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Note on Discovering Doily in PG(2,5)

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Abstract: W. L. Edge proved that the internal points of a conic in PG(2,5), together with the collinear triples on the non-secant lines, form the Desargues configuration. M. Saniga showed an intimate connection between Desargues configurations and the generalized quadrangles of order 2, GQ(2,2), whose representation was dubbed "the doily" by Stan Payne in 1973. In this note, we prove that the external points of a conic in PG(2,5), together with the collinear and non-collinear triples on the non-tangent lines, form the generalized quadrangle of order 2.

Keywords: Desargues configuration; generalized quadrangle of order two; projective plane of order five

MSC: 51E20

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1. Introduction and Motivation

W. L. Edge [1] proved that the internal points of a conic in PG(2,5) together with the non-secant lines form a Desargues configuration. M. Saniga [2] showed an intimate connection between Desargues configurations and the generalized quadrangle of order 2, GQ(2,2). The two results motivate the writing of this note. By using the Singer representation of PG(2,5), we provide a short proof of W. L. Edge's result and, believing it is novel, we prove that the external points of a conic of PG(2,5) define the generalized quadrangle of order 2, GQ(2,2). The reason for deciding to conduct a detailed investigation of this special case is the charm of small projective planes, cf. [3–8].

2. The Singer Representation of PG(2,5)

Let ω be a primitive element of F_{5^3} over F_5 and let $f(x) = a_0 + a_1x + a_2x^2 + x^3$ be its minimal polynomial over F_5. The companion matrix $T := C(f)$ of f is given by

$$T := C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$$

and it induces a Singer cycle γ of PG(2,5), cf. [9]. Let us consider the minimal polynomial $f(x) = 1 + x + x^3$ over F_5. The companion matrix $T := C(f)$ of f

$$T := C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & 0 \end{pmatrix}$$

gives the 31 points of this plane as follows, cf. [10]. Let the point ω^0 be represented by the vector $(x_0, x_1, x_2) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. Then, we get $\omega^i = \omega^{i-1}T$ $i = 1, 2, \ldots, 30$. The 31 points of PG(2,5) are given in the Table 1.

Table 1. The points of PG(2,5).

ω^0 = (1,0,0)					
ω^1 = (0,1,0)	ω^2 = (0,0,1)	ω^3 = (1,1,0)	ω^4 = (0,1,1)	ω^5 = (1,1,4)	ω^6 = (1,2,1)
ω^7 = (1,0,3)	ω^8 = (1,4,0)	ω^9 = (0,1,4)	ω^{10} = (1,1,1)	ω^{11} = (1,0,4)	ω^{12} = (1,2,0)
ω^{13} = (0,1,2)	ω^{14} = (1,1,2)	ω^{15} = (1,3,2)	ω^{16} = (1,3,1)	ω^{17} = (1,0,2)	ω^{18} = (1,3,0)
ω^{19} = (0,1,3)	ω^{20} = (1,1,3)	ω^{21} = (1,4,3)	ω^{22} = (1,4,2)	ω^{23} = (1,3,3)	ω^{24} = (1,4,4)
ω^{25} = (1,2,4)	ω^{26} = (1,2,2)	ω^{27} = (1,3,4)	ω^{28} = (1,2,3)	ω^{29} = (1,4,1)	ω^{30} = (1,0,1)

Let us denote the points represented by ω^i simply by i. Therefore, the Singer group is isomorphic to the additive group Z_{31}, the integers modulo 31. Now select any line: for example, we choose the line $x_1 = x_2$, which contains the points: ℓ_0 = {0,4,10,23,24,26}. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_0 and using addition modulo 31. For convenience, we represent the projective plane of order 5 displaying its lines in arrays via the six parallelism classes of the affine plane of order 5 together with their point at infinity which appear on the right or at the bottom of the array representing the parallel class. We do this by using the Singer difference set defining PG(2,5) as the line at infinity, designated by ℓ_∞. Thus, let ℓ_∞ = {0,4,10,23,24,26}. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_∞ as ℓ_0 and using addition modulo 31. The pencil of lines on point 4 is then intersected by the pencil of lines on point 0 to form the first array. Thus, each row (column) plus its point at infinity represents a line of the plane. Now, let us take into account the Singer representation.

1	2	9	13	19		1	6	16	29	30		1	5	11	25	27	
3	11	15	21	6		7	8	15	19	25		14	18	9	6	7	
8	30	28	14	27	4	20	5	2	14	3	10	15	30	20	12	13	24
12	7	5	16	22		21	17	27	12	9		17	19	16	28	3	
18	17	29	25	20		28	13	18	11	22		22	21	8	2	29	
		0						23						26			

Moreover, by the Singer representation, since all conics in PG(2,5) are projectively equivalent, see [10], let us consider the conic $C = -\ell_\infty$ = {−0,−4,−10,−23,−24,−26} = {0,5,7,8,21,27}. By taking into account the points not on C and not on the tangent lines, we get the 10-set I of the internal points of the conic I = {1,12,13,17,19,22,25,28,29,30}. Now, taking into account the triples on the external lines,

1	2	9	13	19		1	6	16	29	30		1	**5**	11	25	**27**	
3	11	15	**21**	6		**7**	**8**	15	19	25		14	18	9	6	**7**	
8	30	28	14	**27**	4	20	**5**	2	14	3	10	15	30	20	12	13	24
12	**7**	**5**	16	22		**21**	17	**27**	12	9		17	19	16	28	3	
18	17	29	25	20		28	13	18	11	22		22	**21**	**8**	2	29	
		0						23						26			

1	2	9	**13**	**19**		**1**	6	16	**29**	**30**		**1**	5	11	**25**	27	
3	11	15	21	6		7	8	15	**19**	**25**		14	18	9	6	7	
8	30	28	14	27	4	20	5	2	14	3	10	15	**30**	20	**12**	**13**	24
12	7	5	16	22		21	17	27	**12**	9		**17**	**19**	16	**28**	3	
18	**17**	**29**	**25**	20		**28**	**13**	18	11	**22**		**22**	21	8	2	29	
		0						23						26			

we get {{1,13,19},{1,17,22},{1,29,30},{12,13,30},{12,19,29},{12,25,28},{13,22,28}, {17,19,28}, {17,25,29},{22,25,30}}. Let us now consider the point-line incidence geometry (I,T) where the point-set I is the 10-set of the internal points of the conic $I = \{1,12,13,17,19,22,25,28,29,30\}$, and the line-set T is the union of the triples of collinear points on the external lines:

$$T = \{\{1,13,19\},\{1,17,22\},\{1,29,30\},\{12,13,30\},\{12,19,29\},\{12,25,28\},\{13,22,28\},\{17,19,28\},\{17,25,29\},\{22,25,30\}\}$$

A brief inspection of the Figure 1 confirms that the geometry (I,T) is the Desargues configuration, as W. L. Edge proved in [1]. Now, by taking into account the points not on C, but on the tangent lines, we get the 15-set of external points of the conic $E = \{2,3,4,6,9,10,11,14,15,16,18,20,23,24,26\}$, cf. [10]. Now, taking into account the triples on the external lines and the triples of non-collinear points of the triangles of the 2-lines,

1	2	9	13	19		1	6	**16**	29	30		1	5	**11**	25	27	
3	**11**	**15**	21	**6**		7	8	**15**	19	25		**14**	**18**	**9**	**6**	7	
8	30	28	14	27	**4**	**20**	5	**2**	**14**	**3**	**10**	15	30	**20**	12	13	**24**
12	7	5	16	22		21	17	27	12	9		17	19	**16**	28	3	
18	17	29	25	20		28	13	**18**	11	22		22	21	8	2	29	
		0						**23**						**26**			

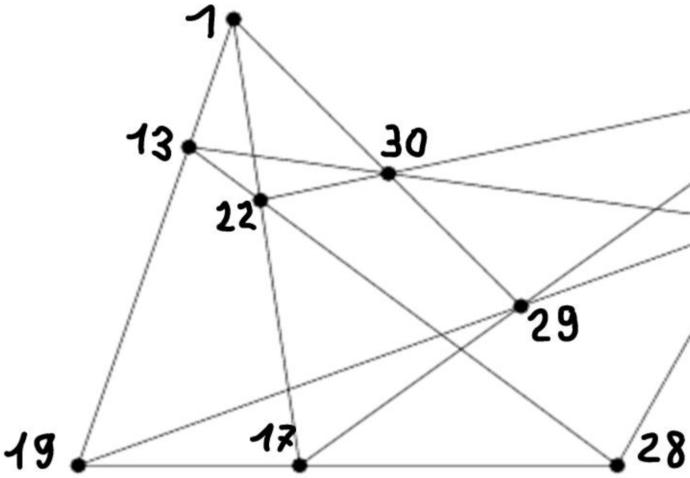

Figure 1. The Desargues configuration.

1	**2**	**9**	13	19		1	**6**	**16**	29	30		1	5	11	25	27	
3	11	15	21	6		7	8	15	19	25		**14**	18	9	**6**	7	
8	30	28	14	27	**4**	20	5	2	**14**	**3**	**10**	**15**	30	**20**	12	13	**24**
12	7	5	16	22		21	17	27	12	**9**		17	19	**16**	28	**3**	
18	17	29	25	**20**		28	13	**18**	**11**	22		22	21	8	**2**	29	
		0						**23**						**26**			

We get the sets $T_1 = \{\{2,4,9\},\{2,6,26\},\{3,9,23\},\{3,16,24\},\{4,18,20\},\{6,10,16\},\{10,11,18\}, \{11,14,23\}, \{14,15,26\},\{15,20,24\}\}$ and $T_2 = \{\{2,11,24\},\{3,18,26\},\{4,14,16\},\{6,20,23\},\{9,10,15\}\}$.

1	**2**	**9**	13	19		1	**6**	16	29	30		1	5	**11**	25	27	
3	**11**	**15**	21	**6**		7	8	**15**	19	25		14	**18**	9	6	7	
8	30	28	**14**	27	**4**	**20**	5	2	14	3	**10**	15	30	20	12	13	**24**
12	7	5	**16**	22		21	17	27	12	**9**		17	19	16	28	**3**	
18	17	29	25	**20**		28	13	18	11	22		22	21	8	**2**	29	
		0						**23**						**26**			

Let us now construct the point-line incidence geometry (E,L) where the point-set E is the 15-set of the external points of the conic, and the line-set $L = T_1 \cup T_2$.

A brief inspection of the Figure 2 confirms that this geometry is isomorphic to GQ(2,2).

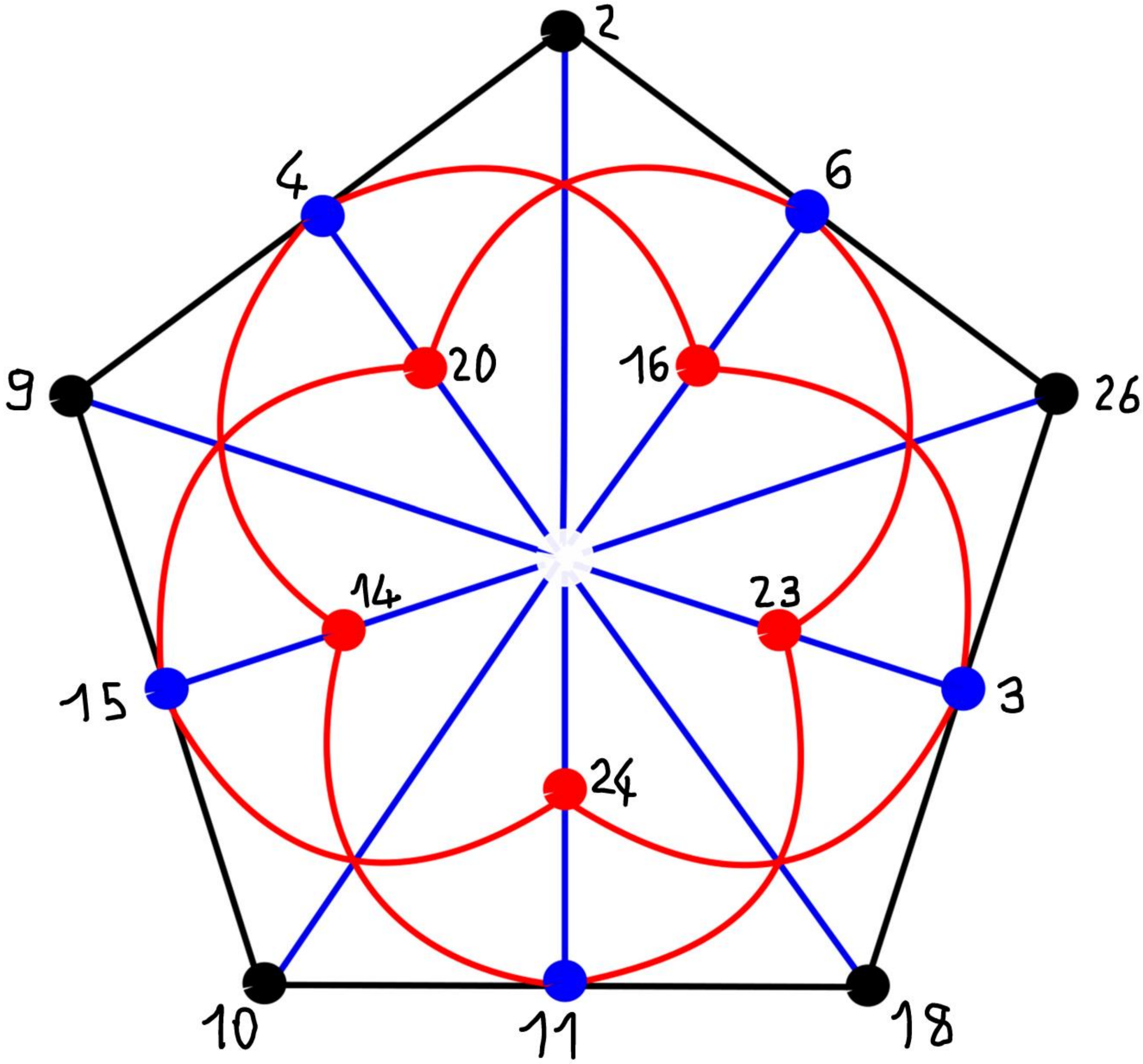

Figure 2. The doily.

3. Conclusions

This note confirms the intimate connection between Desargues configurations and the generalized quadrangles of order two. The representation of the doily found by the author and that proposed by Saniga [2] using the concept of the Veldkamp space of the Desargues configuration share more than meets the eye. Using the fact that the doily is a self-dual geometry, we can swap the roles of points and lines to get (isomorphically) the same geometry. In this case the two different sets T_1 and T_2 correspond to two different types of geometric hyperplanes of the Desargues configuration in Saniga's model [2], namely, to the ten polar point-line pairs and the five Pasch configurations, respectively; moreover, the points of the doily represented by Pasch configurations form an ovoid, which corresponds to the fact that the five lines of the set T_2 form a spread in the author's model.

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