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Periodic Solutions for a Neutral System with Two Volterra Terms

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Abstract: In this paper, we propose a system of equations containing two kernels. In our transformation of the system, we use the integrable dichotomy condition, where we extract the term of the integration matrix from one of the kernels. We then use the fixed-point theory to prove that the system has periodic solutions that are unique under sufficient conditions. An illustrative example at the end of the article is given.

Keywords: dichotomy condition; Volterra term; periodic solutions; fixed point theorem; neutral system

MSC: 47H1; 034K13; 34K40; 34A34

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1. Introduction

In actuarial science, Volterra integral equations are used in ruin theory, which analyzes the risk of insolvency, and several researchers have directed their interest to equations containing Volterra terms, see for instance [1–3].

In the applied sciences, neutral equations play an important role. A positive periodic solution for two neutral functional differential equations was investigated by Luo et al. [4] using Krasnoselskii's fixed-point theorem. Various mathematical ecological and population models are covered by these functional differential equations, including hematopoietic models [5,6], the models of Nicholson's blowflies [7,8] and the models of blood-cell production [9].

The properties of exponential dichotomies and trichotomies have been studied by many researchers over the last decades due to their importance in the theory of differential and integro-differential equations. Some fundamental works on the theory of periodic solutions related to this subject are here [10–19].

The differential system, including many delay terms (Sa Ngiamsunthorn [20]), has been studied for the periodicity of solutions under an integrable dichotomy. Similar systems have been studied in [21,22] under an exponential-type condition.

Motivated by the abovementioned references, a periodic solution to the following nonlinear neutral differential equations interested us:

$$\left(y(\xi)-\int_{\xi-\tau(\xi)}^{\xi}C(\xi,s)y(s)ds\right)'=\int_{\xi-\sigma(\xi)}^{\xi}B(\xi,s)y(s)ds+Q(\xi,y(\xi),y(\xi-\sigma(\xi))),\qquad(1)$$

in which $y:\mathbb{R}\to\mathbb{R}^n$, $\tau,\sigma:\mathbb{R}\to\mathbb{R}^+$, and $Q:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are Υ-periodic continuous functions on $\mathbb{R}$, $\Upsilon>0$. $C(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are Υ-periodic continuous matrix functions with respect to ξ defined on $\mathbb{R}\times\mathbb{R}$ such that $B(\xi,\xi)$ is nonsingular.

This paper is arranged as follows: In the second part of our paper, we provide definitions and fixed-point theorems for integrable dichotomies and previous results. In Section 3, we establish some criteria for determining whether periodic solutions of the system (1) exist and whether they are unique. Section 4 illustrates the main results with an example. Finally, we end the paper with a conclusion.

2. Preliminaries

Several results and definitions of integrable dichotomies are presented in this section and will be crucial for proving our results, see [13,14].

First, we recall some basic facts about integrable dichotomies. Consider the system

$$z'(\xi) = A(\xi)z(\xi), \tag{2}$$

where $A(\xi)$ is $n \times n$ continuous matrix function defined on $\mathbb{R}$. Let $\Psi(\xi)$ be the fundamental matrix of (2) that satisfies $\Psi(0) = I$.

Let $BC(\mathbb{R}, \mathbb{R}^n)$ be the set of continuous and bounded functions. Let P be the projection matrix and $\Theta = \Theta_P$, the Green matrix associated with P given by

$$\Theta(\xi, s) = \begin{cases} -\Psi(\xi)(I-P)\Psi^{-1}(\xi) & \text{for } \xi < s, \\ \Psi(\xi)P\Psi^{-1}(\xi) & \text{for } \xi \geq s. \end{cases}$$

Definition 1 ([14] Definition 1, p. 3)**.** *We say that the linear system (2) has an integrable dichotomy if there exists a constant $\mu > 0$ and a projection matrix P such that the associated Green matrix $\Theta = \Theta_P$ satisfies*

$$\mu = \sup_{\xi \in \mathbb{R}} \int_{-\infty}^{\infty} \|\Theta(\xi, s)\| ds.$$

Hence, under an integrable dichotomy condition, we consider the nonhomogeneous linear system

$$z'(\xi) = A(\xi)z(\xi) + f(\xi), \tag{3}$$

and we need the following results. See for instance [14].

Proposition 1 ([14] Proposition 1, p. 4)**.** *Assume that there is an integrable dichotomy of (2). Then the trivial solution $z(\xi) = 0$ is the only bounded solution to (2).*

Proposition 2 ([14] Proposition 2, p. 4)**.** *Suppose that there is an integrable dichotomy for the homogeneous system (2). If $f \in BC(\mathbb{R}, \mathbb{R}^n)$, then system (3) has a bounded unique solution $z \in BC(\mathbb{R}, \mathbb{R}^n)$. Furthermore,*

$$z(\xi) = \int_{-\infty}^{\infty} \Theta(\xi, s) f(s) ds. \tag{4}$$

Proposition 3 ([14] Propositions 4 and 5, p. 5)**.** *Assume that there is an integrable dichotomy for the system (2) such that $\Psi(\xi)P\Psi^{-1}(\xi)$ is bounded. If $A(\xi + \Upsilon) = A(\xi)$, then $\Psi(\xi)P\Psi^{-1}(\xi)$ is also Υ-periodic. Furthermore, if $f \in BC(\mathbb{R}, \mathbb{R}^n)$ is Υ-periodic, then (3) has a unique periodic solution satisfying (4).*

Our objective is to demonstrate the existence and uniqueness of periodic solutions for system (1) by using the following fixed point theorems; see [2,23].

Theorem 1 (**Banach**)**.** *For any complete metric space (Y, ρ) and $\mathcal{S} : Y \to Y$. If there is a constant $\gamma \in (0, 1)$ such that for $a, b \in Y$,*

$$\rho(\mathcal{S}a, \mathcal{S}b) \leq \gamma \rho(a, b),$$

then there a unique point $z \in Y$ with $\mathcal{S}z = z$.

Theorem 2 (Krasnoselskii). *Let Λ be a nonempty bounded, convex and closed subset of a Banach space Y. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ map Λ into Y such that*

(1) *$\mathcal{S}_1$ is a contraction mapping on Λ,*

(2) *$\mathcal{S}_2$ is completely continuous on Λ, and*

(3) *$a, b \in \Lambda$, gives $\mathcal{S}_1 a + \mathcal{S}_2 b \in \Lambda$.*

Then there is $z \in \Lambda$, which satisfies $z = \mathcal{S}_1 z + \mathcal{S}_2 z$.

Let a constant $M > 0$ and denote

$$\Lambda = \{u \in BC(\mathbb{R}, \mathbb{R}^n) : u(\xi + \Upsilon) = u(\xi) \text{ and } \|u\| \leq M \,\forall \xi \in \mathbb{R}\},$$

it is easily seen that Λ is a nonempty-bounded, convex and closed subset of $BC(\mathbb{R}, \mathbb{R}^n)$.

Assume that for all $u, v \in \Lambda$, there exists $q_1, q_2 > 0$ such that the function Q satisfies

$$\begin{aligned} &|Q(\xi, u(\xi), u(\xi - \sigma(\xi))) - Q(\xi, v(\xi), v(\xi - \sigma(\xi)))| \\ &\leq q_1|u(\xi) - v(\xi)| + q_2|u(\xi - \sigma(\xi)) - v(\xi - \sigma(\xi))|. \end{aligned} \tag{5}$$

Let

$$\sup_{\xi \in [0,\Upsilon]} |\tau(\xi)| = \alpha, \ \sup_{\xi \in [0,\Upsilon]} |\sigma(\xi)| = \beta, \ \sup_{\xi \in [0,\Upsilon]} |Q(\xi, 0, 0)| = \gamma,$$

$$\sup_{\xi, s \in [0,\Upsilon]} \|C(\xi, s)\| = c, \ \sup_{\xi, s \in [0,\Upsilon]} \|B(\xi, s)\| = b,$$

and assume

$$\alpha c M + \mu((\alpha b c + b + \beta b + q_1 + q_2)M + \gamma) \leq M. \tag{6}$$

3. Main Results

Under the conditions stated previously, we will show in this section the existence and the uniqueness of the solution for (1), which can then be written as

$$\begin{aligned} z'(\xi) = {} & B(\xi, \xi)z(\xi) - B(\xi, \xi)z(\xi) \\ & + \int_{\xi - \sigma(\xi)}^{\xi} B(\xi, s)y(s)ds + Q(\xi, y(\xi), y(\xi - \sigma(\xi))), \end{aligned}$$

where

$$z(\xi) = y(\xi) - \int_{\xi - \tau(\xi)}^{\xi} C(\xi, s)y(s)ds.$$

By Proposition 2, system (1) holds the integral equation

$$\begin{aligned} z(\xi) = {} & \int_{-\infty}^{\infty} \Theta(\xi, s)B(s, s)\left(\int_{s - \tau(s)}^{s} C(s, r)y(r)dr - y(s)\right)ds \\ & + \int_{-\infty}^{\infty} \Theta(\xi, s)\left(\int_{s - \sigma(s)}^{s} B(s, r)y(r)dr + Q(s, y(s), y(s - \sigma(s)))\right)ds, \end{aligned}$$

which is

$$\begin{aligned} y(\xi) = {} & \int_{\xi - \tau(\xi)}^{\xi} C(\xi, s)y(s)ds \\ & + \int_{-\infty}^{\infty} \Theta(\xi, s)B(s, s)\left(\int_{s - \tau(s)}^{s} C(s, r)y(r)dr - y(s)\right)ds \\ & + \int_{-\infty}^{\infty} \Theta(\xi, s)\left(\int_{s - \sigma(s)}^{s} B(s, r)y(r)dr + Q(s, y(s), y(s - \sigma(s)))\right)ds. \end{aligned} \tag{7}$$

We define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ by

$$(\mathcal{S}_1 u)(\xi) = \int_{\xi-\tau(\xi)}^{\xi} C(\xi,s)u(s)ds, \tag{8}$$

and

$$\begin{aligned}(\mathcal{S}_2 u)(\xi) &= \int_{-\infty}^{\infty} \Theta(\xi,s)B(s,s)\left(\int_{s-\tau(s)}^{s} C(s,r)u(r)dr - u(s)\right)ds \\ &\quad + \int_{-\infty}^{\infty} \Theta(\xi,s)\left(\int_{s-\sigma(s)}^{s} B(s,r)u(r)dr + Q(s,u(s),u(s-\sigma(s)))\right)ds,\end{aligned} \tag{9}$$

for $u \in BC(\mathbb{R},\mathbb{R}^n)$. Clearly, if the operator $\mathcal{S}_1 + \mathcal{S}_2$ has a fixed point, then it is a periodic solution of (1).

Lemma 1. *The operators $\mathcal{S}_1$ and $\mathcal{S}_2$ defined by (8) and (9) are, respectively, from Λ into $BC(\mathbb{R},\mathbb{R}^n)$; that is, $\mathcal{S}_1, \mathcal{S}_2 : \Lambda \to BC(\mathbb{R},\mathbb{R}^n)$.*

Proof. Let $u \in \Lambda$, we have

$$\begin{aligned}|(\mathcal{S}_1 u)(\xi)| &= \left|\int_{\xi-\tau(\xi)}^{\xi} C(\xi,s)u(s)ds\right| \le \int_{\xi-\tau(\xi)}^{\xi} \|C(\xi,s)\||u(s)|ds \\ &\le \alpha c\|u\| \\ &\le \alpha c M.\end{aligned}$$

Secondly, for $u \in \Lambda$, we get

$$\begin{aligned}|(\mathcal{S}_2 u)(\xi)| &\le \int_{-\infty}^{\infty} \|\Theta(\xi,s)\|\|B(s,s)\|\left(\int_{s-\tau(s)}^{s} \|C(s,r)\||u(r)|dr + |u(s)|\right)ds \\ &\quad + \int_{-\infty}^{\infty} \|\Theta(\xi,s)\| \int_{s-\sigma(s)}^{s} \|B(s,r)\|\|u(r)\|drds \\ &\quad + \int_{-\infty}^{\infty} \|\Theta(\xi,s)\|(q_1|u(s)| + q_2|u(s-\sigma(s))| + |Q(s,0,0)|)ds \\ &\le ((\alpha bc + b + \beta b + q_1 + q_2)M + \gamma) \int_{-\infty}^{\infty} \|\Theta(\xi,s)\|ds \\ &= \mu((\alpha bc + b + \beta b + q_1 + q_2)M + \gamma).\end{aligned} \tag{10}$$

Since all quantities in $\mathcal{S}_1$ and $\mathcal{S}_2$ are periodic, then $\mathcal{S}_1, \mathcal{S}_2 : \Lambda \to BC(\mathbb{R},\mathbb{R}^n)$. □

Lemma 2. *The operator $\mathcal{S}_1 : \Lambda \to BC(\mathbb{R},\mathbb{R}^n)$ given by (8) is a contraction if $\alpha c \in (0,1)$.*

Proof. Let $u, v \in \Lambda$, we get

$$\begin{aligned}|(\mathcal{S}_1 u)(\xi) - (\mathcal{S}_1 v)(\xi)| &= \left|\int_{\xi-\tau(\xi)}^{\xi} C(\xi,s)u(s)ds - \int_{\xi-\tau(\xi)}^{\xi} C(\xi,s)v(s)ds\right| \\ &\le \int_{\xi-\tau(\xi)}^{\xi} \|C(\xi,s)\||u(s)-v(s)|ds \\ &\le \alpha c\|u-v\|.\end{aligned}$$

Then

$$\|\mathcal{S}_1 u - \mathcal{S}_1 v\| \le \alpha c\|u-v\|.$$

Therefore, $\mathcal{S}_1$ is a contraction. □

Lemma 3. *If we assume (5) holds, then the operator $\mathcal{S}_2 : \Lambda \to BC(\mathbb{R},\mathbb{R}^n)$ is completely continuous.*

Proof. We need to prove $\mathcal{S}_2$ is continuous, so let $u_n \in \Lambda$ such that $u_n \to u$ as $n \to \infty$ with n as a positive integer. Hence, by condition (5) we have

$$\begin{aligned}
&|(\mathcal{S}_2 u_n)(\xi) - (\mathcal{S}_2 u)(\xi)| \\
&\leq \int_{-\infty}^{\infty} \|\Theta(\xi,s)\| \|B(s,s)\| \left(\int_{s-\tau(s)}^{s} \|C(s,r)\| |u_n(r) - u(r)| dr + |u_n(s) - u(s)| \right) ds \\
&+ \int_{-\infty}^{\infty} \|\Theta(\xi,s)\| \int_{s-\sigma(s)}^{s} \|B(s,r)\| |u_n(r) - u(r)| ds \\
&+ \int_{-\infty}^{\infty} \|\Theta(\xi,s)\| (q_1 |u_n(s) - u(s)| + q_2 |u_n(s-\sigma(s)) - u(s-\sigma(s))|) ds \\
&\leq \mu(\alpha bc + b + \beta b + q_1 + q_2) \|u_n - u\|.
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} |(\mathcal{S}_2 u_n)(\xi) - (\mathcal{S}_2 u)(\xi)| = 0$$

and by the Dominated Convergence Theorem, we conclude the continuity of $\mathcal{S}_2$.

Next we will prove that the image of the operator $\mathcal{S}_2$ is relatively compact. For any $u_n \in \Lambda$, by (10) we have

$$\|\mathcal{S}_2 u_n\| \leq \mu((\alpha bc + b + \beta b + q_1 + q_2)M + \gamma).$$

A simple calculation of $(\mathcal{S}_2 u_n)'(\xi)$ gives

$$\begin{aligned}
(\mathcal{S}_2 u_n)'(\xi) &= \left(\int_{-\infty}^{\infty} \Theta(\xi,s) B(s,s) \left(\int_{s-\tau(s)}^{s} C(s,r) u_n(r) dr - u_n(s) \right) ds \right)' \\
&+ \left(\int_{-\infty}^{\infty} \Theta(\xi,s) \left(\int_{s-\sigma(s)}^{s} B(s,r) u_n(r) dr + Q(s, u_n(s), u_n(s-\sigma(s))) \right) ds \right)' \\
&= \left(u_n(\xi) - \int_{\xi-\tau(\xi)}^{\xi} C(\xi,s) u_n(s) ds \right)' \\
&= \int_{\xi-\sigma(\xi)}^{\xi} B(\xi,s) u_n(s) ds + Q(\xi, u_n(\xi), u_n(\xi-\sigma(\xi))).
\end{aligned}$$

Then

$$\left\| (\mathcal{S}_2 u_n)' \right\| \leq (\beta b + q_1 + q_2)M + \gamma.$$

Hence, $(\mathcal{S}_2 u_n)$ is equicontinuous and uniformly bounded. According to the Ascoli–Arzela theorem $\mathcal{S}_2(\Lambda)$ is relatively compact. □

In the following Lemma, we prove for any $u, v \in \Lambda$ that $\mathcal{S}_1 u + \mathcal{S}_2 v \in \Lambda$.

Lemma 4. *For any $u, v \in \Lambda$, we have $\mathcal{S}_1 u + \mathcal{S}_2 v \in \Lambda$ since (5) and (6) hold.*

Proof. Let $u, v \in \Lambda$. Then $\|u\|, \|v\| \leq M$. By conditions (5) and (6), we have

$$\begin{aligned}
|(\mathcal{S}_1 u)(\xi) + (\mathcal{S}_2 v)(\xi)| &\leq \alpha c M + \mu((\alpha bc + b + \beta b + q_1 + q_2)M + \gamma) \\
&\leq M,
\end{aligned}$$

it follows that

$$\|\mathcal{S}_1 u + \mathcal{S}_2 v\| \leq M$$

for all $u, v \in \Lambda$. Hence $\mathcal{S}_1 u + \mathcal{S}_2 v \in \Lambda$. □

Theorem 3. *Assume that there is an integrable dichotomy for the system $z'(\xi) = B(\xi,\xi) z(\xi)$, and suppose conditions (5) and (6) hold. Then (1) has a Υ-periodic solution.*

Proof. The assumptions in the Krasnoselskii theorem are satisfied by Lemmas 1–4, so there exists a fixed point $x \in \Lambda$, which is a solution of (1) such that $x = \mathcal{S}_1 x + \mathcal{S}_2 x$. Hence, (1) has a Y-periodic solution. $\square$

Theorem 4. *Assume that system $z'(\xi) = B(\xi, \xi)z(\xi)$ has an integrable dichotomy. If*

$$\alpha c + \mu(b(\alpha c + \beta + 1) + q_1 + q_2) < 1, \tag{11}$$

then system (1) has a unique Y*-periodic solution.*

Proof. Consider the operator $\mathcal{S}$ as the following

$$\begin{aligned}(\mathcal{S}u)(\xi) = & \int_{\xi-\tau(\xi)}^{\xi} C(\xi, s)u(s)ds \\ & + \int_{-\infty}^{\infty} \Theta(\xi, s)B(s, s)\left(\int_{s-\tau(s)}^{s} C(s, r)u(r)dr - u(s)\right)ds \\ & + \int_{-\infty}^{\infty} \Theta(\xi, s)\left(\int_{s-\sigma(s)}^{s} B(s, r)u(r)dr + Q(s, u(s), u(s - \sigma(s)))\right)ds.\end{aligned}$$

For $u_1, u_2 \in BC(\mathbb{R}, \mathbb{R}^n)$, we obtain

$$\begin{aligned}& |(\mathcal{S}u_1)(\xi) - (\mathcal{S}u_2)(\xi)| \\ & \le \int_{\xi-\tau(\xi)}^{\xi} \|C(\xi, s)\||u_1(s) - u_2(s)|ds \\ & + \int_{-\infty}^{\infty} \|\Theta(\xi, s)\|\|B(s, s)\|\left(\int_{s-\tau(s)}^{s} |C(s, r)||u_1(r) - u_2(r)|dr + |u_1(s) - u_2(s)|ds\right) \\ & + \int_{-\infty}^{\infty} \|\Theta(\xi, s)\| \int_{s-\sigma(s)}^{s} \|B(s, r)\||u_1(r) - u_2(r)|drds \\ & + \int_{-\infty}^{\infty} \|\Theta(\xi, s)\|(q_1|u_1(s) - u_2(s)| + q_2|u_1(s - \sigma(s)) - u_2(s - \sigma(s))|) \\ & = (\alpha c + \mu(b(\alpha c + \beta + 1) + q_1 + q_2))\|u_1 - u_2\|.\end{aligned}$$

Since (11) holds, then $\mathcal{S}$ is contraction. Therefore, system (1) has a unique Y-periodic solution. $\square$

4. An Example

Consider the system

$$\left(y(\xi) - \int_{\xi-\frac{1}{2}\sin\frac{\xi}{2}}^{\xi} C(\xi, s)y(s)ds\right)' = \int_{\xi-\frac{1}{3}\cos\frac{\xi}{4}}^{\xi} B(\xi, s)y(s)ds + Q(\xi, y(\xi), y(\xi - \sigma(\xi))) \tag{12}$$

with $n = 2, \mathrm{Y} = 2\pi, y = (y_1, y_2)^{\xi}$,

$$Q(\xi, y(\xi), y(\xi - \sigma(\xi))) = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} + \begin{pmatrix} y_1^2(\xi) \\ y_2^2(\xi) \end{pmatrix} + \frac{1}{15}\begin{pmatrix} y_1\left(\xi - \frac{1}{3}\cos\frac{\xi}{4}\right) \\ y_2\left(\xi - \frac{1}{3}\cos\frac{\xi}{4}\right) \end{pmatrix},$$

$$B(\xi, s) = \begin{pmatrix} \frac{1}{5}\sin(s)\sin(\xi) & \frac{1}{7}\cos(\xi)\cos(s) \\ \frac{1}{5} & -\frac{1}{7} \end{pmatrix},$$

and

$$C(\xi, s) = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2}\sin(\xi - s) & \frac{1}{3}\cos(\xi - s) \end{pmatrix}.$$

If we take the set

$$\Lambda = \{u \in BC(\mathbb{R}, \mathbb{R}^n) : \|u\| \leq 3, u(\xi + 2\pi) = u(\xi) \text{ for all } \xi \in \mathbb{R}\}.$$

Then, we have $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{5}$, $q_1 = 6$, $q_2 = \frac{1}{15}$, and we use $\|A\| = \max_{1\leq j\leq 2} \sum_{i=1}^{2} |a_{ij}|$ to get

$$\begin{aligned} \|C(\xi, s)\| &= \left\| \begin{matrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2}\sin(\xi - s) & \frac{1}{3}\cos(\xi - s) \end{matrix} \right\| \\ &= \max\left\{ \left|\frac{1}{3}\cos(\xi)\right| + \frac{1}{3}, \left|\frac{1}{2}\sin(\xi)\right| + \frac{1}{2} \right\} \\ &\leq 1. \end{aligned}$$

$$\begin{aligned} \|B(\xi, s)\| &= \left\| \begin{pmatrix} \frac{1}{5}\sin(s)\sin(\xi) & \frac{1}{7}\cos(\xi)\cos(s) \\ \frac{1}{5} & -\frac{1}{7} \end{pmatrix} \right\| \\ &= \max\left\{ \left|\frac{1}{5}\sin(s)\sin(\xi)\right| + \frac{1}{5}, \left|\frac{1}{7}\cos(\xi)\cos(s)\right| + \frac{1}{7} \right\} \\ &\leq \frac{2}{5}. \end{aligned}$$

Then $c = 1$ and $b = \frac{2}{5}$.

For $\mu \leq \frac{13}{204}$, Theorem 3 give us 2π-periodic solution not necessarily unique for the system (12).

Now, if $\mu < \frac{15}{204}$, then condition (11) holds, and Theorem 3 gives us the uniqueness of a 2π periodic solution for the system (12).

5. Conclusions

This manuscript dealt with the study of neutral systems containing Volterra terms. The notable thing is that there is no explicit term that we use for an integrable dichotomy, so the kernel $B(\xi, \xi)$ has been assumed to be nonsingular for this purpose. The fixed point theorems of Banach and Krasnoselskii played a pivotal role in proving the existence and uniqueness of periodic solutions.

By the analysis in this paper, our work generalized some previous papers such as [24].

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