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Linear Maps Preserving the Set of Semi-Weyl Operators

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Abstract: Let H be an infinite-dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. In this paper, we characterized the linear maps $\phi : B(H) \to B(H)$, which are surjective up to compact operators preserving the set of left semi-Weyl operators in both directions. As an application, we proved that ϕ preserves the essential approximate point spectrum if and only if the ideal of all compact operators is invariant under ϕ and the induced map φ on the Calkin algebra is an automorphism. Moreover, we have $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm.

Keywords: left semi-Weyl operator; Calkin algebra; linear preservers

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1. Introduction

Let H be an infinite-dimensional separable complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on H, and $\mathcal{K}(H) \subseteq B(H)$ the closed ideal of all compact operators. For an operator $T \in B(H)$, we write T^* for the conjugate operator of T, $N(T)$ for its kernel, and $R(T)$ for its range. The dimension, codimension, and index of T are denoted by $dimT, codimT$, and $indT$, respectively.

An operator $T \in B(H)$ is called upper semi-Fredholm if $R(T)$ is closed and $N(T)$ is finite- dimensional. If $R(T)$ is closed and finite-codimensional, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if $R(T)$ is closed and finite-codimensional and $N(T)$ is finite-dimensional. For a semi-Fredholm operator (upper semi-Fredholm operator or lower semi-Fredholm operator), let $n(T) = dimN(T)$ and $d(T) = dimH/R(T) = codimR(T)$. The index of a semi-Fredholm operator $T \in B(H)$ is given by $ind(T) = n(T) - d(T)$. The operator T is Weyl if it is Fredholm of index zero. $T \in B(H)$ is called left (right) semi-Weyl if T is upper (lower) semi-Fredholm with $ind(T) \leq 0$ ($ind(T) \geq 0$). Let $SF_+^-(H)$ denote the set of all left semi-Weyl operators. For an operator $T \in B(H)$, the spectrum $\sigma(T)$, the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the essential approximate point spectrum $\sigma_{ea}(T)$ of T are defined by $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$, $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$, $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$, and $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left semi} - \text{Weyl}\}$, respectively.

Let $\Phi(H) \subseteq B(H)$ be the set of all Fredholm operators. We denote the Calkin algebra $B(H)/\mathcal{K}(H)$ by $\mathcal{C}(H)$. Let $\pi : B(H) \to \mathcal{C}(H)$ be the quotient map. It is well known that $T \in \Phi(H)$ if and only if $\pi(T)$ is invertible in $\mathcal{C}(H)$.

A bijective linear map $\phi : B(H) \to B(H)$ is called a Jordan isomorphism if $\phi(A^2) = (\phi(A))^2$ for every $A \in B(H)$ or, equivalently, $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$ for all A and B in $B(H)$. It is obvious that every isomorphism and every anti-isomorphism is a Jordan isomorphism. For further properties of Jordan homomorphisms, we refer the reader to [1,2].

In the last two decades, there has been considerable interest in the so-called linear preserver problems (see the survey articles [3–5]). The goal of studying linear preservers is to give structural characterizations of linear maps on algebras having some special properties such as leaving invariant a certain subset of the algebra or leaving invariant a certain function on the algebra. One of the most-famous problems in this direction is Kaplansky's problem ([6]): Let ϕ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that ϕ is a Jordan isomorphism? This problem was first solved in the finite-dimensional case. Dieudonné ([7]) and Marcus and Purves ([8]) proved that every unital invertibility preserving linear map on a complex matrix algebra is either an inner automorphism or a linear anti-automorphism. This result was later extended to the algebra of all bounded linear operators on a Banach space by Sourour ([9]) and to von Neumann algebra by Aupetit ([10]). Many linear preserver problems have been of interest for infinite-dimensional cases. For the most-significant partial results relevant to our discussions, we refer the reader to [9–11]. New contributions to the study of the linear preserver problem have been recently made by Mbekhta in [12], Alizadeh and Shakeri in [13], Bueno, Furtado, and Sivakumar in [14], Buenoa, Furtadob, Klausmeierc, and Veltrid in [15], and Bendaoud, Bourhim and Sarih in [16].

In this article, we studied linear maps preserving left (right) semi-Weyl operators in both directions. We characterized the linear maps $\phi : B(H) \to B(H)$, which are surjective up to compact operators preserving the set of semi-Weyl operators in both directions. As an application, we proved that ϕ preserves the essential approximate point spectrum if and only if the ideal of all compact operators is invariant under ϕ, the induced map φ on the Calkin algebra is an automorphism, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm.

2. Linear Maps Preserving the Set of Left (Right) Semi-Weyl Fredholm Operators

We say that a linear map ϕ preserves property X in both directions, which means that if T is in the domain, then T has property X if and only if $\phi(T)$ has property X. Therefore, a linear map $\phi : B(H) \to B(H)$ preserves the set of left semi-Weyl operators in both directions if $T \in SF_+^-(H) \Leftrightarrow \phi(T) \in SF_+^-(H)$.

A linear map $\phi : B(H) \to B(H)$ is said to be surjective up to compact operators if, for every $T \in B(H)$, there exists $T' \in B(H)$ such that $T - \phi(T') \in \mathcal{K}(H)$. It is clear that if ϕ is surjective, then it is surjective up to compact operators.

In order to prove the theorem and the corollaries, we need some known results.

Lemma 1 (Theorem 4.2 in [5])**.** *Let H be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ be a linear map surjective up to compact operators. Then, the following are equivalent:*

(1) ϕ preserves upper semi-Fredholm operators in both directions;
(2) ϕ preserves lower semi-Fredholm operators in both directions;
(3) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$, and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H), \varphi \circ \pi = \pi \circ \phi$ is an automorphism multiplied by an invertible element $a \in \mathcal{C}(H)$.

Lemma 2 (Theorem 2.1 in [12])**.** *Let H be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ be a linear map surjective up to compact operators. Then, the following are equivalent:*

(1) ϕ preserves the set of Fredholm operators in both directions;
(2) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$, and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$, is the composition of either an automorphism or an anti-automorphism and left multiplication by an invertible element in $\mathcal{C}(H)$.

Lemma 3 (Theorem 4.8 in [3])**.** *Let A be a factor, and let B be a primitive Banach algebra. For a surjective up to inessential elements linear map $\phi : A \to B$, the following are equivalent:*

(1) ϕ preserves Fredholm elements in both directions and $\phi(I)$ is the Weyl element of B;
(2) ϕ preserves Weyl elements in both directions;

(3) *Let $\mathcal{I}(A)$ and $\mathcal{I}(B)$ be the ideal of the inessential elements of A and B. Then, $\phi(\mathcal{I}(A)) \subseteq \mathcal{I}(B)$, and the induced map $\varphi : \mathcal{C}(A) \to \mathcal{C}(B)$ is either an isomorphism or an anti-isomorphism multiplied by an invertible element $a \in B$.*

Lemma 4 (Theorem 3.1 in [4]). *Let A be a unital C*-algebra of real rank zero and B a unital semi-simple complex Banach algebra. Let $\Delta(\cdot)$ denote any one of the spectral functions $\sigma(\cdot), \sigma_l(\cdot), \sigma_r(\cdot), \sigma_l(\cdot) \cap \sigma_r(\cdot), \partial\sigma(\cdot)$, and $\eta\sigma(\cdot)$. Suppose $\phi : A \to B$ is a surjective linear map. If $\Delta(\phi(T)) \subseteq \Delta(T)$ for every $T \in A$, then ϕ is a Jordan homomorphism. Furthermore, if B is prime, then ϕ is either a homomorphism or an anti-homomorphism.*

Theorem 1. *Let H be an infinite-dimensional Hilbert space, and let $\phi : B(H) \to B(H)$ be a linear map preserving left (or right) semi-Weyl operators in both directions. Assume that ϕ is surjective up to compact operators and $\phi(I)$ is Weyl, then $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$, and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$ is an automorphism multiplied by an invertible element $[B] \in \mathcal{C}(H)$.*

Proof. Suppose that $\phi : B(H) \to B(H)$ is a linear map preserving left semi-Weyl operators in both directions. Let $\phi(I) = G + K_0$, where $G \in B(H)$ is invertible and $K_0 \in \mathcal{K}(H)$. There exists $B_0 \in B(H)$ such that $GB_0 = B_0G = I$.

The linear map $\phi_1 : B(H) \to B(H)$ is defined by:

$$\phi_1(T) = B_0\phi(T), \forall T \in B(H).$$

Then, ϕ_1 preserves the left semi-Weyl operators in both directions and $\phi_1(I) = I + K_1$, where $K_1 \in \mathcal{K}(H)$. Let us give some properties for the linear map ϕ_1: (i) ϕ_1 is surjective up to compact operators.

In fact, for any $T \in B(H)$, there exists $T' \in B(H)$ and $K_2 \in \mathcal{K}(H)$ such that $GT = \phi(T') + K_2$. Then, $T = B_0GT = B_0\phi(T') + K_3 = \phi_1(T') + K_3$, where $K_3 = B_0K_2 \in \mathcal{K}(H)$.

(ii) For any $T \in B(H)$, $\sigma_{ea}(T) = \sigma_{ea}(\phi_1(T))$.

Since $T - \lambda I \in SF_+^-(H) \Leftrightarrow \phi_1(T - \lambda I) = \phi_1(T) - \lambda\phi_1(I) = \phi_1(T) - \lambda I - \lambda K_1 \in SF_+^-(H) \Leftrightarrow \phi_1(T) - \lambda I \in SF_+^-(H)$, it follows that $\sigma_{ea}(T) = \sigma_{ea}(\phi_1(T))$ for any $T \in B(H)$.

(iii) ϕ_1 preserves compact operators in both directions.

First, we claim that

$$\mathcal{K}(H) = \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\}$$

$$= \{K \in B(H) : \sigma_{ea}(T + K) = \sigma_{ea}(T) \text{ for all } T \in SF_+^-(H)\}.$$

From the stability properties of the index function, it is clear that $\mathcal{K}(H) \subseteq \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\} = \{K \in B(H) : \sigma_{ea}(T + K) = \sigma_{ea}(T) \text{ for all } T \in SF_+^-(H)\}$.

Let ∂E and ηE denote the boundary and the polynomial convex hull of a compact subset E of $\mathbb{C}$, respectively. For any $T \in B(H)$, since

$$\partial\sigma_w(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_e(T) \subseteq \sigma_w(T) \text{ and } \partial\sigma_w(T) \subseteq \partial\sigma_{ea}(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_w(T),$$

it follows that $\eta\sigma_{ea}(T) = \eta\sigma_w(T) = \eta\sigma_e(T)$.

Now, let $K \in B(H)$ such that $\sigma_{ea}(T + K) = \sigma_{ea}(T)$ for all $T \in B(H)$. Then, $\eta\sigma_e(T + K) = \eta\sigma_e(T)$ for all $T \in B(H)$. Taking into account the semisimplicity of $\mathcal{C}(H)$ and the spectral characterization of the radical, it is not difficult to prove that $\mathcal{K}(H) = \{K \in B(H) : K + SF_+^-(H) \in SF_+^-(H)\} = \{K \in B(H) : \sigma_{ea}(T + K) = \sigma_{ea}(T) \text{for all } T \in SF_+^-(H)\}$.

Let $K \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$; since ϕ_1 preserves left semi-Weyl operators in both directions, there exists $T' \in SF_+^-(H)$ and $K' \in \mathcal{K}(H)$ for which $T = \phi_1(T') + K'$. Hence, $T + \phi_1(K) = \phi_1(T') + K' + \phi_1(K) = \phi_1(T' + K) + K' \in SF_+^-(H)$. Then, $\phi_1(K) \in \mathcal{K}(H)$. For the converse, let $\phi_1(K) \in \mathcal{K}(H)$, for any $T \in SF_+^-(H)$, $\phi_1(T + K) = \phi_1(T) + \phi_1(K) \in SF_+^-(H)$, then $T + K \in SF_+^-(H)$. It follows that $K \in \mathcal{K}(H)$. Now, we prove that ϕ_1 preserves compact operators in both directions.

(iv) $N(\phi_1) \subseteq \mathcal{K}(H)$, and consequently, $N(\phi) \subseteq \mathcal{K}(H)$.

If $K \in N(\phi_1)$ and $T \in SF_+^-(H)$, then $\phi_1(T+K) = \phi_1(T) \in SF_+^-(H)$. Thus, for all $T \in SF_+^-(H)$, $T + K \in SF_+^-(H)$. From the proof of (iii), we know that $K \in \mathcal{K}(H)$.

(v) Let $\varphi_1 : \mathcal{C}(H) \to \mathcal{C}(H)$ be an induced linear map such that $\phi_1 \circ \pi = \pi \circ \phi_1$, then φ_1 is an isomorphism or an anti-isomorphism.

From the fact that $\mathcal{K}(H)$ is invariant under ϕ_1, then ϕ_1 induces a linear map $\varphi_1 : \mathcal{C}(H) \to \mathcal{C}(H)$ such that $\varphi_1 \circ \pi = \pi \circ \phi_1$. Clearly, φ_1 is surjective, since ϕ_1 is surjective up to compact operators. We prove now that φ_1 is injective. Since $N(\varphi_1) = \pi(N(\phi_1))$ and $N(\phi_1) \subseteq \mathcal{K}(H)$, we can obtain that φ_1 is injective.

From (ii), we know that, for any $T \in B(H)$, $\eta\sigma_{ea}(T) = \eta\sigma_{ea}(\phi_1(T))$. Then, from (iii), $\eta\sigma_e(T) = \eta\sigma_e(\phi_1(T))$. This shows that ϕ_1 is an $\eta\sigma_e$-preserving map. Thus, the induced mapping φ_1 is an $\eta\sigma$-preserving map. By Lemma 4, φ_1 is either an isomorphism or an anti-isomorphism.

(vi) φ_1 is an isomorphism.

First, we will prove that ϕ_1 preserves upper semi-Fredholm operators in both directions. By Lemma 2, we know that ϕ_1 preserves Fredholm operators in both directions. Let $T \in B(H)$ be an upper semi-Fredholm; there are two cases to consider: $d(T) = \infty$ and $d(T) < \infty$. If $d(T) = \infty$, using the fact that $\phi_1 : B(H) \to B(H)$ is a linear map preserving left semi-Weyl operators in both directions, we know that $\phi_1(T)$ is upper semi-Fredholm. If $d(T) < \infty$, then T is Fredholm; thus, $\phi_1(T)$ is Fredholm since ϕ_1 preserves Fredholm operators in both directions. Using the same way, we can prove that T is upper semi-Fredholm if $\phi_1(T)$ is upper semi-Fredholm. By Lemma 1, φ_1 is an isomorphism.

From the definition of ϕ_1, we know that ϕ preserves compact operators in both directions, and hence, $\mathcal{K}(H)$ is invariant under ϕ. Let ϕ induce a linear map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ such that $\varphi \circ \pi = \pi \circ \phi$. Then, $\varphi = [B]^{-1}\varphi_1$.

Similar to the above proof, the result is true if ϕ is a linear map preserving right semi-Weyl operators in both directions. The proof is completed. $\square$

Under the same hypothesis and notation as in Theorem 1, we obtain that ϕ_1 preserves the essential spectrum ([12], Theorem 3.2). Then, $ind(\phi(T)) = ind(T)$ or $ind(\phi(T)) = -ind(T)$ for any $T \in \Phi(H)$. Since ϕ_1 preserves left semi-Weyl operators in both directions, it follows that $ind(\phi(T)) \cdot ind(T) \geq 0$ for any $T \in \Phi(H)$. Thus, $ind(\phi(T)) = ind(T)$ for any $T \in \Phi(H)$. Furthermore, we can prove that $ind(\phi(T)) = ind(T)$ for any upper (lower) semi-Fredholm operator $T \in B(H)$. By Lemma 1, Lemma 2, and Lemma 3, we can obtain:

Corollary 1. *Let $\phi : B(H) \to B(H)$ be a linear map preserving left (right) semi-Weyl operators in both directions. Assume that ϕ is surjective up to compact operators and $\phi(I)$ is Weyl, then:*

(1) ϕ preserves Fredholm operators in both directions;
(2) ϕ preserves Weyl operators in both directions;
(3) ϕ preserves upper semi-Fredholm operators in both directions;
(4) ϕ preserves lower semi-Fredholm operators in both directions;
(5) ϕ preserves semi-Fredholm operators in both directions;
(6) For any $T \in \Phi(H)$, $ind(\phi(T)) = ind(T)$;
(7) For any upper (lower) semi-Fredholm operator T, $ind(\phi(T)) = ind(T)$.

Remark 1. *If $\phi : B(H) \to B(H)$ is a linear map preserving Fredholm operators (or upper semi-Fredholm operators, or lower semi-Fredholm operators, or semi-Fredholm operators) in both directions, we cannot induce that ϕ is a linear map preserving left semi-Weyl operators in both directions. For example, let $A, B \in B(\ell_2)$ be defined by:*

$$A(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots), \quad B(x_1, x_2, x_3, \cdots) = (0, 0, 0, x_1, x_2, \cdots),$$

then there exists $A_1, B_1 \in B(\ell_2)$ such that $AA_1 = B_1B = I$. Define $\phi : B(\ell_2) \to B(\ell_2)$ as $\phi(T) = ATB$, $T \in B(\ell_2)$. We can see that ϕ is surjective and preserves Fredholm operators *(upper semi-Fredholm operators, lower semi-Fredholm operators, semi-Fredholm operators) in both directions, but ϕ is not a linear map preserving left semi-Weyl operators in both directions.*

From Remark 1, we have the question: If $\phi : B(H) \to B(H)$ is a linear map preserving Fredholm operators (or upper semi-Fredholm operators, or lower semi-Fredholm operators, or semi-Fredholm operators) in both directions, when does ϕ preserve left semi-Weyl operators in both directions. To answer this question, let us begin by a Lemma (Lemma 2.4 in [5]).

Lemma 5. *Let $A \in B(H)$ be a lower (respectively upper) semi-Fredholm. If A is not Fredholm, then there exists a lower (respectively upper) semi-Fredholm operator B such that every non-trivial linear combination $\lambda A + \mu B$, $\lambda \neq 0$ or $\mu \neq 0$, is lower (respectively upper) semi-Fredholm, but not Fredholm.*

Corollary 2. *Let $\phi : B(H) \to B(H)$ be a linear map preserving left (right) semi-Weyl operators in both directions. Assume that ϕ is surjective up to compact operators, then $\phi(I)$ is a Fredholm operator.*

Proof. Denote $\phi(I) = T$. We will prove that T is Fredholm. On the contrary, we assumed that this is not the case. Since I is a left semi-Weyl operator, T must be a left semi-Weyl operator. Then, by Lemma 5, there exists $S \in B(H)$ such that $\lambda T - S$ is upper semi-Fredholm, but not Fredholm, which means that $\lambda T - S$ is left semi-Weyl. We can further find $R \in B(H)$ such that $\phi(R) = S + K$ for some $K \in \mathcal{K}(H)$. Any compact perturbation of a left semi-Weyl operator is a left semi-Weyl operator; thus, $\lambda T - \phi(R) = \phi(\lambda I - R)$ is left semi-Weyl for every $\lambda \in \mathbb{C}$. As $\phi : B(H) \to B(H)$ is a linear map preserving left semi-Weyl operators in both directions, we obtain that $\sigma_{ea}(R) = \emptyset$, a contradiction. □

Corollary 3. *Let linear map $\phi : B(H) \to B(H)$ be surjective up to compact operators, then the following statements are equivalent:*

(1) *ϕ preserves left semi-Weyl operators in both directions, and $\phi(I)$ is Weyl;*
(2) *ϕ preserves left semi-Weyl operators in both directions, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm;*
(3) *ϕ preserves right semi-Weyl operators in both directions, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm;*
(4) *ϕ preserves Fredholm operators in both directions, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are semi-Fredholm;*
(5) *ϕ preserves upper semi-Fredholm operators in both directions, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are upper semi-Fredholm;*
(6) *$\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$; the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$ is an automorphism multiplied by an invertible element $[B] \in \mathcal{C}(H)$, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm.*

Proof. By the proof of Theorem 1 and Corollary 1, we only need to prove that (6) $\Rightarrow$ (1). By Lemma 1, we know that ϕ preserves upper semi-Fredholm operators and Fredholm operators in both directions. Let $T \in SF_+^-(H)$, then $\phi(T)$ is upper semi-Fredholm. If $d(T) = \infty$, then $d(\phi(T)) = \infty$ because ϕ preserves Fredholm operators in both directions, thus $\phi(T) \in SF_+^-(H)$. If $d(T) < \infty$, then $\phi(T)$ is Fredholm, and hence, $ind(\phi(T)) = ind(T) \leq 0$, again $\phi(T) \in SF_+^-(H)$. Using the same way, we can prove that $T \in SF_+^-(H)$ if $\phi(T) \in SF_+^-(H)$. This proves that ϕ preserves left semi-Weyl operators in both directions. Thus, $\phi(I)$ is Fredholm. Since both $\phi(I)$ and I are Fredholm, it follows that $ind(\phi(I)) = ind(I) = 0$. Then, $\phi(I)$ is Weyl. □

Let $\phi : B(H) \to B(H)$ be surjective up to compact operators. If ϕ preserves left semi-Weyl operators in both directions and $\phi(I)$ is Weyl, we cannot induce that ϕ is σ_{ea}-preserving. For example, let $A_1, B_1 \in B(\ell_2)$ be defined by:

$$A_1(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots), \quad B_1(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots),$$

and define $A = \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}$ and $B = \begin{pmatrix} I & 0 \\ 0 & B_1 \end{pmatrix}$. Let $\chi : B(\ell_2 \oplus \ell_2) \to \mathcal{K}(\ell_2 \oplus \ell_2)$ be a linear map, and consider the linear map $\phi : B(\ell_2 \oplus \ell_2) \to B(\ell_2 \oplus \ell_2)$ defined by $\phi(T) = ATB + \chi(T)$. Then, ϕ is surjective up to compact operators and preserves the set of left semi-Weyl operators in both directions; also, $\phi(I)$ is Weyl. According to the calculation, we obtain that $\sigma_{ea}(I) = \{1\}$, while $\sigma_{ea}(\phi(T)) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. This says that ϕ is not σ_{ea}-preserving. There is a question: When does a map satisfying the hypothesis of Theorem 1 preserve the essential approximate point spectrum?

Corollary 4. *Let H be an infinite-dimensional Hilbert space, and let $\phi : B(H) \to B(H)$ be a linear map. Assume that ϕ is surjective up to compact operators, then the following statements are equivalent:*

(1) ϕ preserves left semi-Weyl operators in both directions and $I - \phi(I) \in \mathcal{K}(H)$;
(2) ϕ preserves right semi-Weyl operators in both directions and $I - \phi(I) \in \mathcal{K}(H)$;
(3) ϕ is σ_{ea}-preserving, i.e., $\sigma_{ea}(\phi(T)) = \sigma_{ea}(T)$ for all $T \in B(H)$;
(4) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$; the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$ is an automorphism, and $ind(\phi(T)) = ind(T)$ if both $\phi(T)$ and T are Fredholm.

Proof. In view of the preceding theorem and corollaries, we only need to prove the equivalence of (1) and (3). Suppose that ϕ preserves the left semi-Weyl operators in both directions and $I - \phi(I) \in \mathcal{K}(H)$. Let $\phi(I) = I + K_0$, $K_0 \in \mathcal{K}(H)$. Since $T - \lambda I \in SF_+^-(H) \Leftrightarrow \phi(T - \lambda I) = \phi(T) - \lambda\phi(I) = \phi(T) - \lambda I - \lambda K_0 \in SF_+^-(H) \Leftrightarrow \phi(T) - \lambda I \in SF_+^-(H)$, it follows that $\sigma_{ea}(T) = \sigma_{ea}(\phi(T))$ for any $T \in B(H)$. For the converse, suppose that $\sigma_{ea}(\phi(T)) = \sigma_{ea}(T)$ for all $T \in B(H)$, then ϕ preserves the left semi-Weyl operators in both directions. We need to prove that $I - \phi(I) \in \mathcal{K}(H)$. Put $K = \phi(I) - I$. Let $T \in B(H)$, $T' \in B(H)$, and $K' \in \mathcal{K}(H)$ for which $T = \phi(T') + K'$ (ϕ is surjective up to compact operators). Then, $\sigma_{ea}(T) = \sigma_{ea}(\phi(T') + K') = \sigma_{ea}(\phi(T')) = \sigma_{ea}(T')$ and

$$\begin{aligned} & \sigma_{ea}(T + K) = \sigma_{ea}(T + \phi(I) - I) = \sigma_{ea}(T + \phi(I)) - 1 \\ = \; & \sigma_{ea}(\phi(T') + \phi(I) + K') - 1 = \sigma_{ea}(\phi(T' + I)) - 1 \\ = \; & \sigma_{ea}(T' + I) - 1 = \sigma_{ea}(T') = \sigma_{ea}(T), \end{aligned}$$

This gives $\sigma_{ea}(T + K) = \sigma_{ea}(T)$ for all $T \in B(H)$. It follows from the proof of Theorem 1 that $K \in B(H)$ is compact. □

Let $SW(H) = \{T \in B(H) : T$ be left semi-Weyl or right semi-Weyl$\}$. Define the semi-Weyl spectrum $\sigma_{SW}(T)$ of an operator $T \in B(H)$ as $\sigma_{SW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SW(H)\}$. Similar to the proof of Theorem 1, we have that $\mathcal{K}(H) = \{K \in B(H) : K + SW(H) \in SW(H)\} = \{K \in B(H) : \sigma_{SW}(T + K) = \sigma_{SW}(T)$ for all $T \in SW(H)\}$. We can prove the following:

Corollary 5. *Let H be an infinite-dimensional Hilbert space, and let $\phi : B(H) \to B(H)$ be a linear map. Assume that ϕ is surjective up to compact operators, then the following statements are equivalent:*

(1) ϕ preserves semi-Weyl operators in both directions, and $I - \phi(I) \in \mathcal{K}(H)$;
(2) ϕ is σ_{SW}-preserving, i.e., $\sigma_{SW}(\phi(T)) = \sigma_{SW}(T)$ for all $T \in B(H)$;
(3) ϕ preserves semi-Fredholm operators in both directions, and $I - \phi(I) \in \mathcal{K}(H)$;
(4) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$; the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$ is an automorphism or an anti-isomorphism.

We conclude this paper by a natural conjecture that we have been unable to answer:

Conjecture 1. *Let H be an infinite-dimensional Hilbert space, and let $\phi : B(H) \to B(H)$ be a linear map. Assume that ϕ is surjective up to compact operators, then the following statements are equivalent:*

(1) *ϕ preserves the essential approximate point spectrum;*
(2) *There exists $\psi : B(H) \to B(H)$ an automorphism and there exists $\chi : B(H) \to \mathcal{K}(H)$ a linear map such that $\phi(T) = \psi(T) + \chi(T)$ for every $T \in B(H)$;*
(3) *$\phi(T) = ATA^{-1} + \chi(T)$ for every $T \in B(H)$, where A is an invertible operator in $B(H)$ and $\chi : B(H) \to \mathcal{K}(H)$ is a linear map.*

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References

1. Herstein, I.N. Jordan homomorphisms. *Trans. Am. Math. Soc.* **1956**, *81*, 331–341. [CrossRef]
2. Jacobson, N.; Rickart, C.E. Jordan homomorphisms of rings. *Trans. Am. Math. Soc.* **1950**, *69*, 479–502. [CrossRef]
3. Bendaoud, M.; Bourhim, A.; Burgos, M.; Sarih, M. Linear maps preservig Fredholm and Atkinson elements of C^*-algebra. *Linear Multilinear Algebra* **2009**, *57*, 823–838. [CrossRef]
4. Cui, J.; Hou, J. Linear maps between Banach algebras compressing certain spectral functions. *Rocky Mt. J. Math.* **2004**, *34*, 565–585.
5. Mbekhta, M.; Šemrl, P. Linear maps preserving semi-Fredholm operators and generalized invertibility. *Linear Multilinear Algebra* **2009**, *57*, 55–64. [CrossRef]
6. Kaplansky, I. Algebraic and analytic aspects of opertors algebras. In *CBMS Regional Conference Series in Math. 1*; American Mathematical Society: Providence, RI, USA, 1970.
7. Dieudonné, J. Sur une généralisation du groupe orthogonal à quatre variables. *Arch. Math.* **1949**, *1*, 282–287. [CrossRef]
8. Marcus, M.; Purves, R. Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions. *Canad. J. Math.* **1959**, *11*, 383–396. [CrossRef]
9. Sourour, A.R. Invertibility preserving linear maps on L(H). *Trans. Am. Math. Soc.* **1996**, *348*, 13–30. [CrossRef]
10. Aupetit, B. Spectrum-preserving linear mapping between Banach algebra or Jordan-Banach algebras. *J. Lond. Math. Soc.* **2000**, *62*, 917–924. [CrossRef]
11. Želozko, W. Characterization of multiplicative linear functionals in complex Banach algebras. *Stud. Math.* **1968**, *30*, 83–85.
12. Mbekhta, M. Linear maps preserving the set of Fredholm operators. *Proc. Am. Math. Soc.* **2007**, *135*, 3613–3619. [CrossRef]
13. Alizadeh, R.; Shakeri, F. Linear maps preserving Pareto eigenvalues. *Linear Multilinear Algebra* **2017**, *65*, 1053–1061. [CrossRef]
14. Bueno, M.I.; Furtado, S.; Sivakumar, K.C. Linear maps preserving the Lorentz-cone spectrum in certain subspaces of M_n. *Banach J. Math. Anal.* **2021**, *15*, 58. [CrossRef]
15. Buenoa, M.I.; Furtadob, S.; Klausmeierc, A.; Veltrid, J. Linear maps preserving the Lorentz spectrum: The 2×2 case. *Electron. J. Linear Algebra* **2022**, *38*, 317–330. [CrossRef]
16. Bendaoud, M.; Bourhim, A.; Sarih, M. Linear maps preservig the essential spectral radius. *Linear Alg. Appl.* **2008**, *428*, 1041–1045. [CrossRef]