaces, was widely investigated by several authors [6,7]. Furthermore, several fixed-point (FP) results were obtained, with applications to nonlinear functional analysis. To learn more about applications of FP theory, please see references [8–10].

Stability is crucial in any dynamical systems. Specifically, the stability of stochastic differential equations is a challenging and still open problem. In this paper, we consider the stochastic differential equation (SDE) of the form:

$$\nu'(\gamma,\mu) = F(\gamma,\mu,\nu(\gamma,\mu)). \tag{1}$$

Using a new fuzzy controller function, constructed based on the fuzzy Mittag–Leffler (FML) function, we are able to stabilize the pseudo SDE (1). Additionally, by adopting the FP technique [11–13] we prove the fuzzy Mittag–Leffler–Hyers–Ulam–Rassias ($MLHUS$) stability of the SDE [14,15]. Our findings extend and improve some existing results [16,17] by using a new fuzzy controller function that allows studying the $MLHUS$ stability of SDEs in fuzzy normed spaces, and by using the alternative of FP-theorem [18,19].

In the subsequent analysis, for simplicity, we use the notions: $\Pi = (0,1)$, $J = (0,1]$, $\Omega = [0,\infty]$ and $\Delta = (0,\infty)$.

Definition 1 ([9,20,21])**.** *Consider that S is a linear space and that η represents a fuzzy set from $S \times \Delta$ to J. Then, the ordered pair (S,η) is a fuzzy normed (FN) space whenever:*

(FN1) $\eta(\zeta,\tau) = 1, \ \forall \tau \in \Delta$ *iff* $\zeta = 0$;
(FN2) $\eta(a\zeta,\tau) = \eta\left(\zeta, \frac{\tau}{|a|}\right), \ \forall \zeta \in S, \forall a \in \mathbb{R}\backslash\{0\}$;
(FN3) $\eta(\xi+\zeta,\tau+\varsigma) \geq \bigwedge(\eta(\xi,\tau),\eta(\zeta,\varsigma)), \ \forall \xi,\zeta \in S, \forall \tau,\varsigma \in \Delta$;
(FN4) $\eta(\zeta,.) : \Delta \to J$ *is continuous.*

A complete FN space is denoted by FB space.

Consider that $(S, \|.\|)$ is a linear normed space. If for all $\varsigma \in \Delta$

$$\eta(\zeta,\varsigma) = \exp\left(-\frac{\|\zeta\|}{\varsigma}\right),$$

then (S,η) is a FN-space.

Consider that $(\digamma,\Sigma,\xi)$ is a probability measure space. Assume that $(T,\mathfrak{B}_T)$ and $(S,\mathfrak{B}_S)$ are Borel measurable spaces, in which T and S are FB spaces. A mapping $F : \digamma \times T \to S$ is called a random operator (RO) if $\{\gamma : F(\gamma,\xi) \in B\} \in \Sigma$ for all ξ in T and $B \in \mathfrak{B}_S$. In addition, F is RO if $F(\gamma,\xi) = \zeta(\gamma)$ is a S-valued random variable for every ζ in T. A RO $F : \digamma \times T \to S$ is called *linear* if $F(\gamma, a\xi_1 + b\xi_2) = aF(\gamma,\xi_1) + bF(\gamma,\xi_2)$ almost everywhere for each ξ_1,ξ_2 in T and a,b are scalars, and *bounded* if there exists a non-negative real-valued random variable $M(\gamma)$ such that

$$\eta(F(\gamma,\xi_1) - F(\gamma,\xi_2), M(\gamma)\tau) \geq \eta(\xi_1 - \xi_2,\tau),$$

almost everywhere for each ξ_1,ξ_2 in T, $\tau \in \Delta$ and $\gamma \in \digamma$.

In this work, we present the FP technique, which is the second most popular tool for proving the stability of functional equations [22,23].

Theorem 1 ([10])**.** (**The alternative of *FP***). *Assume that (T,ρ) is a complete generalized metric space and that $\Lambda : T \to T$ is a strictly contractive function with the Lipschitz constant $\iota < 1$. Then, for every $\xi \in T$, either*

$$\rho(\Lambda^n\xi, \Lambda^{n+1}\xi) = \infty,$$

for each $n \in \mathbb{N}$, or there is a $n_0 \in \mathbb{N}$ for which:

(i) $\rho(\Lambda^n\xi,\Lambda^{n+1}\xi) < \infty, \ \forall n \geq n_0$;
(ii) *the FP ξ^* of Λ is the convergent point of the sequence $\{\Lambda^n\xi\}$;*
(iii) *in the set $V = \{\zeta \in T \mid \rho(\Lambda^{n_0}\xi,\zeta) < \infty\}$, ξ^* is the unique FP of Λ;*
(iv) $(1-\iota)\rho(\zeta,\xi^*) \leq \rho(\zeta,\Lambda\zeta)$ *for every $\zeta \in V$.*

Definition 2 ([24])**.** *The Mittag–Leffler function is given by the series:*

$$E_q(\mu) = \sum_{k=0}^{\infty} \frac{\mu^k}{\Gamma(qk+1)},$$

where $q \in \mathbb{C}$, $Re(q) > 0$ and $\Gamma(\mu)$ is a gamma function:

$$\Gamma(\mu) = \int_0^{\infty} e^{-t}t^{\mu-1}dt,$$

with $Re(\mu) > 0$. In particular, if $q = 1$, we get:

$$E_1(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{\Gamma(j+1)} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} = e^{\mu}.$$

Using Definition 2, we introduce the FML function as:

$$E_q(\mu,\tau) = \frac{\tau}{\tau + E_q(\mu)}, \ \forall \tau > 0.$$

2. Fuzzy *MLHUS* Stability

The norm $L^1(\digamma \times \Xi, S)$ is written $\eta(.,\tau)_{L^1(\digamma \times \Xi)}$. We prove the fuzzy *MLHUS* stability for the *SDE* $\nu'(\gamma,\mu) = F(\gamma,\mu,\nu(\gamma,\mu))$.

Theorem 2. *Consider that* $c \in \mathbb{R}$, $r > 0$,

$$\Xi = \{\mu \in \mathbb{R} \mid |\mu - c| \leq r\},$$

and $F : \digamma \times \Xi \times \mathbb{R} \to \mathbb{R}$ *is a continuous RO which satisfies a Lipschitz condition:*

$$\eta(F(\gamma,\mu,\nu(\gamma,\mu)) - F(\gamma,\mu,\omega(\gamma,\mu)),\tau) \geq \eta\Big(\nu - \omega, \frac{\tau}{L}\Big), \tag{2}$$

for any $\mu \in \Xi$, $\tau \in \Delta$, $\gamma \in \digamma$ *and* $\nu,\omega \in \mathbb{R}$, *where L is a constant with* $rL \in \Pi$. *If a continuously differentiable operator* $\nu : \digamma \times \Xi \to \mathbb{R}$ *satisfies the differential inequality:*

$$\eta\bigg(\int_c^\mu [F(\gamma,\xi,\nu(\gamma,\xi)) - \nu'(\gamma,\xi)]d\xi, \tau\bigg) \geq E_q(\mu,\tau), \tag{3}$$

for any $\mu \in \Xi$, $\tau \in \Delta$ *and* $\gamma \in \digamma$, *where* E_q *is a FML function,*

$$\inf_{\xi \in \Xi} E_q(\xi, \frac{\tau}{2r}) \geq E_q\Big(\mu, \frac{\tau}{r}\Big), \tag{4}$$

for any $\mu \in \Xi$, $\tau \in \Delta$, $\gamma \in \digamma$, *then there exists a unique continuous RO* $\nu_0 : \digamma \times \Xi \to \mathbb{R}$ *such that*

$$\nu_0(\gamma,\mu) = \nu(\gamma,c) + \int_c^\mu F(\gamma,\xi,\nu_0(\gamma,\xi))d\xi.$$

Furthermore, ν_0 *is a solution of* (1) *and*

$$\eta(\nu(\gamma,\mu) - \nu_0(\gamma,\mu),\tau) \geq E_q(\mu,(1-rL)\tau), \tag{5}$$

for any $\mu \in \Xi$, $\tau \in \Delta$, $\gamma \in \digamma$.

Proof. Consider the space of continuous *RO*s

$$\Upsilon = \{\alpha : \digamma \times \Xi \to \mathbb{R} \mid \alpha \text{ is a continuous } RO\}. \tag{6}$$

Introduce the below function on Υ^2 as,

$$\begin{aligned}&\rho(\alpha,\beta)\\ &= \inf\Big\{\lambda \in \Delta \mid \eta(\alpha(\gamma,\mu) - \beta(\gamma,\mu),\tau) \geq E_q\Big(\mu,\frac{\tau}{\lambda}\Big), \forall \mu \in \Xi, \tau \in \Delta, \gamma \in \digamma\Big\}.\end{aligned} \tag{7}$$

Mihet and Radu [25] proved that (Υ,ρ) is a complete generalized metric (see also [26]). We introduce the *RO* $\Lambda : \Upsilon \to \Upsilon$ by:

$$(\Lambda\alpha)(\gamma,\mu) = \nu(\gamma,c) + \int_c^\mu F(\gamma,\xi,\alpha(\gamma,\xi))d\xi, \tag{8}$$

for every $\alpha \in \Upsilon$, $\gamma \in \digamma$ and $\mu \in \Xi$. The continuity of *RO* α implies the continuity of $\Lambda\alpha$ and well-defined Λ.

Consider $\alpha,\beta \in \Upsilon$ and $\gamma \in \digamma$. Additionally, consider $\lambda_{\alpha,\beta} \in \Delta$ such that:

$$\eta(\alpha(\gamma,\mu) - \beta(\gamma,\mu),\tau) \geq E_q\bigg(\mu, \frac{\tau}{\lambda_{\alpha\beta}}\bigg). \tag{9}$$

Assume that $c = \varpi_1 < \varpi_2 < \cdots < \varpi_k = \mu$, $\Delta\mu_i = \varpi_i - \varpi_{i-1} = \frac{|\mu - c|}{k}$, $i = 1, 2, \cdots, k$, and $\|\Delta\mu\| = \max_{1 \le i \le k}(\Delta\mu_i)$, for every $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in F$. Utilizing (2), (4), (8), and (9) we have the following

$$\eta\big((\Lambda\alpha)(\gamma,\mu) - (\Lambda\beta)(\gamma,\mu), \tau\big),$$

by equality (8)

$$= \eta\left(\int_c^\mu \big(F(\gamma,\varpi_i,\alpha(\gamma,\varpi_i)) - F(\gamma,\varpi_i,\beta(\gamma,\varpi_i))\big)d\mu, \tau\right),$$

by integral definition

$$= \eta\left(\lim_{\|\Delta\mu\|\to 0}\sum_{i=1}^{k}\big(F(\gamma,\varpi_i,\alpha(\gamma,\varpi_i)) - F(\gamma,\varpi_i,\beta(\gamma,\varpi_i))\big)\Delta\mu_i, \tau\right),$$

by continuity property of η

$$= \lim_{\|\Delta\mu\|\to 0}\eta\left(\sum_{i=1}^{k}\big(F(\gamma,\varpi_i,\alpha(\gamma,\varpi_i)) - F(\gamma,\varpi_i,\beta(\gamma,\varpi_i))\big)\Delta\mu_i, \tau\right),$$

by triangular inequality

$$\ge \lim_{\|\Delta\mu\|\to 0}\bigwedge\eta\left(\big(F(\gamma,\varpi_i,\alpha(\gamma,\varpi_i)) - F(\gamma,\varpi_i,\beta(\gamma,\varpi_i))\big)\Delta\mu_i, \frac{\tau}{k}\right),$$

by property of infimum

$$\ge \inf_{\xi\in\Xi}\eta\left(\big(F(\gamma,\xi,\alpha(\gamma,\xi)) - F(\gamma,\xi,\beta(\gamma,\xi))\big), \frac{\tau}{k\Delta\mu_i}\right)$$

$$\ge \inf_{\xi\in\Xi}\eta\left(\big(F(\gamma,\xi,\alpha(\gamma,\xi)) - F(\gamma,\xi,\beta(\gamma,\xi))\big), \frac{\tau}{k\|\Delta\mu\|}\right)$$

$$\ge \inf_{\xi\in\Xi}\eta\left(\big(F(\gamma,\xi,\alpha(\gamma,\xi)) - F(\gamma,\xi,\beta(\gamma,\xi))\big), \frac{k\tau}{k|\mu - c|}\right),$$

by equality (2)

$$\ge \inf_{\xi\in\Xi}\eta\left(\alpha(\gamma,\xi) - \beta(\gamma,\xi), \frac{\tau}{(2rL)}\right),$$

by equality (9)

$$\ge \inf_{\xi\in\Xi}E_q\left(\xi, \frac{\tau}{(2rL)\lambda_{\alpha,\beta}}\right),$$

by equality (4)

$$\ge E_q\left(\mu, \frac{\tau}{(rL)\lambda_{\alpha,\beta}}\right),$$

for every $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in F$, that is, $\rho(\Lambda\alpha, \Lambda\beta) \le (rL)\lambda_{\alpha,\beta}$. Therefore, we can conclude that $\rho(\Lambda\alpha, \Lambda\beta) \le (rL)\rho(\alpha, \beta)$ for any $\alpha, \beta \in \Upsilon$, in which $(rL) \in \Pi$. Therefore, Λ is a strictly contraction mapping.

By (3), (5), (8), and $\nu \in \Upsilon$, we obtain:

$$\begin{aligned}
&\eta\big((\Lambda\nu)(\gamma,\mu)-\nu(\gamma,\mu),\tau\big),\\
&\text{by equality (8)}\\
&=\eta\left(\nu(\gamma,c)+\int_c^\mu F(\gamma,\xi,\nu(\gamma,\xi))d\xi-\nu(\gamma,\mu),\tau\right),\\
&\text{by property of integral}\\
&=\eta\left(\int_c^\mu [F(\gamma,\xi,\nu(\gamma,\xi))-\nu'(\gamma,\xi)]d\xi,\tau\right),\\
&\text{by equality (3)}\\
&\geq E_q\left(\mu,\frac{\tau}{1}\right),
\end{aligned}$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. Thus, (7) implies that

$$\rho(\Lambda\nu,\nu) < 1, \tag{10}$$

and hence,

$$\rho(\Lambda^{n+1}\nu,\Lambda^n\nu) < 1 < \infty.$$

Now, Theorem 1 implies that:

(i) there is a continuous *RO* $\nu_0 : \digamma \times \Xi \to \mathbb{R}$ where $\Lambda\nu_0 = \nu_0$, that is, ν_0 is *FP* of Λ, which is uniqu in the set

$$V = \{\alpha \in \Upsilon : \rho(\alpha,\nu) < \infty\}.$$

(ii) $\Lambda^n\nu \to \nu_0$ in (Υ,ρ) as $n \to \infty$.

(iii) using (10) we obtain:

$$\rho(\nu,\nu_0) \leq \frac{1}{1-rL}\rho(\Lambda\nu,\nu) \leq \frac{1}{1-rL},$$

which implies the validity of (5) for each $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. $\square$

Consider that (Υ,η) is a *FN* space. We introduce the fuzzy set η_B as:

$$\eta_B(\alpha(\gamma,\xi),\tau) := \inf_{\xi\in\Xi}\left\{\eta\left(\alpha(\gamma,\xi),\frac{\tau}{e^{\theta\xi}}\right) : \theta \in \Delta, \Xi \subset \mathbb{R}_+\right\},$$

for every $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. Then, (Υ,η_B) is a *FN* space (Bielecki *FN* space). In fact, (FN1), (FN2), and (FN4) are obvious. Now, we prove only (FN3). Observe that:

$$\begin{aligned}
&\bigwedge(\eta_B(\alpha(\gamma,\xi),\tau),\eta_B(\beta(\gamma,\xi),\varsigma))\\
=\;&\bigwedge\left(\inf_{\xi\in\Xi}\left\{\eta\left(\alpha(\gamma,\xi),\frac{\tau}{e^{\theta\xi}}\right)\right\},\inf_{\xi\in\Xi}\left\{\eta\left(\beta(\gamma,\xi),\frac{\varsigma}{e^{\theta\xi}}\right)\right\}\right)\\
\leq\;&\bigwedge\left(\inf_{\xi\in\Xi}\left\{\eta\left(\alpha(\gamma,\xi),\frac{\tau}{e^{\theta\xi}}\right),\eta\left(\beta(\gamma,\xi),\frac{\varsigma}{e^{\theta\xi}}\right)\right\}\right)\\
\leq\;&\inf_{\xi\in\Xi}\left\{\eta\left((\alpha+\beta)(\gamma,\xi),\frac{(\tau+\varsigma)}{e^{\theta\xi}}\right)\right\}\\
=\;&\inf_{\xi\in\Xi}\left\{\eta\left((\alpha+\beta)(\gamma,\xi),\frac{(\tau+\varsigma)}{e^{\theta\xi}}\right)\right\}\\
=\;&\eta_B((\alpha+\beta)(\gamma,\xi),(\tau+\varsigma)),
\end{aligned}$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$, which proves the triangle inequality (FN3).

Now, we prove the fuzzy *MLHUS* stability of the random Equation (1) via the Bielecki fuzzy norm.

Theorem 3. *Assume that $c \in \mathbb{R}$, $r > 0$ and $\Xi = \{\mu \in \mathbb{R} \mid |\mu - c| \leq r\}$. Consider that $F : \digamma \times \Xi \times \mathbb{R} \to \mathbb{R}$ is a continuous RO which satisfies in the Lipschitz condition:*

$$\eta(F(\gamma, \mu, \nu(\gamma, \mu)) - F(\gamma, \mu, \omega(\gamma, \mu)), \tau) \geq \eta\left(\nu - \omega, \frac{\tau}{L}\right),$$

for any $\mu \in \Xi$, $\tau \in \Delta$, $\gamma \in \digamma$ and where L is a constant with $rL \in \Pi$. If a continuously differentiable function $\nu : \digamma \times \Xi \to \mathbb{R}$ satisfies the differential inequality:

$$\eta\left(\int_c^\mu [F(\gamma, \xi, \nu(\gamma, \xi)) - \nu'(\gamma, \xi)]d\xi, \tau\right) \geq E_q(\mu, \tau),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$, where E_q is a FML function; then, with the Bielecki fuzzy norm, the fuzzy MLHUS stability is verified for the Equation (1).

Proof. By the same method used in the proof of Theorem 2, we assume that $c = \varpi_1 < \varpi_2 < \cdots < \varpi_k = \mu$, $\Delta\mu_i = \varpi_i - \varpi_{i-1} = \frac{|\mu - c|}{k}$, $i = 1, 2, \cdots, k$ and $\|\Delta\mu\| = \max_{1 \leq i \leq k}(\Delta\mu_i)$. Now, we show the contraction of Λ on Y with respect to the Bielecki fuzzy norm introduced in (6):

$$\begin{aligned}
&\eta\big((\Lambda\alpha)(\gamma, \mu) - (\Lambda\beta)(\gamma, \mu), \tau\big) \\
&\text{by equality (8)} \\
&= \eta\left(\int_c^\mu \big(F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))\big)d\mu, \tau\right) \\
&\text{by integral definition} \\
&= \eta\left(\lim_{\|\Delta\mu\| \to 0} \sum_{i=1}^k \big(F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))\big)\Delta\mu_i, \tau\right) \\
&\text{by continuity property of } \eta \\
&= \lim_{\|\Delta\mu\| \to 0} \eta\left(\sum_{i=1}^k \big(F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))\big)\Delta\mu_i, \tau\right) \\
&\text{by triangular inequality} \\
&\geq \lim_{\|\Delta\mu\| \to 0} \bigwedge \eta\left(\big(F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))\big)\Delta\mu_i, \frac{\tau}{k}\right) \\
&\text{by property of infimum} \\
&\geq \inf_{\xi \in \Xi} \eta\left(\big(F(\gamma, \xi, \alpha(\gamma, \xi)) - F(\gamma, \xi, \beta(\gamma, \xi))\big), \frac{\tau}{k\Delta\mu_i}\right) \\
&\geq \inf_{\xi \in \Xi} \eta\left(\big(F(\gamma, \xi, \alpha(\gamma, \xi)) - F(\gamma, \xi, \beta(\gamma, \xi))\big), \frac{\tau}{k\|\Delta\mu\|}\right) \\
&\geq \inf_{\xi \in \Xi} \eta\left(\big(F(\gamma, \xi, \alpha(\gamma, \xi)) - F(\gamma, \xi, \beta(\gamma, \xi))\big), \frac{k\tau}{k|\mu - c|}\right) \\
&\text{by equality (2)} \\
&\geq \inf_{\xi \in \Xi} \eta\left(\alpha(\gamma, \xi) - \beta(\gamma, \xi), \frac{\tau}{(rL)}\right) \\
&\geq \inf_{\xi \in \Xi} \eta\left(\alpha(\gamma, \xi) - \beta(\gamma, \xi), \frac{\tau}{(rL)e^{\theta\xi}}\right) \\
&\text{by definition } \eta_B \\
&\geq \eta_B\left(\alpha - \beta, \frac{\tau}{(rL)}\right).
\end{aligned}$$

then,

$$\eta((\Lambda\alpha)(\gamma,\mu)-(\Lambda\beta)(\gamma,\mu),\tau)\geq\eta_B\left(\alpha-\beta,\frac{\tau}{(rL)}\right),$$

for any $\mu\in\Xi,\tau\in\Delta,\gamma\in F$, that is, $\rho(\Lambda\alpha,\Lambda\beta)\leq\eta_B\left(\alpha-\beta,\frac{\tau}{(rL)}\right)$. Hence, we can conclude that $\rho(\Lambda\alpha,\Lambda\beta)\leq(rL)\rho(\alpha,\beta)$ for any $\alpha,\beta\in\Upsilon$. By letting $(rL)\in\Pi$, we obtain the strict continuity.Furthermore, by Theorem 1, we obtain:

$$\rho(\nu,\nu_0)\leq\frac{1}{1-rL}\rho(\Lambda\nu,\nu)\leq\frac{1}{1-rL},$$

so, the fuzzy $MLHUS$ stability of Equation (1) is verified. □

Theorem 4. *Suppose that a and b are real numbers such that $a<b$. Let $\Xi=[a,b]$ and $c\in\Xi$. Assume that K and L are positive constants such that $LK\in\Pi$. Consider that $F:F\times\Xi\times\mathbb{R}\to\mathbb{R}$ is a continuous RO which satisfies a Lipschitz condition:*

$$\eta(F(\gamma,\mu,\nu)-F(\gamma,\mu,\omega),\tau)\geq\eta\left(\nu-\omega,\frac{\tau}{L}\right),\tag{11}$$

for any $\mu\in\Xi,\tau\in\Delta,\gamma\in F$ and $\nu,\omega\in\mathbb{R}$. If a continuously differentiable operator $\nu:F\times\Xi\to\mathbb{R}$ satisfies the differential inequality:

$$\eta\left(\int_0^\mu[F(\gamma,\xi,\nu(\gamma,\xi))-\nu'(\gamma,\xi)]d\xi,\tau\right)\geq E_q(\mu,\tau),\tag{12}$$

for any $\mu\in\Xi,\tau\in\Delta$ and $\gamma\in F$, where E_q is a FML function,

$$\inf_{\xi\in\Xi}E_q(\xi,\frac{\tau}{(b-a)})\geq E_q\left(\mu,\frac{\tau}{K}\right),\tag{13}$$

for any $\mu\in\Xi,\tau\in\Delta,\gamma\in F$, then there exists a unique continuous RO $\nu_0:F\times\Xi\to\mathbb{R}$ such that:

$$\nu_0(\gamma,\mu)=\nu(\gamma,c)+\int_0^\mu F(\gamma,\xi,\nu_0(\gamma,\xi))d\xi.$$

Furthermore, ν_0 is a solution of (1) and

$$\eta(\nu(\gamma,\mu)-\nu_0(\gamma,\mu),\tau)\geq E_q(\mu,(1-rL)\tau),$$

for any $\mu\in\Xi,\tau\in\Delta,\gamma\in F$.

Proof. Consider the space of continuous *RO*s:

$$\Upsilon=\{\alpha:F\times\Xi\to\mathbb{R}\mid\alpha\text{ is a continuous }RO\}.$$

Introduce the below function on Υ^2 as,

$$\rho(\alpha,\beta)=\inf\left\{\lambda\in\Delta\mid\eta(\alpha(\gamma,\mu)-\beta(\gamma,\mu),\tau)\geq E_q\left(\mu,\frac{\tau}{\lambda}\right),\forall\mu\in\Xi,\tau\in\Delta,\gamma\in F\right\}.\tag{14}$$

Further, introduce the *RO* $\Lambda:\Upsilon\to\Upsilon$ by:

$$(\Lambda\alpha)(\gamma,\mu)=\nu(\gamma,c)+\int_0^\mu F\big(\mu,\xi,\alpha(\gamma,\xi)\big)d\xi,\tag{15}$$

for every $\alpha\in\Upsilon,\gamma\in F$ and $\mu\in\Xi$. The continuity of *RO* α implies the continuity of $\Lambda\alpha$ and well defining Λ.

Consider $\alpha, \beta \in \Upsilon$ and $\gamma \in \digamma$. Let $\lambda_{\alpha,\beta} \in \Delta$ such that:

$$\eta(\alpha(\gamma, \mu) - \beta(\gamma, \mu), \tau) \geq E_q\left(\mu, \frac{\tau}{\lambda_{\alpha,\beta}}\right). \tag{16}$$

In addition, let $a = \varpi_1 < \varpi_2 < \cdots < \varpi_k = b$, $\Delta\mu_i = \varpi_i - \varpi_{i-1} = \frac{b-a}{k}$, $i = 1, 2, \cdots, k$ and $\|\Delta\mu\| = \max_{1 \leq i \leq k}(\Delta\mu_i)$, for every $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. Utilizing (11), (13), (14) and (15) we have the following

$$\eta((\Lambda\alpha)(\gamma, \mu) - (\Lambda\beta)(\gamma, \mu), \tau),$$

by equality (15)

$$= \eta\left(\int_0^\mu (F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i)))d\mu, \tau\right),$$

by integral definition

$$= \eta\left(\lim_{\|\Delta\mu\| \to 0} \sum_{i=1}^{k} (F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i)))\Delta\mu_i, \tau\right),$$

by continuity property of η

$$= \lim_{\|\Delta\mu\| \to 0} \eta\left(\sum_{i=1}^{k} (F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i)))\Delta\mu_i, \tau\right),$$

by triangular inequality

$$\geq \lim_{\|\Delta\mu\| \to 0} \bigwedge \eta\left((F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i)))\Delta\mu_i, \frac{\tau}{k}\right),$$

by property of infimum

$$\geq \inf_{\xi \in \Xi} \eta\left((F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))), \frac{\tau}{k\Delta\mu_i}\right)$$

$$\geq \inf_{\xi \in \Xi} \eta\left((F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))), \frac{\tau}{k\|\Delta\mu\|}\right)$$

$$\geq \inf_{\xi \in \Xi} \eta\left((F(\gamma, \varpi_i, \alpha(\gamma, \varpi_i)) - F(\gamma, \varpi_i, \beta(\gamma, \varpi_i))), \frac{k\tau}{k(b-a)}\right),$$

by equality (11)

$$\geq \inf_{\xi \in \Xi} \eta\left(\alpha(\gamma, \xi) - \beta(\gamma, \xi), \frac{\tau}{L(b-a)}\right),$$

by equality (16)

$$\geq \inf_{\xi \in \Xi} E_q\left(\xi, \frac{\tau}{L(b-a)\lambda_{\alpha,\beta}}\right),$$

by equality (13)

$$\geq E_q\left(\mu, \frac{\tau}{LK\lambda_{\alpha,\beta}}\right),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$, that is, $\rho(\Lambda\alpha, \Lambda\beta) \leq (LK)\lambda_{\alpha,\beta}$. Hence, we can conclude that $\rho(\Lambda\alpha, \Lambda\beta) \leq (LK)\rho(\alpha, \beta)$ for all $\alpha, \beta \in \Upsilon$, in which $(LK) \in \Pi$. Therefore, Λ is a strict contraction mapping.

By (3), (5), (7) and $\nu \in \Upsilon$, we obtain:

$$\begin{aligned}
&\eta\big((\Lambda\nu)(\gamma,\mu)-\nu(\gamma,\mu),\tau\big),\\
&\text{by equality (15)}\\
&=\eta\bigg(\nu(\gamma,c)+\int_c^\mu F(\gamma,\xi,\nu(\gamma,\xi))d\xi-\nu(\gamma,\mu),\tau\bigg),\\
&\text{by property of integral}\\
&=\eta\bigg(\int_c^\mu [F(\gamma,\xi,\nu(\gamma,\xi))-\nu'(\gamma,\xi)]d\xi,\tau\bigg),\\
&\text{by equality (12)}\\
&\geq E_q\Big(\mu,\frac{\tau}{1}\Big),
\end{aligned}$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. Thus, (7) implies that:

$$\rho(\Lambda\nu,\nu) < 1, \tag{17}$$

and hence,

$$\rho(\Lambda^{n+1}\nu,\Lambda^n\nu) < 1 < \infty.$$

Now, Theorem 1 implies that:

(i) there is a continuous *RO* $\nu_0 : \digamma \times \Xi \to \mathbb{R}$ where $\Lambda\nu_0 = \nu_0$, that is, ν_0 is *FP* of Λ, which is unique in the set

$$V = \{\alpha \in \Upsilon : \rho(\alpha,\nu) < \infty\}.$$

(ii) $\Lambda^n\nu \to \nu_0$ in (Υ,ρ) as $n \to \infty$.

(iii) using (17) we obtain:

$$\rho(\nu,\nu_0) \leq \frac{1}{1-rL}\rho(\Lambda\nu,\nu) \leq \frac{1}{1-rL},$$

which implies the validity of (5) for each $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. □

3. Application

Example 1. *Consider positive real numbers K and L such that $LK \in \Pi$, $K < \frac{b(b-a)}{a}$. Assume that $\Xi = [a,b]$. For an arbitrary polynomial $p(\gamma,\mu)$, we let a continuously differentiable RO $\nu : \digamma \times \Xi \to \mathbb{R}$ to satisfy:*

$$\eta\bigg(\int_0^\mu [F(\gamma,\xi,\nu(\gamma,\xi))-\nu^{'}(\gamma,\xi)]d\xi,\tau\bigg) \geq E_q(\mu,\tau),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$. If we set $F(\gamma,\mu,\nu) = L\nu(\gamma,\mu) + p(\gamma,\mu)$, where E_q is a FML function,

$$\inf_{\xi\in\Xi} E_q(\xi,\frac{\tau}{(b-a)}) \geq E_q\Big(\mu,\frac{\tau}{K}\Big),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ and $\gamma \in \digamma$, then, by Theorem 4, there is a unique continuous RO $\nu_0 : \digamma \times \Xi \to \mathbb{R}$ such that:

$$\nu_0(\gamma,\mu) = \nu(\gamma,0) + \int_0^\mu (L\nu(\gamma,\xi) + p(\gamma,\xi))d\xi,$$

and

$$\eta(\nu(\gamma,\mu)-\nu_0(\gamma,\mu),\tau) \geq E_q(\mu,(1-rL)\tau),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ *and* $\gamma \in \digamma$.

Example 2. *Consider that* r *and* L *are positive constants with* $rL \in \Pi$ *and*

$$\Xi = \{\mu \in \mathbb{R} \mid |\mu - c| \leq r, \ \textit{for some } c \in \mathbb{R}\}.$$

Let a continuous discrete random function $\nu : \digamma \times \Xi \to \mathbb{R}$ *satisfy the following inequality:*

$$\eta\left(\int_c^\mu [F(\gamma, \xi, \nu(\gamma, \xi)) - \nu'(\gamma, \xi)]d\xi, \tau\right) \geq E_q(\mu, \tau),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ *and* $\gamma \in \digamma$, *where* $p(\gamma, \mu)$ *is a polynomial. If we set* $F(\gamma, \mu, \nu) = L\nu(\gamma, \mu) + p(\gamma, \mu)$, *where* E_q *is a FML function,*

$$\inf_{\xi \in \Xi} E_q(\xi, \frac{\tau}{2r}) \geq E_q\left(\mu, \frac{\tau}{r}\right),$$

for any $\mu \in \Xi$, $\tau \in \Delta$, $\gamma \in \digamma$, *then, by Theorem 2 there exists a unique random operator* $\nu_0 : \digamma \times \Xi \to \mathbb{R}$ *such that:*

$$\nu_0(\gamma, \mu) = \nu(\gamma, 0) + \int_0^\mu (L\nu(\gamma, \xi) + p(\gamma, \xi))d\xi,$$

and

$$\eta(\nu(\gamma, \mu) - \nu_0(\gamma, \mu), \tau) \geq E_q(\mu, (1 - rL)\tau),$$

for any $\mu \in \Xi$, $\tau \in \Delta$ *and* $\gamma \in \digamma$.

4. Conclusions

In this paper we introduced a new fuzzy controller function to stabilize the SDE of the form $\nu'(\gamma, \mu) = F(\gamma, \mu, \nu(\gamma, \mu))$. By adopting the FP technique, we proved the fuzzy $MLHUS$ stability of the SDE. Some examples were given to illustrate the theoretical findings and to show the effectiveness of the method. Extension of the method to SDEs of different types will be further investigated.

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