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*Article*

# Quarter-Symmetric Metric Connection on a Cosymplectic Manifold

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**Abstract:** We study the quarter-symmetric metric $A$-connection on a cosymplectic manifold. Observing linearly independent curvature tensors with respect to the quarter-symmetric metric $A$-connection, we construct the Weyl projective curvature tensor on a cosymplectic manifold. In this way, we obtain new conditions for the manifold to be projectively flat. At the end of the paper, we define $\eta$-Einstein cosymplectic manifolds of the $\theta$-th kind and prove that they coincide with the $\eta$-Einstein cosymplectic manifold.

**Keywords:** almost-contact manifold; cosymplectic manifold; co-Kähler manifold; quarter-symmetric connection; $\eta$-Einstein manifold

**MSC:** 53B05; 53B35; 53C05; 53C15

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## 1. Introduction

This paper deals with almost-contact metric manifolds and cosymplectic manifolds. A cosymplectic manifold is an almost-contact metric manifold with a normality condition and with the 2-form $F$ and 1-form $\eta$ both closed, according to Blair's definition [1]. Lately, the name co-Kähler manifolds has also been used for such manifolds, since they are odd-dimensional analogs of Kähler manifolds [2]. A trivial example of cosymplectic manifolds is given by a product of a Kähler manifold with a circle or line (for instance, see [1,3]). Moreover, there is an example of a compact cosymplectic manifold that is not a global product of a compact Kähler manifold with a circle [4]. In [5], the author studied contact, concircular, recurrent, and torse-forming vector fields on cosymplectic manifolds. Note that a different definition of cosymplectic manifolds was used in some papers (for instance, see [2,6,7]).

Here, we investigate the application of quarter-symmetric metric connections on almost-contact metric manifolds and cosymplectic manifolds. The quarter-symmetric connection in differentiable manifolds was introduced by S. Golab [8]. The systematic study of the quarter-symmetric metric connection was continued by S. C. Rastogi in [9,10]. Many authors studied the quarter-symmetric metric connection on almost-contact metric manifolds and their special manifolds. The properties of the torsion tensor of the quarter-symmetric metric connection on almost-contact metric manifolds were studied in [11]. In [12], the authors studied the existence of almost-pseudo-symmetric and Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection. The existence of weakly symmetric and weakly Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection was studied in [13]. The papers [14–16] are devoted to studying some special types of $K$-contact manifolds with respect to the quarter-symmetric metric connection.

Among other things, the $*$-conformal $\eta$-Ricci–Yamabe soliton admitting a quarter-symmetric metric connection on $\alpha$-cosymplectic manifolds was studied in [17]. If $\alpha = 0$, then the $\alpha$-cosymplectic manifold reduces to the cosymplectic manifold. Additionally, if the characteristic vector field $\xi$ is projective on an $\alpha$-cosymplectic manifold, then it is a cosymplectic manifold [18]. On the other hand, if $\alpha \in \mathbb{R} \setminus \{0\}$, then the $\alpha$-cosymplectic manifold is $\alpha$-Kenmotsu (see [19]). On the $\alpha$-Kenmotsu manifold, the characteristic vector field $\xi$ is never projective [18].

If $\mathcal{M}$ is an $n$-dimensional locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection, then the scalar curvature of the Levi–Civita connection of $\mathcal{M}$ is a negative constant (see Theorem 2 in [20]). In the 3-dimensional Kenmotsu manifold, the $\eta$-parallel and cyclic-parallel Ricci tensor with respect to the quarter-symmetric metric connection and the Levi–Civita connection are equivalent [21].

The present paper can be considered a continuation of [22,23], where some curvature properties of quarter-symmetric metric connections on a generalized Riemannian manifold and Kähler manifold were studied. Our goal is to continue to determine new results and geometric structures on cosymplectic manifolds, as well as to apply these results to obtain special examples of the mentioned manifold.

## 2. Almost-Contact Metric Manifolds

Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold, where $\mathcal{M}$ is an $n$-dimensional differentiable manifold, $G$ is a non-symmetric (0,2) tensor (the so-called generalized Riemannian metric), $g$ is the symmetric part of $G$, and $F$ is the skew-symmetric part of $G$. The tensor $A$ is defined as a tensor associated with the tensor $F$, i.e.,

$$F(X,Y) = g(AX,Y). \tag{1}$$

Depending on the properties of (1,1) tensor $A$, we can observe various examples of the generalized Riemannian manifold, such as the almost-Hermitian, almost-para-Hermitian, almost-contact, and almost-para-contact manifolds (see [24]).

An almost-contact metric manifold$(\mathcal{M}, g, A, \eta, \xi)$ is an $n$-dimensional differentiable manifold $\mathcal{M}$ (where $n = 2k+1$) equipped with an almost-contact structure $A$ and a characteristic (or Reeb) vector field $\xi$ dual to $\eta$ with respect to $g$, $\eta(\xi) = 1$, $\eta(X) = g(X,\xi)$, which satisfies

$$A^2 = -I + \eta \otimes \xi, \quad A\xi = 0, \quad \eta \circ A = 0 \tag{2}$$

and

$$g(AX,AY) = g(X,Y) - \eta(X)\eta(Y). \tag{3}$$

The symmetric metric $g$ that satisfies the previous relationship is called compatibly metric with the almost-contact structure. The fundamental 2-form $F$, defined by (1), is a degenerate of $F(X,\xi) = 0$ and has a rank of $2k$. It can be easily shown that the generalized metric $G = g + F$ and the fundamental 2-form $F$ satisfy the following relationships:

$$\begin{aligned}
G(X,\xi) = \eta(X), \quad & G(\xi,\xi) = 1, \\
F(AX,Y) = -F(X,AY), \quad & F(AX,AY) = F(X,Y), \\
G(AX,Y) = -G(X,AY), \quad & G(AX,AY) = G(X,Y) - \eta(X)\eta(Y).
\end{aligned}$$

An almost-contact manifold is said to be normal if the corresponding complex structure on $\mathcal{M} \times \mathbb{R}$ is integrable, which is equivalent to the condition $N^{ac} = N + \mathrm{d}\eta \otimes \xi = 0$, where $N$ denotes the Nijenhuis tensor of structure tensor $A$ and d denotes the exterior derivative. An almost-contact metric manifold is said to be an almost-cosymplectic manifold if the 2-form $F$ and 1-form $\eta$ are both closed, i.e., $\mathrm{d}F = 0$ and $\mathrm{d}\eta = 0$ [3]. If an almost-cosymplectic manifold is normal, then it is called a cosymplectic (or a co-Kähler) manifold [2,3]. An almost-contact metric manifold is cosymplectic if and only if $\overset{g}{\nabla} A = 0$ (for instance, see p. 95 in [25]).

The present paper deals with applying quarter-symmetric connections on almost-contact metric manifolds. A linear connection $\overset{1}{\nabla}$ is said to be quarter-symmetric if its torsion tensor is of the form

$$\overset{1}{T}(X,Y) = \eta(Y)AX - \eta(X)AY.$$

The quarter-symmetric connection $\overset{1}{\nabla}$ preserving the generalized Riemannian metric $G$, $\overset{1}{\nabla}G = 0$, is called the *quarter-symmetric G-metric connection*, and it is determined by the following equations (see [22]):

$$\overset{1}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \eta(X)AY \tag{4}$$

and

$$\overset{1}{\nabla} g = 0, \quad \overset{1}{\nabla} A = \overset{g}{\nabla} A = 0, \tag{5}$$

where $\overset{g}{\nabla}$ is a Levi–Civita connection. The symmetric connection $\overset{0}{\nabla}$ and the dual connection $\overset{2}{\nabla}$ of the quarter-symmetric connection (4) are given by

$$\overset{0}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \frac{1}{2}\eta(X)AY - \frac{1}{2}\eta(Y)AX, \tag{6}$$

$$\overset{2}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \eta(Y)AX. \tag{7}$$

In [24], it is proved that for the $G$-metric connection $\overset{1}{\nabla}$ on the almost-contact metric manifold, $\overset{1}{\nabla}\eta = \overset{1}{\nabla}\xi = 0$ holds. Taking into account Equation (5), it follows that on the almost-contact metric manifold, the torsion tensor is parallel to the connection $\overset{1}{\nabla}$, i.e., $\overset{1}{\nabla}\overset{1}{T} = 0$. In the paper [11], quarter-symmetric metric connections (4) are studied on almost-contact metric manifolds, and the properties of the torsion tensor $\overset{1}{T}$ are presented.

From Equation (5), we see that structure tensor $A$ is parallel with respect to the Levi–Civita connection, and it implies the following statement.

**Theorem 1.** *The almost-contact metric manifold $(\mathcal{M}, g, A, \eta, \xi)$ with a quarter-symmetric connection (4) preserving the generalized Riemannian metric G is a cosymplectic (co-Kähler) manifold.*

Following the previous theorem, further consideration can be given to the cosymplectic (i.e., co-Kähler) manifold. The term "generalized metric (i.e., $G$-metric) connection" is equivalent to the term "metric $A$-connection".

In the cosymplectic manifold, it also holds that $\overset{g}{\nabla}\eta = \overset{g}{\nabla}\xi = 0$ (see [3]). Moreover, the Reeb vector $\xi$ is Killing, and its dual 1-form $\eta$ is harmonic (see Lemma 1.2 in [26]). The Riemannian curvature tensor $\overset{g}{R}$ of the Levi–Civita connection on the cosymplectic manifold $(\mathcal{M}, g, A, \eta, \xi)$ satisfies the following relationships (for instance see [3,7,27,28]):

$$\overset{g}{R}(X,Y)AZ = A\overset{g}{R}(X,Y)Z, \quad \overset{g}{R}(AX,AY)Z = \overset{g}{R}(X,Y)Z, \tag{8}$$

$$\eta(\overset{g}{R}(X,Y)Z) = 0, \quad \overset{g}{R}(X,Y)\xi = \overset{g}{R}(X,\xi)Z = 0, \tag{9}$$

$$\overset{g}{Ric}(AX,AY) = \overset{g}{Ric}(X,Y), \quad \overset{g}{Ric}(X,\xi) = 0, \quad \overset{g}{Q}\xi = 0, \tag{10}$$

where $\overset{g}{Ric}(Y,Z) = Trace\{X \to \overset{g}{R}(X,Y)Z\}$ is the Ricci tensor and $\overset{g}{Q}$ is the Ricci operator defined by $\overset{g}{Ric}(X,Y) = g(\overset{g}{Q}X,Y)$. Additionally, the Ricci operator $\overset{g}{Q}$ commutes with the structure tensor $A$, i.e., $A\overset{g}{Q} = \overset{g}{Q}A$ (see [7] or [27]).

## 3. Curvature Properties of Quarter-Symmetric Metric $A$-Connection on Cosymplectic Manifold

The six linearly independent curvature tensors can be observed with respect to a non-symmetric connection [29]. The curvature tensors $\overset{0}{R}, \overset{1}{R}, \dots, \overset{5}{R}$ with respect to the quarter-symmetric connection (4) on the generalized Riemannian manifold are presented in [22] by Equations (2.6), (2.9)–(2.13). Considering cosymplectic manifold properties (more precisely, Equation (2) and $\overset{g}{\nabla}\eta = 0$), the curvature tensors with respect to the quarter-symmetric connection (4) take the following form:

$$\overset{\gamma}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z, \ \gamma = 1,2,3, \tag{11}$$

$$\overset{0}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{4}\eta(Z)(\eta(Y)X - \eta(X)Y), \tag{12}$$

$$\overset{4}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \eta(Z)(\eta(Y)X - \eta(X)Y), \tag{13}$$

$$\overset{5}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{2}\eta(Y)(\eta(Z)X - \eta(X)Z). \tag{14}$$

Since the curvature tensors $\overset{1}{R}$ and $\overset{2}{R}$ of the connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively, coincide with the Riemannian curvature tensor $\overset{g}{R}$ of the Levi–Civita connection (see Equation (11)), the following theorem holds.

**Theorem 2.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold, let $\overset{g}{\nabla}$ be a Levi-Civita connection, let $\overset{1}{\nabla}$ be a quarter-symmetric metric $A$-connection (4), and let $\overset{2}{\nabla}$ be its dual connection given by (7). The Riemannian curvature tensor $\overset{g}{R}$ is invariant under connection transformations $\overset{g}{\nabla} \to \overset{1}{\nabla}$ and $\overset{g}{\nabla} \to \overset{2}{\nabla}$.*

On the other hand, for the transformation of connections $\overset{g}{\nabla} \to \overset{0}{\nabla}$, we will prove the following theorem.

**Theorem 3.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold, let $\overset{g}{\nabla}$ be a Levi–Civita connection, and let $\overset{0}{\nabla}$ be a symmetric connection given by (6). The Riemannian curvature tensor $\overset{g}{R}$ cannot be invariant under the connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$.*

**Proof.** If we assume that $\overset{g}{R}$ is invariant under the connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$, then $\overset{0}{R} = \overset{g}{R}$ holds. Based on Equation (12), we have $\eta(Y)X - \eta(X)Y = 0$. Furthermore, by contracting, we obtain $(n-1)\eta(Y) = 0$, which is impossible. □

From Equations (11)–(14), we can easily conclude that all curvature tensors are skew-symmetric by $X$ and $Y$, except tensor $\overset{5}{R}$. On the other hand, all curvature tensors $\overset{0}{R}, \overset{1}{R}, \dots, \overset{5}{R}$ have the cyclic-symmetry property. Since tensors $\overset{1}{R}$, $\overset{2}{R}$ and $\overset{3}{R}$ coincide with $\overset{g}{R}$, it is clear that they have the same properties. In the further discussion, we will study only the properties of the curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$. Using the properties satisfied by the Riemannian curvature tensor $\overset{g}{R}$, the following relationships can be easily proven:

$$\eta(\overset{\theta}{R}(X,Y)Z) = 0, \quad \overset{\theta}{R}(AX,AY)Z = \overset{g}{R}(X,Y)Z, \quad \theta = 0,4,5,$$
$$\overset{0}{R}(X,Y)AZ = \overset{4}{R}(X,Y)AZ = \overset{g}{R}(X,Y)AZ, \quad \overset{5}{R}(X,AY)Z = \overset{g}{R}(X,AY)Z$$

and

$$\begin{aligned} A\overset{0}{R}(X,Y)Z &= \overset{0}{R}(X,Y)AZ + \frac{1}{4}\eta(Z)\overset{1}{T}(X,Y), \\ A\overset{4}{R}(X,Y)Z &= \overset{4}{R}(X,Y)AZ + \eta(Z)\overset{1}{T}(X,Y), \\ A\overset{5}{R}(X,Y)Z &= \overset{5}{R}(X,Y)AZ + \frac{1}{2}\eta(Y)\eta(Z)AX. \end{aligned}$$

The curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ and the Reeb vector field $\xi$ satisfy

$$\begin{aligned} 4\overset{0}{R}(X,Y)\xi &= \overset{4}{R}(X,Y)\xi = 2\overset{5}{R}(X,\xi)Y = \eta(Y)X - \eta(X)Y, \\ 4\overset{0}{R}(X,\xi)Y &= \overset{4}{R}(X,\xi)Y = 2\overset{5}{R}(X,Y)\xi = -\eta(Y)A^2X, \\ \overset{0}{R}(\xi,\xi)X &= \overset{4}{R}(\xi,\xi)X = \overset{5}{R}(\xi,X)\xi = 0. \end{aligned}$$

By contracting with respect to $X$ in Equations (12)–(14), we obtain the corresponding Ricci tensors, as follows:

$$\overset{0}{Ric} = \overset{g}{Ric} + \frac{n-1}{4}\eta\otimes\eta, \tag{15}$$
$$\overset{4}{Ric} = \overset{g}{Ric} + (n-1)\eta\otimes\eta, \tag{16}$$
$$\overset{5}{Ric} = \overset{g}{Ric} + \frac{n-1}{2}\eta\otimes\eta. \tag{17}$$

We see that all Ricci tensors are symmetric and satisfy the following properties:

$$\overset{\theta}{Ric}(AX,AY) = \overset{g}{Ric}(X,Y), \quad \theta = 0,4,5, \tag{18}$$
$$4\overset{0}{Ric}(X,\xi) = \overset{4}{Ric}(X,\xi) = 2\overset{5}{Ric}(X,\xi) = (n-1)\eta(X), \tag{19}$$
$$4\overset{0}{Ric}(\xi,\xi) = \overset{4}{Ric}(\xi,\xi) = 2\overset{5}{Ric}(\xi,\xi) = n-1. \tag{20}$$

Using the above results, we prove the following theorem.

**Theorem 4.** *Let $(\mathcal{M},g,A,\eta,\xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). Then, the Ricci operators $\overset{\theta}{Q}$, $\overset{\theta}{Ric}(X,Y) = g(\overset{\theta}{Q}X,Y)$ commute with the structure tensor $A$.*

**Proof.** From Equations (15)–(17), we obtain the corresponding Ricci operators,

$$\overset{0}{Q} = \overset{g}{Q} + \frac{n-1}{4}\eta\otimes\xi, \quad \overset{4}{Q} = \overset{g}{Q} + (n-1)\eta\otimes\xi, \quad \overset{5}{Q} = \overset{g}{Q} + \frac{n-1}{2}\eta\otimes\xi. \tag{21}$$

Taking into account Equation (2), we have

$$A\overset{\theta}{Q} = A\overset{g}{Q} \quad \text{and} \quad \overset{\theta}{Q}A = \overset{g}{Q}A.$$

Considering that the Ricci operator $\overset{g}{Q}$ commutes with $A$, we have thus proved the statement. □

From the relationship (21), we obtain corresponding curvature scalars, which satisfy

$$4(\overset{0}{r} - \overset{g}{r}) = \overset{4}{r} - \overset{g}{r} = 2(\overset{5}{r} - \overset{g}{r}) = n - 1 \tag{22}$$

and with this, we have proved the following theorem.

**Theorem 5.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). Then, the differences $\overset{\theta}{r} - \overset{g}{r}$ are constant, where $\overset{\theta}{r}$ and $\overset{g}{r}$ denote curvature scalars,* $\theta = 0, 4, 5$.

Using the equations of the curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ and the properties of the Riemannian curvature tensor $\overset{g}{R}$, we can easily prove that these tensors cannot be zero.

**Theorem 6.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The curvature tensors $\overset{0}{R}$, $\overset{4}{R}$, and $\overset{5}{R}$ given by (12)–(14) are non-zero.*

**Proof.** If we assume that $\overset{0}{R} = 0$, then from Equation (12), we have

$$\overset{g}{R}(X, Y)Z = \frac{1}{4}\eta(Z)(\eta(X)Y - \eta(Y)X).$$

If we use the equation $\overset{g}{R}(X, Y)\xi = 0$, we obtain that $\eta(X)Y - \eta(Y)X = 0$, which is impossible. The same is proved for the tensors $\overset{4}{R}$, $\overset{5}{R}$. □

## 4. Projectively Flat Cosymplectic Manifold

In this section, we will study the Weyl projective curvature tensor of the Levi–Civita connection on a cosymplectic manifold. Namely, using curvature tensors of a quarter-symmetric connection (4), we will construct tensors that coincide with the Weyl projective curvature tensor.

**Theorem 7.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric A-connection (4). The following holds:*

$$\overset{\theta}{W}(X, Y)Z = \overset{g}{W}(X, Y)Z, \quad \theta = 0, 4, \tag{23}$$

$$\overset{5}{W}(X, Y)Z = \overset{g}{W}(X, Z)Y + \overset{g}{R}(Z, Y)X, \tag{24}$$

*where $\overset{g}{W}$ is the Weyl projective curvature tensor of the Levi–Civita connection given by*

$$\overset{g}{W}(X, Y)Z = \overset{g}{R}(X, Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X, Z)Y - \overset{g}{Ric}(Y, Z)X), \tag{25}$$

*and* $\overset{0}{W}$, $\overset{4}{W}$, *and* $\overset{5}{W}$ *are given by*

$$\overset{\theta}{W}(X,Y)Z = \overset{\theta}{R}(X,Y)Z + \frac{1}{n-1}(\overset{\theta}{Ric}(X,Z)Y - \overset{\theta}{Ric}(Y,Z)X), \quad \theta = 0,4, \tag{26}$$

$$\overset{5}{W}(X,Y)Z = \overset{5}{R}(X,Y)Z + \frac{1}{n-1}(\overset{5}{Ric}(X,Y)Z - \overset{5}{Ric}(Z,Y)X). \tag{27}$$

**Proof.** We prove the equality for tensor $\overset{5}{W}$. From Equation (17), we have

$$\frac{1}{2}\eta \otimes \eta = \frac{1}{n-1}(\overset{5}{Ric} - \overset{g}{Ric}).$$

By substituting the previous equation in (14), the curvature tensor of the fifth kind takes the form

$$\overset{5}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{5}{Ric}(Y,Z)X - \overset{g}{Ric}(Y,Z)X - \overset{5}{Ric}(X,Y)Z + \overset{g}{Ric}(X,Y)Z),$$

and after rearranging, we obtain

$$\begin{aligned}
\overset{5}{W}(X,Y)Z &= \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Z,Y)X) \\
&= \overset{g}{R}(X,Z)Y + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Z,Y)X) - \overset{g}{R}(X,Z)Y + \overset{g}{R}(X,Y)Z \\
&= \overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,X)Y + \overset{g}{R}(X,Y)Z = \overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,Y)X,
\end{aligned}$$

where $\overset{5}{W}$ is given by (27) and where we used the skew-symmetry and cyclic-symmetry properties of $\overset{g}{R}$. □

Tensor $\overset{0}{W}$ is the projective curvature tensor with respect to the connection $\overset{0}{\nabla}$, and since it coincides with the Weyl projective curvature tensor of the Levi–Civita connection, we can formulate the following statement.

**Theorem 8.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold, let $\overset{g}{\nabla}$ be a Levi–Civita connection, and let $\overset{0}{\nabla}$ be a symmetric connection given by (6). The Weyl projective curvature tensor $\overset{g}{W}$ is invariant under the connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$.*

Additionally, since the Riemannian curvature tensor $\overset{g}{R}$ coincides with $\overset{1}{R}$ and $\overset{2}{R}$, the Weyl projective curvature tensor is invariant under connection transformations $\overset{g}{\nabla} \to \overset{1}{\nabla}$ and $\overset{g}{\nabla} \to \overset{2}{\nabla}$.

Given that we have constructed the Weyl projective curvature tensor $\overset{g}{W}$ on a cosymplectic manifold, we will examine what happens when this manifold is projectively flat. If we assume that $\overset{g}{W} = 0$, then it holds that

$$\overset{g}{R}(X,Y)Z = \frac{1}{n-1}(\overset{g}{Ric}(Y,Z)X - \overset{g}{Ric}(X,Z)Y). \tag{28}$$

Based on the properties of the Riemannian curvature tensor $\overset{g}{R}$ and the Ricci tensor $\overset{g}{Ric}$ on the cosymplectic manifold, i.e., using Equations (9) and (10), we have

$$0 = \overset{g}{R}(X,\xi)Z = \frac{1}{n-1}(\overset{g}{Ric}(\xi,Z)X - \overset{g}{Ric}(X,Z)\xi) = -\frac{1}{n-1}\overset{g}{Ric}(X,Z)\xi$$

from where we obtain $\overset{g}{Ric} = 0$. By substituting the last equality in (28), we obtain $\overset{g}{R} = 0$. In this way, we have proved the following assertion.

**Theorem 9.** *A cosymplectic manifold is projectively flat if and only if it is flat.*

Considering the previous results for the quarter-symmetric metric $A$-connection (4), we have the following corollary.

**Corollary 1.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric $A$-connection (4). The manifold is projectively flat if and only if the tensors $\overset{\theta}{W}$, $\theta = 0,4,5$, given by (26) and (27), vanish.*

**Proof.** Considering that $\overset{0}{W}$ and $\overset{4}{W}$ coincide with $\overset{g}{W}$, the statement is clear for those two tensors. Let us now prove the statement for tensor $\overset{5}{W}$. If the manifold is projectively flat, then it is also flat, so it follows that $\overset{g}{W} = \overset{g}{R} = 0$. Furthermore, from Equation (24), we obtain $\overset{5}{W} = 0$.

On the other hand, if $\overset{5}{W} = 0$, then from Equation (24), we have $\overset{g}{W}(X,Z)Y + \overset{g}{R}(Z,Y)X = 0$, from where it follows that

$$\overset{g}{R}(X,Y)Z = \frac{1}{n-1}(\overset{g}{Ric}(Z,Y)X - \overset{g}{Ric}(X,Y)Z). \tag{29}$$

Taking into account Equations (9) and (10), we obtain

$$0 = \overset{g}{R}(X,Y)\xi = \frac{1}{n-1}(\overset{g}{Ric}(\xi,Y)X - \overset{g}{Ric}(X,Y)\xi) = -\frac{1}{n-1}\overset{g}{Ric}(X,Y)\xi,$$

from which it follows that $\overset{g}{Ric} = 0$. By substituting this equality in (29), we obtain that the manifold is flat, which implies that it is also projectively flat. This completes the proof of the theorem. □

**Remark 1.** *Theorem 9 can be considered as a consequence of the statements from [1,30]. Namely, the manifold is projectively flat if and only if it is of constant curvature (see pp. 84–85 in [30]). A cosymplectic manifold of constant curvature is flat (see [1,3]). Therefore, we can conclude that a projectively flat cosymplectic manifold is flat. Here, we have given explicit proof.*

## 5. $\eta$-Einstein Cosymplectic Manifold

A cosymplectic manifold is $\eta$-Einstein if

$$\overset{g}{Ric} = ag + b\eta \otimes \eta, \tag{30}$$

where $a$, $b$ are smooth functions. If we use the Ricci tensor property $\overset{g}{Ric}(X,\xi) = 0$, then from the previous equation, we have $a + b = 0$, and the Ricci tensor takes the form

$\overset{g}{Ric} = a(g - \eta \otimes \eta)$. By contracting the last equality, we obtain that $\overset{g}{r} = a(n-1)$. Thus, the $\eta$-Einstein cosymplectic manifold satisfies the following equation (see [31,32]):

$$\overset{g}{Ric} = \frac{\overset{g}{r}}{n-1}(g - \eta \otimes \eta). \tag{31}$$

We see that the $\eta$-Einstein cosymplectic manifold is Ricci flat if and only if $\overset{g}{r} = 0$ [31]. Moreover, in [32], it has been proved that the curvature scalar $\overset{g}{r}$ is constant in the case of the $\eta$-Einstein cosymplectic manifold of a dimension of 5 or higher. An example of the $\eta$-Einstein cosymplectic manifold is the cosymplectic manifold of constant $\phi$-sectional curvature $c$, whose Ricci tensor is given by $\overset{g}{Ric} = \frac{c(k+1)}{2}(g - \eta \otimes \eta)$ (see Equation (2.4) in [33]). Additionally, any 3-dimensional cosymplectic manifold is $\eta$-Einstein [34,35] and of quasi-constant curvature [31]. For $b = 0$ in (30), we have an Einstein manifold. Any Einstein cosymplectic manifold is Ricci flat [36].

Considering that the Ricci tensors $\overset{0}{Ric}$, $\overset{4}{Ric}$, $\overset{5}{Ric}$ given by (15)–(17) are symmetric, we will now define special classes of the cosymplectic manifold with a quarter-symmetric metric $A$-connection (4).

**Definition 1.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric $A$-connection (4). The manifold is $\eta$-Einstein of the $\theta$-th kind, $\theta = 0, 4, 5$ if*

$$\overset{\theta}{Ric} = \overset{\theta}{a}g + \overset{\theta}{b}\eta \otimes \eta, \quad \theta = 0, 4, 5,$$

*where $\overset{\theta}{a}$ and $\overset{\theta}{b}$ are smooth functions.*

By contracting the previous equation, we obtain

$$\overset{\theta}{r} = \overset{\theta}{a}n + \overset{\theta}{b}, \quad \theta = 0, 4, 5.$$

Based on the properties of the Ricci tensor $\overset{0}{Ric}$ (see Equation (20)), we have $4(\overset{0}{a} + \overset{0}{b}) = n - 1$. By solving the system of equations

$$\overset{0}{r} = \overset{0}{a}n + \overset{0}{b}, \quad 4(\overset{0}{a} + \overset{0}{b}) = n - 1,$$

we obtain

$$\overset{0}{a} = \frac{4\overset{0}{r} - n + 1}{4(n-1)}, \quad \overset{0}{b} = \frac{n(n-1) - 4\overset{0}{r}}{4(n-1)}.$$

Similarly,

$$\overset{4}{a} = \frac{\overset{4}{r} - n + 1}{n-1}, \quad \overset{4}{b} = \frac{n(n-1) - \overset{4}{r}}{n-1},$$
$$\overset{5}{a} = \frac{2\overset{5}{r} - n + 1}{2(n-1)}, \quad \overset{5}{b} = \frac{n(n-1) - 2\overset{5}{r}}{2(n-1)}.$$

Consequently, the $\eta$-Einstein cosymplectic manifold of the $\theta$-th kind $\theta = 0, 4, 5$ takes the form

$$\overset{0}{Ric} = \frac{4\overset{0}{r} - n + 1}{4(n-1)} g + \frac{n(n-1) - 4\overset{0}{r}}{4(n-1)} \eta \otimes \eta,$$
$$\overset{4}{Ric} = \frac{\overset{4}{r} - n + 1}{n-1} g + \frac{n(n-1) - \overset{4}{r}}{n-1} \eta \otimes \eta,$$
$$\overset{5}{Ric} = \frac{2\overset{5}{r} - n + 1}{2(n-1)} g + \frac{n(n-1) - 2\overset{5}{r}}{2(n-1)} \eta \otimes \eta.$$

Using Equation (22), the $\eta$-Einstein cosymplectic manifold of the $\theta$-th kind $\theta = 0, 4, 5$ can be written in terms of the scalar curvature $\overset{g}{r}$

$$\overset{0}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 4\overset{g}{r}}{4(n-1)} \eta \otimes \eta, \tag{32}$$
$$\overset{4}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - \overset{g}{r}}{n-1} \eta \otimes \eta, \tag{33}$$
$$\overset{5}{Ric} = \frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 2\overset{g}{r}}{2(n-1)} \eta \otimes \eta. \tag{34}$$

**Theorem 10.** *Let $(\mathcal{M}, g, A, \eta, \xi)$ be a cosymplectic manifold with a quarter-symmetric metric $A$-connection (4). The manifold is $\eta$-Einstein if and only if it is $\eta$-Einstein of the $\theta$-th kind,* $\theta = 0, 4, 5$.

**Proof.** We prove the theorem for $\theta = 0$. If the manifold is $\eta$-Einstein, then by substituting Equation (31) in (15), we obtain Equation (32), and therefore the manifold is $\eta$-Einstein of the zeroth kind. On the other hand, if the manifold is $\eta$-Einstein of the zeroth kind, then from Equations (15) and (32), we have

$$\frac{\overset{g}{r}}{n-1} g + \frac{(n-1)^2 - 4\overset{g}{r}}{4(n-1)} \eta \otimes \eta = \overset{g}{Ric} + \frac{n-1}{4} \eta \otimes \eta,$$

from which we obtain

$$\overset{g}{Ric} = \frac{\overset{g}{r}}{n-1} (g - \eta \otimes \eta),$$

which means that the cosymplectic manifold is $\eta$-Einstein with respect to the Riemannian metric $g$. □

## 6. Results and Discussion

The paper discussed the application of a quarter-symmetric connection on almost-contact metric manifolds. We proved that an almost-contact metric manifold with a quarter-symmetric $G$-metric connection is actually a cosymplectic manifold. Based on the properties of the Riemannian curvature tensor, we also observed the properties of the curvature tensor with respect to the quarter-symmetric metric $A$-connection. Invariants for certain connection transformations were also found. For example, the Riemannian curvature tensor is invariant under the transformation of the Levi–Civita connection to a quarter-symmetric metric $A$-connection (4) or to its dual connection (7).

The Weyl projective curvature tensor $\overset{g}{W}$ is well known as a geodesic mapping invariant of Riemannian manifolds (for instance, see [37]). We found a way to construct it on a cosymplectic manifold. More precisely, using the curvature tensors with respect to the quarter-symmetric metric $A$-connection on the cosymplectic manifold, we constructed tensors that coincide with the Weyl projective curvature tensor $\overset{g}{W}$. We proved that a cosymplectic manifold is projectively flat if and only if it is flat.

At the end of the paper, we constructed examples of $\eta$-Einstein cosymplectic manifolds. Namely, with respect to the Ricci tensors on the cosymplectic manifold with a quarter-symmetric metric $A$-connection, we defined the $\eta$-Einstein manifold of the $\theta$-th kind, $\theta = 0, 4, 5$, and we demonstrated that these manifolds coincide with the $\eta$-Einstein (with respect to the Riemannian metric).

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