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On Extended L^r-Norm-Based Derivatives to Intuitionistic Fuzzy Sets

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Abstract: The study of differential equation theory has come a long way, with applications in various fields. In 1961, Zygmund and Calderón introduced the notion of derivatives to metric L^r, which proved to be better in applications than approximate derivatives. However, most of the studies available are on Fuzzy Set Theory. In view of this, intuitionistic fuzzy L^r-norm-based derivatives deserve study. In this study, the L^r-norm-based derivative for intuitionistic fuzzy number valued functions is introduced. Some of its basic properties are also discussed, along with numerical examples. The results obtained show that the proposed derivative is not dependent on the existence of the Hukuhara difference. Lastly, the Cauchy problem for the intuitionistic fuzzy differential equation is discussed.

Keywords: fuzzy set; intuitionistic fuzzy number; Hukuhara differentiable; generalized Hukuhara differentiable

MSC: 34A07; 54A40

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1. Introduction

The fuzzy set theory introduced by L A Zadeh in 1965 [1] paved the way for dealing with vagueness or uncertainty in mathematical models where each element $u \in U$ (universal set) is defined in terms of A(u) [membership function] $\{A : U \to [0, 1]\}$. Subsequently, many researchers expanded the theory [2–5], one of which is Atanassov, who introduced the intuitionistic fuzzy set (IFS), in which each element is defined in terms of A(u), membership, and B(u), non-membership, functions such that A + B $\leq$ 1 and 1 $-$ (A + B) is the degree of hesitation [2]; this is a generalization of Zadeh's [1] fuzzy set theory. Atanassov later conducted additional research to improve intuitionistic fuzzy set theory (IFST) [2,6–8]. In recent years IFST has been increasingly used in applications of decision-making problems [9,10], medical diagnosis [11–13], software selection [14], environmental management [15], transport problems [16], predator prey [17], etc. Susanto et al. [18] generated fuzzy interval data from crisp data using the Cheng et al. [19] correlation method to determine the relationship between students' anxiety and mathematical self-efficacy, based on the concept of α-cut from a fuzzy set. In [20], an extensive study of IFST was presented, from its introduction and along its evolution, including applications in real-life scenarios. One of the extensions of IFST is the Pythagorean fuzzy set (PFS); this concept was put forward by Yager [21], satisfying the condition 0 $\leq$ A^2 + B^2 $\leq$ 1. Later, Zhang et al. [22] gave the mathematical form and presented the concept of Pythagorean fuzzy number. Since then, the theory has been studied extensively [23–25]. In [26], the authors presented a bibliometric analysis of PFS for 2013–2020.

The fuzzy differential equation (FDE) is one of the many extensions of fuzzy set theory to classical mathematics. Many researchers have studied FDE with different approaches, such as differential inclusions [27], Hukuhara derivatives [28], and Zadeh's extension principle [29,30]. In 1967 [28], Hukuhara put forth the idea of the Hukuhara difference to eliminate the problem of the inverse element of Minkowski's sum [31]. With α-cut being a compact and convex set, Madan et al. [32] proposed an approach to solve fuzzy differential equations using the concept of the Hukuhara derivative [28]. But this approach has a weak point which is that the solution becomes fuzzier, and so the length of the support of the fuzzy solution increases [33]. So, in order to overcome the disadvantages, Bede et al. [34] introduced the concept of generalized Hukuhara derivatives. Later, Stefanini [35] proposed a generalized Hukuhara difference, which was applied to Bede's generalized Hukuhara derivative [36]. This yielded a better result, as it made problem solving less complicated; since then, the study of the GH derivative has been expanded further by the authors [37–39]. Akin et al. [40] proposed solving the second-order initial value problem with intuitionistic fuzzy initial values under the intuitionistic Zadeh extension principle. In [30], Akin et al. extended the GH difference to the intuitionistic fuzzy set by using the properties of α and β cuts and proposed the concept of strongly generalized Hukuhara derivatives for the intuitionistic fuzzy environment. They further extended the concept to solve differential equations in the intuitionistic fuzzy environment with the use of (i) GH and (ii) GH differentiability [41].

Differential equations in metric spaces, presented by [42], resulted in the introduction of a new concept called the metric derivative. This concept was further developed to solve the Cauchy problem for fuzzy as well as intuitionistic fuzzy differential equations [42,43] and set valued functions [44]. Khastan et al. [45] introduced new metric-based derivatives and compared these with other existing metric-based derivatives. L^r-norm-based derivatives were introduced by Caldern and Zygmund in 1961 [46] in order to solve elliptic partial differential equations. This was further extended to integral theory, such as Perron's integral for derivatives in L^r [47], the L^r-Henstock-Kurzweil integral [48], and the fuzzy Henstock integral [49]. Accordingly, Shao et al. [50] introduced fuzzy L^r-derivatives and the fuzzy L^r-Henstock–Kurzweil integral based on the derivative. In [51–53], convex inequalities are studied using Hermite–Hadamard-type inequalities for fuzzy number valued mappings.

After the review of literature, it was found that numerous studies have been conducted on fuzzy derivatives using various techniques, but the same cannot be said for IFST. There are very few studies available on intuitionistic metric-based derivatives, with most done using GH differentiability. But, the existence of GH difference is not true for all intuitionistic fuzzy number valued functions; this paper can help to address this limitation. Firstly, the objective of this paper is to extend the derivative to intuitionistic fuzzy set valued functions based on L^r-norm, which was proposed in [50]. Secondly, we propose to study its properties, such as continuity, uniqueness, and many more. Thirdly, we study the idea that GH differentiability implies IFL^r-differentiable, but the converse is not always true. Finally, it is proposed to solve the intuitionistic fuzzy differential equation using the extended derivative.

In this paper, Section 2 includes fundamental definitions and theorems of intuitionistic fuzzy sets and definitions of L^r-norm-based derivatives for fuzzy sets. Section 3 contains definitions and theorems related to L^r-norm-based derivatives, which are extended from the fuzzy case to the intuitionistic fuzzy case by using the definitions and theorems in Section 2 along with numerical examples. In Section 4, the initial value problem is solved for the intuitionistic differential equation. And lastly, Section 5 includes a summary and results, with limitations and lines for future work.

2. Preliminaries

Definition 1. *[2] Let $P = \{(u, A_P(u), B_P(u)) \mid u \in U\}$ be an intuitionistic fuzzy set (IFS)*

where A_P, B_P: $U \to [0, 1]$,
$A_P(u)$ is the membership function of u,
$B_P(u)$ is the non-membership function of u,

and the condition $0 \leq A_P(u) + B_P(u) \leq 1$ holds true.
Atanassov's intuitionistic fuzzy set is the generalization of Zadeh's fuzzy set. Then, Zadeh's fuzzy set can be written as $P = \{(u, A_P(u), 0) \mid u \in U\}$, where the non-membership function $B_P(u) = 0$. IF(U) will be used to denote the set of all intuitionistic fuzzy sets in U.

Definition 2. *[2] The α-cut of $P \in IF(\mathbb{R}^n)$ is defined as*
$P(\alpha) = \{u: u \in \mathbb{R}^n, A_P(u) \geq \alpha\}$, *for* $0 < \alpha \leq 1$, *and*
$P(0) = cl\left(\bigcup_{\alpha \in (0,1]} P(\alpha)\right)$, *for* $\alpha = 0$.

Definition 3. *[2] The β-cut of $P \in IF(\mathbb{R}^n)$ is defined as*
$P^*(\beta) = \{u: u \in \mathbb{R}^n, B_P(u) \leq \beta\}$, *for* $0 < \beta < 1$, *and*
$P^*(1) = cl\left(\bigcup_{\beta \in [0,1)} P^*(\beta)\right)$, *for* $\beta = 1$.

Definition 4. *[30] If $P \in IF(\mathbb{R}^n)$ satisfies the following conditions, then it is called an intuitionistic fuzzy number in $\mathbb{R}^n$:*

1. *P is a normal set, i.e., $\exists\, u_0 \in \mathbb{R}^n$, such that $A_P(u_0) = 1$ (hence, $B_P(u_0) = 0$).*
2. *$P(0)$ and $P^*(1)$ are bounded sets in $\mathbb{R}^n$.*
3. *A_P: $\mathbb{R}^n \to [0, 1]$ is upper semi-continuous: $\forall\, k \in [0, 1]$, the set $\{u: u \in \mathbb{R}^n, A_P(u) < k\}$ is open.*
4. *B_P: $\mathbb{R}^n \to [0, 1]$ is lower semi-continuous: $\forall\, k \in [0, 1]$, the set $\{u: u \in \mathbb{R}^n, B_P(u) > k\}$ is open.*
5. *The membership function A_P is quasi-concave:*
   $A_P(\pi u + (1 - \pi) v) \geq \min\{A_P(u), A_P(v)\}, \forall\, u, v \in \mathbb{R}^n, \pi \in [0, 1]$.
6. *The non-membership function B_P is quasi-convex:*
   $B_P(\pi u + (1 - \pi) v) \leq \max\{B_P(u), B_P(v)\}, \forall\, u, v \in \mathbb{R}^n, \pi \in [0, 1]$.

   We will denote the set of intuitionistic fuzzy numbers of $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

Theorem 1. *[31] The family of all compact and convex subsets of $\mathbb{R}^n$ is closed under Minkowski's addition and scalar multiplication.*

Definition 5. *[30] Let $P, Q \in IF_N(\mathbb{R}^n)$ and $c \in \mathbb{R} - \{0\}$. Then, the addition and scalar multiplication of fuzzy numbers in $IF_N(\mathbb{R}^n)$ are defined as follows:*

(i) $P + Q = D \Leftrightarrow D(\alpha) = P(\alpha) + Q(\alpha)$ *and* $D^*(\beta) = P^*(\beta) + Q^*(\beta)$.
(ii) $c(P) = D \Leftrightarrow D(\alpha) = cP(\alpha)$ *and* $D^*(\beta) = cP^*(\beta)$.

Theorem 2. *[30] Let $P, Q \in IF_N(\mathbb{R}^n)$. Let us define the following distance functions as*

$\mathfrak{D}_1(P, Q) = \sup\{d_H(P(\alpha), Q(\alpha)): \alpha \in [0, 1]\}$
$\mathfrak{D}_2(P, Q) = \sup\{d_H(P^*(\beta) + Q^*(\beta)): \beta \in [0, 1]\}$,

where d_H is the Hausdorff metric. The function
$\mathfrak{D}(P, Q) = \max\{\mathfrak{D}_1(P, Q), \mathfrak{D}_2(P, Q)\}$

defines a metric on $IF_N(\mathbb{R}^n)$. Hence, $(IF_N(\mathbb{R}^n), \mathfrak{D})$ is a metric space.

Definition 6. *[30] Let $P, Q \in IF_N(\mathbb{R}^n)$, then*

- *The Hukuhara difference of P and Q, if it exists, is given by*

$$P \ominus_H Q = R \Leftrightarrow P = Q + R.$$

- *The generalized Hukuhara difference of P and Q, if it exists, is given by*

$$P \ominus_{GH} Q = R \Leftrightarrow P = Q + R or Q = P + (-1) R.$$

Definition 7. *[30] Let $\mathfrak{F}$: $(m, n) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and u, $u + h \in (m, n)$. $\mathfrak{F}$ is called the Hukuhara differentiable at u if there exists an element $\mathfrak{F}'_H(u) \in IF_N(\mathbb{R})$ such that for all $h > 0$ the following is satisfied:*

$$\lim_{h\to 0^+} \frac{\mathfrak{F}(u+h) \ominus_H \mathfrak{F}(u)}{h} = \lim_{h\to 0^+} \frac{\mathfrak{F}(u) \ominus_H \mathfrak{F}(u-h)}{h} = \mathfrak{F}'_H(u).$$

Definition 8. *[30] The intuitionistic fuzzy number valued function $\mathfrak{F}$ is called the generalized Hukuhara differentiable at u if there exists an element $\mathfrak{F}'_{GH}(u) \in IF_N(\mathbb{R})$ such that for all $h > 0$ at least one of the following conditions is satisfied:*

i)

$$\lim_{h\to 0^+} \frac{\mathfrak{F}(u\ ) \ominus_H \mathfrak{F}(u+h)}{-h} = \lim_{h\to 0^+} \frac{\mathfrak{F}(u-h) \ominus_H \mathfrak{F}(x\ )}{-h} = \mathfrak{F}'_{GH}(u).$$

ii)

$$\lim_{h\to 0^+} \frac{\mathfrak{F}(u+h) \ominus_H \mathfrak{F}(u)}{h} = \lim_{h\to 0^+} \frac{\mathfrak{F}(u-h) \ominus_H \mathfrak{F}(u\ )}{-h} = \mathfrak{F}'_{GH}(u).$$

iii)

$$\lim_{h\to 0^+} \frac{\mathfrak{F}(u+h) \ominus_H \mathfrak{F}(u)}{h} = \lim_{h\to 0^+} \frac{\mathfrak{F}(u) \ominus_H \mathfrak{F}(u-h)}{h} = \mathfrak{F}'_{GH}(u).$$

iv)

$$\lim_{h\to 0^+} \frac{\mathfrak{F}(u) \ominus_H \mathfrak{F}(u+h)}{-h} = \lim_{h\to 0^+} \frac{\mathfrak{F}(u) \ominus_H \mathfrak{F}(u-h)}{h} = \mathfrak{F}'_{GH}(u).$$

Definition 9. *[50] Let $F_N(\mathbb{R})$ be a set of fuzzy subsets, $X \in F_N(\mathbb{R})$ is said to be a fuzzy number if X is normal, convex, upper semi-continuous and*

$X_\alpha = \{u\colon u \in \mathbb{R}, A_X(u) \geq \alpha\}$, for $0 < \alpha \leq 1$, and $X_0 = cl\left(\bigcup_{\alpha \in (0,1]} X(\alpha)\right)$, for $\alpha = 0$ is bounded. The distance $d_H(X, Y)$ is defined as

$$d_H(X, Y) = sup_{\alpha \in [0,1]}\left\{\left|X_\alpha{}^- - Y_\alpha{}^-\right|, \left|X_\alpha{}^+ - Y_\alpha{}^+\right|\right\}.$$

Obviously, $(F_N(\mathbb{R}); d_H)$ is a complete metric space.

Definition 10. *[50] $\mathfrak{F}$ is fuzzy L^r-differentiable (FL^r-differentiable) at $u \in [m, n]$ if there exists $\mathfrak{F}' \in F_N(\mathbb{R})$ such that the following four situations hold:*

i)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[d_H\left(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 \left[d_H\left(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = 0.$$

ii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[d_H\left(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 \left[d_H\left(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = 0.$$

iii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[d_H\left(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 \left[d_H\left(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = 0.$$

iv)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l\left[d_H\left(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0\left[d_H\left(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} = 0.$$

If $\mathfrak{F}$ satisfies case (i), then $\mathfrak{F}$ is (i)-FL^r-differentiable. Similarly, it is same for the other cases as well.

3. Proposed Definitions

Definition 11. *The distance of intuitionistic fuzzy numbers P, Q ∈ IF_N ($\mathbb{R}^n$) with respect to their α-cut and β-cut is denoted by $\mathfrak{D}$(P, Q) and is defined as*

$\mathfrak{D}_1(P, Q) = sup_{\alpha\,\in\,[0,1]} max\left\{\left|P(\alpha)^- - Q(\alpha)^-\right|,\ \left|P(\alpha)^+ - Q(\alpha)^+\right|\right\}$.
$\mathfrak{D}_2(P, Q) = sup_{\beta\,\in\,[0,1]}\ max\ \left\{\left|P^*(\beta)^- - Q^*(\beta)^-\right|,\ \left|P^*(\beta)^+ - Q^*(\beta)^+\right|\right\}$.
$\mathfrak{D}(P, Q) = max\ \{\mathfrak{D}_1(P,\ Q),\ \mathfrak{D}_2(P,\ Q)\}$.

Lemma 1. *For P, Q, R, S ∈ $IF_N(\mathbb{R}^n)$*

(i) $\mathfrak{D}(P+R,\ Q+R) = \mathfrak{D}(P,\ Q)$.
(ii) $\mathfrak{D}(c\cdot P,\ c\cdot Q) = |c|\mathfrak{D}(P,\ Q)$, $c\in\mathbb{R}$.
(iii) $\mathfrak{D}(P+Q,\ R+S\ )\ \le \mathfrak{D}(P,\ R) + \mathfrak{D}(Q,\ S)$.
(iv) $\mathfrak{D}(\tau\cdot P,\ \omega\cdot P) = |\tau-\omega|\mathfrak{D}(P, 0)$, *for* $\tau\omega > 0$.
(v) $P \le Q$ *iff* $P(\alpha) \le Q(\alpha)$, $\alpha \in [0, 1]$ *iff* $P(\alpha)^+ \le Q(\alpha)^+$; $P(\alpha)^- \le Q(\alpha)^-$ *and* $P^*(\beta)^+ \ge Q^*(\beta)^+$; $P^*(\beta)^- \ge Q^*(\beta)^-$; $\alpha,\ \beta \in [0, 1]$.

Definition 12. *For* $1 \le r \le \infty$

(1) *$\mathfrak{F}$ is right-hand upper Intuitionistic fuzzy L^r-differentiable (IFL^r-differentiable) if there exists $\mathfrak{F}'$ ∈ IF_N ($\mathbb{R}^n$) such that*

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\ \mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

Similarly, $\mathfrak{F}$ is left-hand upper Intuitionistic fuzzy L^r-differentiable (IFL^r-differentiable) if there exists $\mathfrak{F}'$ ∈ IF_N ($\mathbb{R}^n$) such that

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

(2) *$\mathfrak{F}$ is right-hand lower Intuitionistic fuzzy L^r-differentiable (IFL^r-differentiable) if there exists $\mathfrak{F}'$ ∈ IF_N ($\mathbb{R}^n$) such that*

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

Similarly, $\mathfrak{F}$ is left-hand lower Intuitionistic fuzzy L^r-differentiable (IFL^r-differentiable) if there exists $\mathfrak{F}'$ ∈ IF_N ($\mathbb{R}^n$) such that

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h),\ \mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

$\mathfrak{F}$ is upper IFL^r-differentiable if two Intuitionistic fuzzy L^r-derivatives in (1) exist and are equal. Similarly, $\mathfrak{F}$ is lower IFL^r-differentiable if two Intuitionistic fuzzy L^r-derivatives in (2) exist and are equal.

Definition 13. *Let $\mathfrak{F}$ be an Intuitionistic fuzzy L^r-differentiable (IFL^r-differentiable) at $u \in [m, n]$, then there exists $\mathfrak{F}' \in IF_N\,(\mathbb{R}^n)$ such that the following four situations hold:*

i)

$$\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\tfrac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

ii)

$$\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\tfrac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

iii)

$$\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\tfrac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

iv)

$$\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\tfrac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0.$$

If $\mathfrak{F}$ satisfies case (i), then $\mathfrak{F}$ is (i)-IFL^r-differentiable. Similarly, it is same for the other cases as well.

Theorem 3. *If $\mathfrak{F}$ is IFL^r-differentiable, then the derivative is unique.*

Proof. Without loss of generality, let us assume $\mathfrak{F}$ is (ii) IFL^r-differentiable at $u \in$ [m, n]. Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be the derivatives of $\mathfrak{F}$. Then, $\forall\epsilon > 0\ \exists\ \delta > 0$, such that

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathbb{D}_1 h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{1}$$

and

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{2}$$

Then, by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{gathered}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathbb{D}_1 h,\ \mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} = \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u-h)+\mathbb{D}_1 h,\ \mathfrak{F}(u-h)+\mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} \\ \le \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u-h)+\mathbb{D}_1 h,\mathfrak{F}(u))+\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} \\ \le \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u-h)+\mathbb{D}_1 h,\mathfrak{F}(u))]^r dh\right\}^{\frac{1}{r}} + \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u-h)+\mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} \\ \le \tfrac{\epsilon}{2}+\tfrac{\epsilon}{2} = \epsilon.\end{gathered} \tag{3}$$

Therefore, the derivatives $\mathbb{D}_1$ and $\mathbb{D}_2$ are equal and hence unique.

Similar results follow for cases (i), (iii) and (iv). □

Theorem 4. *If $\mathfrak{F}$ is IFL^r-differentiable, then it is continuous.*

Proof. Assume that $\mathfrak{F}$ is (i) IFL^r-differentiable. Then, $\forall \epsilon > 0\ \exists\ \delta > 0$, for $|l| < \delta$

$$\lim_{l \to 0^+} \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{4}$$

Then, by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{aligned} \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u))]^r dh \right\}^{\frac{1}{r}} &= \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h) + \mathfrak{F}'(u)h, \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}} \\ &\le \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h) + \mathfrak{F}(u)h, \mathfrak{F}(u)) + \mathfrak{D}(\mathfrak{F}(u),\ \mathfrak{F}(u) + \mathfrak{F}'(u)h))]^r dh \right\}^{\frac{1}{r}} \\ &\le \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h) + \mathfrak{F}'(u)h, \mathfrak{F}(u))]^r \right\}^{\frac{1}{r}} + \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r \right\}^{\frac{1}{r}} \\ &\le \epsilon + \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}'(u)h,\ 0)]^r \right\}^{\frac{1}{r}} \\ &\le \epsilon + \left\{ \tfrac{l^r}{r+1} \left\| \mathfrak{F}'^{(u)} \right\|^r \right\}^{\frac{1}{r}}. \end{aligned} \tag{5}$$

$$\Rightarrow \lim_{l \to 0^+} \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u))]^r dh \right\}^{\frac{1}{r}} = 0. \tag{6}$$

Similarly,

$$\lim_{l \to 0^+} \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h))]^r dh \right\}^{\frac{1}{r}} = 0. \tag{7}$$

Therefore, f is continuous.

Similar results follow for cases (ii), (iii) and (iv). □

Example 1. *Suppose T(u) = T_0 is a constant function, then T(u) is IFL^r-differentiable and T$'$(u) = 0.*

Solution. From the case (i) of Definition 13.

$$\begin{aligned} \lim_{l \to 0^+} \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}} &= \lim_{l \to 0^+} \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(T(u+h), T(u) + 0h)]^r dh \right\}^{\frac{1}{r}} \\ &= \lim_{l \to 0^+} \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(T_0, T_0 + 0)]^r dh \right\}^{\frac{1}{r}} \\ &= \lim_{l \to 0^+} \left\{ \tfrac{1}{l} \textstyle\int_0^l [\mathfrak{D}(T_0, T_0)]^r dh \right\}^{\frac{1}{r}} = 0. \end{aligned} \tag{8}$$

and

$$\begin{aligned} \lim_{l \to 0^-} \left\{ \tfrac{1}{l} \textstyle\int_l^0 [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}} &= \lim_{l \to 0^-} \left\{ \tfrac{1}{l} \textstyle\int_l^0 [\mathfrak{D}(T(u), T(u-h) + 0h)]^r dh \right\}^{\frac{1}{r}} \\ &= \lim_{l \to 0^-} \left\{ \tfrac{1}{l} \textstyle\int_l^0 [\mathfrak{D}(T_0, T_0 + 0)]^r dh \right\}^{\frac{1}{r}} = 0. \end{aligned} \tag{9}$$

Therefore, the fuzzy constant function is (i) IFL^r-differentiable.

Similar results follow for cases (ii), (iii) and (iv). □

Example 2. *For $\lambda > 0$, consider a function T(u) = $e^{\lambda u} T_0$ satisfying T$'$(u) = $\lambda e^{\lambda u} T_0$, where T_0 is a constant function.*

Solution. From case (i) of Definition 13.

$$\begin{aligned}
&\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'^{(u)}h\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(T(u+h),T(u)+T'(u)h)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(e^{\lambda(u+h)}T_0,e^{\lambda u}T_0+\lambda e^{\lambda u}hT_0\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(e^{\lambda u}e^{\lambda h}T_0,e^{\lambda u}T_0(1+\lambda h)\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&\qquad\text{[From Lemma 1]}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(e^{\lambda h}T_0,T_0(1+\lambda h)\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(e^{\lambda h}T_0,T_0(1+\lambda h)\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&\qquad\text{[From Lemma 1]}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\left|e^{\lambda h}-(1+\lambda h)\right|\mathfrak{D}(T_0,T_0)\right]^r dh\right\}^{\frac{1}{r}}=0.
\end{aligned}\tag{10}$$

Similarly for left end limit,

$$\begin{aligned}
&\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'^{(u)}h\right)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(T(u),T(u-h)+T'(u)h)\right]^r dh\right\}^{\frac{1}{r}}\\
&=\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}\left(e^{\lambda u}T_0,e^{\lambda(u-h)}T_0+\lambda e^{\lambda u}hT_0\right)\right]^r dh\right\}^{\frac{1}{r}}=0.
\end{aligned}\tag{11}$$

Therefore, $T(u) = e^{\lambda u}T_0$ is (i) IFL^r-differentiable.

Similar results follow for (ii), (iii) and (iv). □

Theorem 5. *Let $\mathfrak{F},\ \mathfrak{G} : [m,n] \to IFN\ (\mathbb{R}^n)$, $1 \le r < \infty$. Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are upper IFL^r-differentiable, then*

(i) $\mathfrak{F}+\mathfrak{G}$ *is upper IFL^r-differentiable and* $(\mathfrak{F}+\mathfrak{G})' = \mathfrak{F}' + \mathfrak{G}'$.
(ii) $(\omega\mathfrak{F})'(u) = \omega\mathfrak{F}'(u)$, *for all* $\omega \in \mathbb{R}$.

Proof. (i) Let $\mathfrak{F}$ and $\mathfrak{G}$ be upper IFL^r-differentiable.

Then, $\forall\epsilon > 0,\ \exists\,\delta > 0$ s. t. for $|l| < \delta$

$$\left\{\frac{1}{l}\int_0^l\left[\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)\right]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2}\tag{12}$$

and

$$\left\{\frac{1}{l}\int_0^l\left[\mathfrak{D}\left(\mathfrak{G}(u),\mathfrak{G}(u-h)+\mathfrak{G}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2}\,.\tag{13}$$

Then, by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{aligned}
&\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(\mathfrak{G}(u)+\mathfrak{F}(u),\mathfrak{G}(u-h)+\mathfrak{F}(u-h)+\mathfrak{G}'(u)h\ \mathfrak{F}'(u)h)\right]^r dh\right\}^{\frac{1}{r}}\\
&\le\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(\mathfrak{G}(u),\mathfrak{G}(u-h)+\mathfrak{G}'(u)h)+\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)\right]^r dh\right\}^{\frac{1}{r}}\\
&\le\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(\mathfrak{G}(u),\mathfrak{G}(u-h)+\mathfrak{G}'(u)h)\right]^r dh\right\}^{\frac{1}{r}}+\left\{\tfrac{1}{l}\int_0^l\left[\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)\right]^r dh\right\}^{\frac{1}{r}}\\
&\le\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.
\end{aligned}\tag{14}$$

Similarly, $\forall \epsilon > 0$, $\exists\, \delta > 0$ s. t. for $|l| < \delta$

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{15}$$

and

$$\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{G}(u+h),\mathfrak{G}(u)+\mathfrak{G}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{16}$$

Then, by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{gathered}\left\{\tfrac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{G}(u+h)+\mathfrak{F}(u+h),\mathfrak{G}(u)+\mathfrak{F}(u)+\mathfrak{G}'(u)h\ \mathfrak{F}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} \\ \leq \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u+h),\mathfrak{G}(u)+\mathfrak{G}'(u)h)+\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ \leq \left\{\tfrac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{G}(u+h),\mathfrak{G}(u)+\mathfrak{G}'(u)h\right)\right]^r dh\right\}^{\frac{1}{r}} + \left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ \leq \tfrac{\epsilon}{2}+\tfrac{\epsilon}{2} = \epsilon.\end{gathered} \tag{17}$$

Thus, $\mathfrak{F}+\mathfrak{G}$ is upper IFL^r-differentiable at u and $(\mathfrak{F}+\mathfrak{G})' = \mathfrak{F}'+\mathfrak{G}'$.

(ii) Let $\mathfrak{F}$ be upper IFL^r-differentiable.
Then, $\forall \epsilon > 0$, $\exists\, \delta > 0$ s. t. for $|l| < \delta$.

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2}. \tag{18}$$

Then, by Lemma 1 and Minkowski's inequality, we obtain

$$\begin{gathered}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}((\omega\mathfrak{F})(u),(\omega\mathfrak{F})(u-h)+(\omega\mathfrak{F})'(u)h)]^r dh\right\}^{\frac{1}{r}},\ \omega\in\mathbb{R} \\ = \left\{\tfrac{1}{l}\int_0^l \omega[\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ = \omega\left\{\tfrac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'^{(u)}h\right)\right]^r dh\right\}^{\frac{1}{r}} < \tfrac{\epsilon}{2} < \epsilon.\end{gathered} \tag{19}$$

Similarly,

$$\begin{gathered}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}((\omega\mathfrak{F})(u+h),(\omega\mathfrak{F})(u)+(\omega\mathfrak{F})'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ = \left\{\tfrac{1}{l}\int_0^l \omega[\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ = \omega\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \tfrac{\epsilon}{2} < \epsilon.\end{gathered} \tag{20}$$

Thus, $(\omega\mathfrak{F})' = \omega\mathfrak{F}'$, at u for all $\omega\in\mathbb{R}$. □

Remark 1.

(i) *If $\mathfrak{F}$ and $\mathfrak{G}$ are lower IFL^r-differentiable, then Theorem 5 holds.*
(ii) *Also, if $\mathfrak{F}$ is upper IFL^r-differentiable and $\mathfrak{G}$ is lower IFL^r-differentiable, then $\mathfrak{F}$ and $\mathfrak{G}$ are IFL^r-differentiable, and Theorem 5 still holds.*

Theorem 6. *Let $\mathfrak{G} : [m,n] \to \mathbb{R}^n$ be differentiable ($\mathfrak{G}^*$ be its derivative), $\mathfrak{F} : [m,n] \to IF_N(\mathbb{R}^n)$ be GH-differentiable. If $H : [m,n] \to IF_N(\mathbb{R}^n)$ s. t. $H(u) = \mathfrak{G}(u)\ \mathfrak{F}(u)$, then H(u) is IFL^r-differentiable and $H'(u) = \mathfrak{G}^*\ (u)\cdot\mathfrak{F}(u) + \mathfrak{G}(u)\cdot\mathfrak{F}'_{GH}(u)$.*

Proof. Let us assume that $H'(u) \in IF_N(\mathbb{R}^n)$ exists; we shall prove that H(u) *is* IFL^r-differentiable

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(H(u+h),H(u)+H\prime(u)h)]^r dh\right\}^{\frac{1}{r}}$$
$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\ \mathfrak{G}(u+h)\mathfrak{F}(u+h),\ \mathfrak{G}(u)\mathfrak{F}(u)+H\prime(u)h)]^r dh\right\}^{\frac{1}{r}}$$
$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\ \mathfrak{G}(u+h)\mathfrak{F}(u+h)-\mathfrak{G}(u)\mathfrak{F}(u+h)+\mathfrak{G}(u)\mathfrak{F}(u+h),\ \mathfrak{G}(u)\mathfrak{F}(u)+H'(u)h)]^r dh\right\}^{\frac{1}{r}}$$
$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\ \mathfrak{G}(u+h)\mathfrak{F}(u+h)-\mathfrak{G}(u)\mathfrak{F}(u+h)+\mathfrak{G}(u)\mathfrak{F}(u+h)\ominus_H\ \mathfrak{G}(u)\mathfrak{F}(u),\ H\prime(u\ )h)]^r dh\right\}^{\frac{1}{r}} \quad (21)$$
$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}((\ \mathfrak{G}(u+h)-\mathfrak{G}(u))\mathfrak{F}(u+h)+\mathfrak{G}(u)(\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u)),H\prime(u))]^r dh\right\}^{\frac{1}{r}}$$
$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{G}^*\ (u)\mathfrak{F}(u+h)+\mathfrak{G}(u)\mathfrak{F}'_{GH}(u),H\prime(u)\right)\right]^r dh\right\}^{\frac{1}{r}}.$$
As when $h\to 0^+$, $H'^{(u)}=\mathfrak{G}^*(u)\mathfrak{F}(u)+\mathfrak{G}(u)\mathfrak{F}'_{GH}(u)$.

Therefore,

$$\begin{aligned}&\underset{l\to 0^+}{lim}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(H(u+h),H(u)+H'(u)h)]^r dh\right\}^{\frac{1}{r}}\\ &=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{G}(u)\mathfrak{F}(u+h)+\mathfrak{G}(u)\mathfrak{F}'_{GH}(u),H\prime(u)\right)\right]^r dh\right\}^{\frac{1}{r}}\\ &=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(H'(u),H\prime(u))]^r dh\right\}^{\frac{1}{r}}=0.\end{aligned} \quad (22)$$

Similarly,

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(H(u),H(u-h)+H\prime(u)h)]^r dh\right\}^{\frac{1}{r}}=0. \quad (23)$$

Thus, $H(u)$ is (i)-IFL^r-differentiable at u and $H(u)=\mathfrak{G}^*\ (u)\cdot\mathfrak{F}(u)+\mathfrak{G}(u)\cdot\mathfrak{F}'_{GH}(u)$. Similar results follow for cases (ii), (iii) and (iv). □

Theorem 7. *IF $\mathfrak{F}$ is GH-differentiable, then it is IFL^r-differentiable.*

Proof. *Without loss of generality, let us assume that $\mathfrak{F}$ is (iii)-GH-differentiable, then for $l>0$, we have*

$$\lim_{h\to 0^+}\frac{\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u)}{h}=\lim_{h\to 0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-h)}{h}=\mathfrak{F}'_{GH}(u). \quad (24)$$

Thus,

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}\left(\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u),\mathfrak{F}'_{GH}(u)h\right)\\ &=\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}\left(\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-h),\mathfrak{F}'_{GH}(u)h\right)\\ &=\ 0.\end{aligned}$$

Then,

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}\left(\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-h),\mathfrak{F}'_{GH}(u)\right)\\ &=\lim_{h\to 0^+}\frac{1}{-h}\mathfrak{D}\left(\mathfrak{F}(u-h)\ominus_H\mathfrak{F}(u),-\mathfrak{F}'_{GH}(u)h\right)\\ &=\lim_{h'\to 0^-}\frac{1}{h'}\mathfrak{D}\left(\mathfrak{F}(u+h')\ominus_H\mathfrak{F}(u),\mathfrak{F}'_{GH}(u)h'\right)\end{aligned}$$

(putting $h'=-h$)

$$=\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}\Big(\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u),\mathfrak{F}'_{GH}(u)h\Big).$$

Therefore, $\forall\epsilon>0,\exists\, T_1>0\ s.t. 0<h<T_1$,

$$\mathfrak{D}\Big(\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u),\mathfrak{F}'_{GH}(u)h\Big)<\epsilon$$

i.e.,

$$D\Big(\mathfrak{F}(u+h),\ \mathfrak{F}(u)+\mathfrak{F}'_{GH}(u)h\Big) < \epsilon. \tag{25}$$

Furthermore, $\forall\epsilon > 0,\ \exists\, T_1 > 0$, we restrict $0 < l < T_1$, then

$$\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\Big(\mathfrak{F}(u+h), \mathfrak{F}(u)+\mathfrak{F}'_{GH}(u)h\Big)\right]^r dh\right\}^{\frac{1}{r}} < \left\{\frac{1}{l}\int_0^l \epsilon^r dh\right\}^{\frac{1}{r}} = \epsilon. \tag{26}$$

Similarly, $\forall\epsilon > 0, \exists\, T_2 < 0$, s. t. $T_2 < l < 0$, then

$$\left\{\frac{1}{l}\int_l^0 \left[\mathfrak{D}\Big(\mathfrak{F}(u+h), \mathfrak{F}(u)+\mathfrak{F}'_{GH}(u)h\Big)\right]^r dh\right\}^{\frac{1}{r}} < \epsilon. \tag{27}$$

This implies that

$$\begin{aligned}&\lim_{l\to 0^+}\left\{\tfrac{1}{l}\textstyle\int_0^l \left[\mathfrak{D}\big(\mathfrak{F}(u+h), \mathfrak{F}(u)+\mathfrak{F}'_{GH}(u)h\big)\right]^r dh\right\}^{\frac{1}{r}}\\ &= \lim_{l\to 0^-}\left\{\tfrac{1}{l}\textstyle\int_0^l \left[\mathfrak{D}\big(\mathfrak{F}(u+h), \mathfrak{F}(u)+\mathfrak{F}'_{GH}(u)h\big)\right]^r dh\right\}^{\frac{1}{r}} = 0.\end{aligned} \tag{28}$$

Therefore, $\mathfrak{F}$ is IFL^r—differentiable at u.
Similar results follow for cases (i), (ii) and (iv). □

Remark 2. *The converse of the above theorem does not hold true. We shall prove this using an example.*

Example 3. *Let* $\mathfrak{F}$*: [−2, 2]* $\to IF_N(\mathbb{R})$

$$[\mathfrak{F}(u)]_\alpha = \Big[2\alpha - 2 + \Big(\tfrac{1}{2} - \tfrac{3}{4}\alpha\Big)|u|,\ 2 - 2\alpha + \Big(\tfrac{3}{4}\alpha - \tfrac{1}{2}\Big)|u|\Big].$$
$$[\mathfrak{F}(u)]_\beta = \Big[-2\beta + \Big(\tfrac{3}{4}\beta - \tfrac{1}{3}\Big)|u|,\ 2\beta + \Big(\tfrac{1}{3} - \tfrac{3}{4}\beta\Big)|u|\Big].$$
$$0 \le \alpha + \beta \le 1$$

In the case of α, $\mathfrak{F}(0) = (-2, 0, 2)$ which is a triangular fuzzy number.
In particular, $\mathfrak{F}(u) = \Big(-1, -\frac{1}{2}, \frac{1}{2}, 1\Big)$, when $u = \pm 2$.

Since $[\mathfrak{F}(0)]_\alpha = [2\alpha - 2, 2 - 2\alpha]$ and $[\mathfrak{F}(2)]_\alpha = \Big[\frac{1}{2}\alpha - 1, 1 - \frac{1}{2}\alpha\Big]$, it is easy to see that for $\delta > 0$, $\forall\, h \in B(0,\ \delta)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(h)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(-h)$, $\mathfrak{F}(h) \ominus_H \mathfrak{F}(0)$ and $\mathfrak{F}(-h) \ominus_H \mathfrak{F}(0)$ do not exist. Thus, $\mathfrak{F}$ is not GH-differentiable.

For $u = 0$,

$$\begin{aligned}&\mathfrak{D}_1(\mathfrak{F}(0+h), \mathfrak{F}(0)+\mathfrak{F}'(0)h)\\ &= \mathfrak{D}_1\Big(\Big[2\alpha - 2 + \Big(\tfrac{1}{2} - \tfrac{3}{4}\alpha\Big)|h|,\ 2 - 2\alpha + \Big(\tfrac{3}{4}\alpha - \tfrac{1}{2}\Big)|h|\Big],\ [2\alpha - 2,\ 2 - 2\alpha]\Big)\\ &= sup_{\alpha\,\in\,[0,1]} max\Big\{\Big|\Big(\tfrac{1}{2} - \tfrac{3}{4}\alpha\Big)h\Big|,\ \Big|\Big(\tfrac{3}{4}\alpha - \tfrac{1}{2}\Big)h\Big|\Big\}\\ &= \tfrac{1}{2}|h|.\end{aligned} \tag{29}$$

In the case of β, $\mathfrak{F}(0) = \{0\}$.
In particular, $\mathfrak{F}(u) = \big(-\frac{7}{6}, -\frac{2}{3}, \frac{2}{3}, \frac{7}{6}\big)$, when $u = \pm 2$.

Since $[\mathfrak{F}(0)]_\beta = [-2\beta, 2\beta]$ and $[\mathfrak{F}(2)]_\beta = \Big[-\frac{1}{2}\beta - \frac{2}{3}, \frac{1}{2}\beta - \frac{2}{3}\Big]$, it is easy to see that for $\delta > 0$, $\forall\, h \in B(0,\ \delta)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(h)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(-h)$, $\mathfrak{F}(h) \ominus_H \mathfrak{F}(0)$ and $\mathfrak{F}(-h) \ominus_H \mathfrak{F}(0)$ do not exist. Thus, $\mathfrak{F}$ is not GH-differentiable.

For $u = 0$,

$$\begin{aligned} &\mathfrak{D}_2(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h) \\ &= \mathfrak{D}_2\left(\left[-2\beta + \left(\tfrac{3}{4}\beta - \tfrac{1}{3}\right)|h|,\ 2\beta + \left(\tfrac{1}{3} - \tfrac{3}{4}\beta\right)|h|\right],\ [-2\beta,\ 2\beta]\right) \\ &= sup_{\alpha\ \in\ [0,1]} max\left\{\left|\left(\tfrac{3}{4}\beta - \tfrac{1}{3}\right)h\right|,\ \left|\left(\tfrac{1}{3} - \tfrac{3}{4}\beta\right)h\right|\right\} \\ &= \tfrac{5}{12}|h|. \end{aligned} \tag{30}$$

Next, we can find that $\mathfrak{F}$ is (i)-IFL^r- differentiable and $\mathfrak{F}'(0) = 0$.
For $u = 0$,

$$\begin{aligned} &\lim_{h\to 0}\mathfrak{D}(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h) \\ &= \lim_{h\to 0} max\{\mathfrak{D}_1(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h),\ \mathfrak{D}_2(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h)\} \\ &= \lim_{h\to 0} max\left\{\tfrac{1}{2}|h|,\ \tfrac{5}{12}|h|\right\} = \lim_{h\to 0}\tfrac{1}{2}|h| = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\forall\, \epsilon > 0,\ \exists\, T_1 > 0\ s.t.\ 0 < h < T_1, \\ &\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h) < \epsilon. \end{aligned} \tag{31}$$

Furthermore, $\forall\, \epsilon > 0,\ \exists\, T_1 > 0$, we restrict $0 < l < T_1$, then

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \left\{\frac{1}{l}\int_0^l \epsilon^r dh\right\}^{\frac{1}{r}} = \epsilon. \tag{32}$$

Similarly, $\forall\, \epsilon > 0,\ \exists\, T_2 < 0,\ s.\ t.\ \ T_2 < l < 0$, then

$$\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \epsilon. \tag{33}$$

This implies that

$$\begin{aligned} &\lim_{l\to 0^+}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} \\ &= \lim_{l\to 0^-}\left\{\tfrac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0. \end{aligned} \tag{34}$$

Therefore, $\mathfrak{F}$ is IFL^r-differentiable at $u = 0$ with $\mathfrak{F}'(0) = 0$ but not GH-differentiable.
Similar results follow for cases (ii), (iii) and (iv).

4. Intuitionistic Fuzzy Cauchy Problem

Lemma 2 *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$: $[m, n] \to IF_N(\mathbb{R}^n)$ be differentiable and assume that its derivatives $\mathfrak{G}_1$, $\mathfrak{G}_2$: $[m, n] \to IF_N(\mathbb{R}^n)$ are integrable over [m, n], then*

$$\mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))\ \le \mathfrak{D}(\mathfrak{F}_1\ (u_0),\ \mathfrak{F}_2\ (u_0)) + \int_{u_0}^{u} \mathfrak{D}(\mathfrak{G}_1(s), \mathfrak{G}_2(s))\mathrm{ds} \text{ for all } s\ \in\ [m,\ n]. \tag{35}$$

Proof.

$$\forall\epsilon >\ 0,\ \exists\, T > 0 \text{ s.t. } 0 <\ h < T_1,\ \mathfrak{D}(\mathfrak{F}(u+h),\ \mathfrak{F}(u) + \mathfrak{G}(u)h)\ <\ \epsilon. \tag{36}$$

Define

$$\zeta(u) = \mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))$$

Then

$$\begin{aligned}
&\zeta(u+h)-\zeta(u)=\mathfrak{D}(\mathfrak{F}_1(u+h),\ \mathfrak{F}_2(u+h))-\mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))\\
&=\mathfrak{D}(\mathfrak{F}_1(u+h),\ \mathfrak{F}_1(u)+h\ \mathfrak{G}_1(u))+\mathfrak{D}(\mathfrak{F}_1(u)+h\ \mathfrak{G}_1(u),\ \mathfrak{F}_2(u)+h\ \mathfrak{G}_1(u))+\\
&\mathfrak{D}(\mathfrak{F}_2(u)+h\ \mathfrak{G}_1(u),\ \mathfrak{F}_2(u))+h\ \mathfrak{G}_2(u))+\mathfrak{D}(\mathfrak{F}_2(u)+h\ \mathfrak{G}_2(u),\ \mathfrak{F}_2(u+h))-\mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))\\
&=\mathfrak{D}(\mathfrak{F}_1(u+h),\ \mathfrak{F}_1(u)+h\ \mathfrak{G}_1(u))+\mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))+h\ \mathfrak{D}(\mathfrak{G}_1(u),\ \mathfrak{G}_2(u))+\\
&\mathfrak{D}(\mathfrak{F}_2(u)+h\ \mathfrak{G}_2(u),\ \mathfrak{F}_2(u+h))-\mathfrak{D}(\mathfrak{F}_1(u),\ \mathfrak{F}_2(u))\\
&\zeta(u+h)-\zeta(u)=\mathfrak{D}(\mathfrak{F}_1(u+h),\ \mathfrak{F}_1(u)+h\ \mathfrak{G}_1(u))+h\ \mathfrak{D}(\mathfrak{G}_1(u),\ \mathfrak{G}_2(u))+\mathfrak{D}(\mathfrak{F}_2(u)+h\ \mathfrak{G}_2(u),\mathfrak{F}_2(u+h)).
\end{aligned}$$

$$\frac{\zeta(u+h)-\zeta(u)}{h}=\frac{\mathfrak{D}(\mathfrak{F}_1(u+h),\mathfrak{F}_1(u)+h\mathfrak{G}_1(u))}{h}+\mathfrak{D}(\mathfrak{G}_1(u),\mathfrak{G}_2(u))+\frac{\mathfrak{D}(\mathfrak{F}_2(u)+h\mathfrak{G}_2(u),\mathfrak{F}_2(u+h))}{h}. \tag{37}$$

Now, $\epsilon > 0$, $\exists\ T_1 > 0$, we restrict $0 < 1 < T_1$,

$$\begin{aligned}
&\frac{\zeta(u+h)-\zeta(u)}{h}\\
&\le \left\{\frac{1}{l}\int_0^l[\mathfrak{D}(\mathfrak{F}_1(u+h),\mathfrak{F}_1(u)+\mathfrak{G}(u)h)]^r dh\right\}^{\frac{1}{r}}+\mathfrak{D}(\mathfrak{G}_1(u),\mathfrak{G}_2(u))\\
&+\left\{\frac{1}{l}\int_0^l[\mathfrak{D}(\mathfrak{F}_2(u+h),\mathfrak{F}_2(u)+\mathfrak{G}(u)h)]^r dh\right\}^{\frac{1}{r}}.
\end{aligned}$$

and as lim tends to 0^+, $\zeta'(u)\le\mathfrak{D}(\mathfrak{G}_1(u),\mathfrak{G}_2(u))$.

$$\Rightarrow \zeta(u)\le\int_{u_0}^{u}\mathfrak{D}(\mathfrak{G}_1(\mathrm{s}),\mathfrak{G}_2(s))ds. \tag{38}$$

$$\mathfrak{D}(\mathfrak{F}_1(u),\mathfrak{F}_2(u))\le\mathfrak{D}(\mathfrak{F}_1(u_0),\mathfrak{F}_2(u_0))+\int_{u_0}^{u}\mathfrak{D}(\mathfrak{G}_1(s),\mathfrak{G}_2(s))ds.$$

Similar results follow for the left end limit. □

Consider an initial value problem for the intuitionistic fuzzy differential equation:

$$\begin{aligned}
&\mathfrak{F}'(u)=\mathfrak{G}(u,\ \mathfrak{F}(u)).\\
&\mathfrak{F}(u_0)=(A(u_0),\ B(u_0)),\ I=[m,n]\text{ and }\mathfrak{G}:\ I\times IF_N(\mathbb{R}^n)\to IF_N(\mathbb{R}^n).
\end{aligned} \tag{39}$$

Let $C(I\times IF_N(\mathbb{R}^n), IF_N(\mathbb{R}^n))$ be the set of all continuous mappings from $I\times IF_N(\mathbb{R}^n)$ to $\mathrm{IF_N}(\mathbb{R}^n)$.

Definition 14. *In [54] $\mathfrak{F}$: $I\to IF_N(\mathbb{R}^n)$ is a solution of the initial value problem if it is continuous and satisfies the integral solution:*

$$\mathfrak{F}(u)=\mathfrak{F}(u_0)\oplus\int_{\mathrm{u_0}}^{\mathrm{u}}\mathfrak{G}(s,\mathfrak{F}(s))ds. \tag{40}$$

Theorem 8. *Let $\mathfrak{G}\in C(I\times\ IF_N(\mathbb{R}^n),IF_N(\mathbb{R}^n))$, such that there exists a constant $k\ge 0$ satisfying*

$$\mathfrak{D}(\mathfrak{G}(u,\mathfrak{F}_1),\mathfrak{G}(u,\mathfrak{F}))\le\ k\,\mathfrak{D}(\mathfrak{F}_1,\mathfrak{F}_2)\ ;(u,\mathfrak{F}_1)\ u,\mathfrak{F}_2)\in I\times\mathrm{IF}_N(\mathbb{R}^n). \tag{41}$$

Then, the initial intuitionistic fuzzy problem (4.5) has a unique solution.

Proof. For $\mathfrak{F}\in C(I, IF_N(\mathbb{R}^n))$, let us consider a mapping G: $X_0\to X_0$, where

$$\mathrm{X_0=C(I,\ IF_N}(\mathbb{R}^n))$$

is defined by

$$G\mathfrak{F}_1(u)=\mathfrak{F}(u_0)\oplus\int_{u_0}^{u}\mathfrak{G}(s,\ \mathfrak{F}(s))ds. \tag{42}$$

Let

$$\varphi(u) = G\mathfrak{F}_1(u+h), \psi(u) = G\mathfrak{F}_1(u).$$

Then

$$\begin{aligned}
\mathfrak{D}(\varphi(u),\ \psi(u)) &= \mathfrak{D}(G\mathfrak{F}_1(u+h),\ G\mathfrak{F}_1(u)) \\
&= \mathfrak{D}(\mathrm{G}\mathfrak{F}_1(u+h),\mathrm{G}\mathfrak{F}_1(u)) \\
&= \mathfrak{D}\left(\mathfrak{F}(u_0) \oplus \int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds,\ \mathfrak{F}(u_0) \oplus \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds\right) \\
&= \mathfrak{D}\left(\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}(s))ds,\ \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}(s))ds\right) \\
&= \mathfrak{D}\left(\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds,\ \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds\right) \\
&= max\left(\mathfrak{D}_1\left(\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}(s))ds,\ \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds\right), \mathfrak{D}_2\left(\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds,\ \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))ds\right)\right) \\
&= max\left(sup_{\alpha\ \in\ [0,1]} max\left\{\left|\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^- ds - \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^- ds\right|,\ \left|\int_{u_0}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^+ ds\right.\right.\right. \\
&\quad \left.\left. - \int_{u_0}^{u} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^+ ds\right|\right\},\ sup_{\alpha\ \in\ [0,1]} max\left\{\left|\int_{u_0}^{u+h} \mathfrak{G}^*(s,\ \mathfrak{F}_1(s))(\beta)^- ds\right.\right. \\
&\quad \left.\left.\left. - \int_{u_0}^{u} \mathfrak{G}^*(s,\ \mathfrak{F}_1(s))(\beta)^- ds\right|,\ \left|\int_{u_0}^{u+h} \mathfrak{G}^*(s,\ \mathfrak{F}(s))(\beta)^+ ds - \int_{u_0}^{u} \mathfrak{G}^*(s,\ \mathfrak{F}_1(s))(\beta)^+ ds\right|\right\}\right) \\
&= max\left(sup_{\alpha\ \in\ [0,1]} max\left\{\left|\int_{u}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^- ds\right|,\ \left|\int_{u}^{u+h} \mathfrak{G}(s,\ \mathfrak{F}_1(s))(\alpha)^+ ds\right|\right\},\ sup_{\alpha\ \in\ [0,1]} max\left\{\left|\int_{u}^{u+h} \mathfrak{G}^*(s,\ \mathfrak{F}_1(s))(\beta)^- ds\right|,\ \left|\int_{u}^{u+h} \mathfrak{G}^*(s,\ \mathfrak{F}_1(s))(\beta)^+ ds\right|\right\}\right).
\end{aligned} \tag{43}$$

when h $\to 0^+$, $\mathfrak{D}(\varphi(u), \psi(u)) \to 0^+$. Therefore,

$$G\mathfrak{F}_1 \in CI\ (\mathbb{R}^n)$$

Now, let $\mathfrak{F}_1,\ \mathfrak{F}_2 \in C(I,\ IF_N(\mathbb{R}^n))$ and by Lemma 2.

$$\begin{aligned}
\mathfrak{D}(\mathrm{G}\mathfrak{F}_1(u),\mathrm{G}\mathfrak{F}_2(u)) &\leq \mathfrak{D}\left(\int_{u_0}^{u} (\mathfrak{G}(s,\ \mathfrak{F}_1(s)), \mathfrak{G}(s,\ \mathfrak{F}_2(s)\ )ds\right) \\
&\leq \left(\mathfrak{D}\int_{u_0}^{u} (\mathfrak{G}(s,\ \mathfrak{F}_1(s)), \mathfrak{G}(s,\ \mathfrak{F}(s))ds\ \right) \\
&\leq k(u-u_0)\mathfrak{D}(\mathfrak{F}_1, \mathfrak{F}_2)[Lipschitz\ condition].
\end{aligned} \tag{44}$$

$$\therefore\ G\ is\ a\ contraction.$$

and hence the initial value problem (39) has a unique solution. □

5. Conclusions

In this paper, we extended the fuzzy L^r-norm-based derivative (FL^r-derivative) to the intuitionistic fuzzy number valued function. We proposed a definition for the intuitionistic fuzzy L^r-norm-based derivative (IFL^r-derivative) using the Hausdorff distance of *α-cuts and β-cuts*. Next, we proved some properties, such as uniqueness of the IFL^r-derivative, and its continuity. Next, we showed that the IFL^r-derivative can be written as a product of the derivative and GH derivative. Furthermore, showed that GH differentiability implies the IFL^r-differentiable, but the converse is not always true, which allows us to determine the derivative for functions without the existence of the GH derivative. Lastly, we solved the Cauchy problem for the intuitionistic fuzzy differential equation with the proposed extended IFL^r-derivative by contraction mapping. In view of the above results we can conclude that the IFL^r-derivative is a generalization of the GH derivative. The solutions of the intuitionistic fuzzy differential equation (IFDE) are determined by the lower and upper *α-cuts and β-cuts* of intuitionistic functions; solving it may seem feasible in the linear case, but the task becomes complicated in non-linear intuitionistic fuzzy differential equations. So, we may conclude the above method cannot be applied to solve non-linear intuitionistic fuzzy differential equations.

Future directions of work may be as follows:

- In future, we plan a device method to solve non-linear intuitionistic fuzzy differential equations.
- In future, we plan to find numerical method for the proposed derivative, applying it to intuitionistic integral theory and the fuzzy partial differential equation based on IFL^r-derivative.

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