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Article

Merging Intuitionistic and De Morgan Logics

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Abstract: We introduce De Morgan Heyting logic for Heyting algebras with De Morgan negation (DH-algebras). The variety $\mathcal{DH}$ of all DH-algebras is congruence distributive. The lattice of all subvarieties of $\mathcal{DH}$ is distributive. We show the discrete dualities between De Morgan frames and DH-algebras. The Kripke completeness and finite approximability of some DH-logics are proven. Some conservativity of DH expansion of a Kripke complete superintuitionistic logic is shown by the construction of frame expansion. Finally, a cut-free terminating Gentzen sequent calculus for the DH-logic of De Morgan Boolean algebras is developed.

Keywords: Heyting algebra; De Morgan algebra; intuitionistic logic

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1. Introduction

Intuitionistic logic provides a basis for the study of foundations of constructive sciences. When the constructive reasoning is concerned in many scenarios, merging intuitionistic logic and other logical operators becomes useful. An interesting direction is the study of intuitionistic modal logic (cf., e.g., [1–4]). From the algebraic perspective, algebras for an intuitionistic modal logic are Heyting algebras with modal operators. This approach emphasizes the interaction between intuitionistic and modal operators. Thus, it differs from various ways of combining logics like fibring, fusion or product (cf., e.g., [5–7]).

The aim of the present work is merging the intuitionistic and De Morgan logics. Sankappanavar [8] proposed Heyting algebras with a dual homomorphism, and the variety of De Morgan Heyting algebras (DH-algebras) was investigated. A DH-algebra is a Heyting algebra $(A, \wedge, \vee, \to, 0)$ with a De Morgan negation $\sim$. It is well known that quasi-Boolean algebras (or De Morgan algebras) are bounded distributive lattices with De Morgan negation (cf., e.g., [9]). The logic of De Morgan algebras is just the Belnap–Dunn four-valued logic (cf., e.g., [10–13]). In this paper, we study the logic of DH-algebras which is the combination of intuitionistic and De Morgan logics.

In the present article, we show some results on the variety $\mathcal{DH}$ of all DH-algebras. First, we show that $\mathcal{DH}$ is congruence distributive and hence it has the congruence extension property. Then, we show that the lattice $\Lambda(\mathcal{DH})$ of all subvarieties of $\mathcal{DH}$ is distributive. This lattice $\Lambda(\mathcal{DH})$ is dually isomorphic to the lattice of all DH-logics. Then, we present discrete dualities between DH-algebras and De Morgan frames. A De Morgan frame (cf., e.g., [14,15]) is an intuitionistic frame with an antitone involution. We note that an involution on a nonempty set W is function $g : W \to W$ such that $g(g(w)) = w$ for all $w \in W$. The involution was used for the representation of quasi-Boolean algebras in [16].

Using De Morgan frames, we offer relational semantics for syntactically defined De Morgan Heyting logics. The method of canonical model is applied to prove Kripke completeness. We modify the Lemmon filtration method to show the finite approximability which yields the decidability of some DH-logics. Furthermore, the conservativity of the DH expansion of a Kripke complete superintuitionistic logic which preserves expansion is

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proven by the construction of expanding rooted intuitionistic frames. Using this construction and the conservativity result, we show that the unique Post-complete DH-logic C is embedded into the minimal DH-logic J via an extended Glivenko-style translation.

Finally, in the last part of this article, the proof-theoretic aspect of the DH-logic B for De Morgan Boolean algebras is investigated. We introduce a structural rule-free Gentzen sequent calculus GB and show that the cut rule is admissible. Then, we show the sequent calculus GB is sound and complete with respect to the class of all De Morgan frames for B. Moreover, GB is terminating in the sense that the proof search of a given sequent terminates. It follows that the derivability of a sequent in GB is decidable.

2. De Morgan Heyting Algebras

A Heyting algebra is algebra $(A, \wedge, \vee, \to, 0)$ such that $(A, \wedge, \vee, 0)$ is a lattice with zero and for all $x, y, z \in A$ the following law of residuation holds:

$$\text{(Res) } x \wedge y \leq z \text{ if and only if } y \leq x \to z.$$

Here, $\leq$ is the lattice order. We use abbreviations $\neg x := x \to 0$ (intuitionistic negation), $1 := \neg 0$ and $x \leftrightarrow y := (x \to y) \wedge (y \to x)$. We let HA be the variety of all Heyting algebras. It is well known that the lattice reduct of a Heyting algebra is bounded distributive.

Definition 1. *Algebra $\mathbb{A} = (A, \wedge, \vee, \to, \sim, 0)$ is called a De Morgan Heyting algebra ('DH-algebra' for short) if $(A, \wedge, \vee, \to, 0)$ is a Heyting algebra and $\sim$ is a De Morgan negation, i.e., for all $x, y \in A$, the following conditions hold:*

(D1) $\sim 0 = 1$.
(D2) $\sim\sim x = x$.
(D3) $\sim(x \wedge y) = \sim x \vee \sim y$.
(D4) $\sim(x \vee y) = \sim x \wedge \sim y$.

We write $(A, \sim)$ for a DH-algebra where A is supposed to be a Heyting algebra. We let $\mathcal{DH}$ be the variety of all DH-algebras.

Lemma 1. *We let $\mathbb{A} = (A, \sim)$ be a DH-algebra. The following hold for all $x, y \in A$:*

(1) $x \leq y$ *if and only if* $\sim y \leq \sim x$.
(2) $x \leq y$ *if and only if* $x \to y = 1$.
(3) $x \wedge (x \to y) = x \wedge y$.
(4) $y \wedge (x \to y) = y$.
(5) $\sim(x \to y) \leq x \to \sim y$.

Proof. For (1), we assume $x \leq y$. Then, $\sim y \wedge \sim x = \sim(y \vee x) = \sim y$ and so $\sim y \leq \sim x$. We assume $\sim y \leq \sim x$. Then, $x = \sim\sim x \leq \sim\sim y = y$. Items (2) – (4) hold in Heyting algebras. For (5), clearly, $y \leq x \to y$. Then, $y \leq \sim x \vee (x \to y)$. By (1), $\sim(\sim x \vee (x \to y)) \leq \sim y$. By (D4), $\sim\sim x \wedge \sim(x \to y) \leq \sim y$. By (D2), $x \wedge \sim(x \to y) \leq \sim y$. By (Res), $\sim(x \to y) \leq x \to \sim y$. $\square$

We let $\mathbb{A} = (A, \sim)$ be a DH-algebra. For every $x \in A$, we define $\boxplus x = \neg\sim x$. For every $n \geq 0$ and unary operator $\odot$ on A, we let $\odot^n x$ be defined inductively by $\odot^0 x = x$ and $\odot^{n+1} = \odot \odot^n x$. For every $x \in A$ and $n \geq 0$, we let $t_n(x) = \bigwedge_{k \leq n} \boxplus^k x$.

Filter F in a DH-algebra $\mathbb{A} = (A, \sim)$ is regular if $\boxplus x \in F$ for every $x \in F$. We let $\mathcal{F}^\circ(A)$ be the set of all regular filters in $\mathbb{A}$. Clearly, $\Delta_F = \{1\}$ is a regular filter which is called the trivial one. The regular filter generated by subset $X \subseteq A$ is denoted by $[X)^\circ_A$. The set $\mathcal{F}^\circ(A)$ is closed under arbitrary intersection and hence $\langle \mathcal{F}^\circ(A), \subseteq \rangle$ forms a complete lattice where $\bigwedge S = \bigcap S$ and $\bigvee S = [\bigcup S)^\circ_A$ for every $S \subseteq \mathcal{F}^\circ(A)$. If $X = \{x\}$, we write $[x)^\circ_A$ for $[\{x\})^\circ_A$. For every $X \subseteq A$ and $x \in A$, elements in regular filters $[X)^\circ_A$ and $[x)^\circ_A$ are characterized in [8] as follows:

(1) $y \in [X)^\circ_A$ if and only if $\bigwedge_{1 \leq i \leq k} t_{n_i}(x_i) \leq y$ for some $n_1, \ldots, n_k \in \omega$ and $x_1, \ldots, x_k \in X$.
(2) $y \in [x)^\circ_A$ if and only if $t_n(x) \leq y$ for some $n \in \omega$.

These characterizing conditions are used in our following proofs.

Fact 1. *We let $(A, \sim)$ be a DH-algebra. For all $G \in \mathcal{F}^\circ(A)$ and $n \geq 0$, if $x \in G$, then $t_n(x) \in G$.*

Lemma 2. *For every DH-algebra $\mathbb{A}$, $\langle \mathcal{F}^\circ(A), \subseteq \rangle$ is a distributive lattice.*

Proof. We let F, G and $H \in \mathcal{F}^\circ(A)$. We assume $x \in F \cap (G \vee H)$. Then, $x \in [G \cup H)^\circ_A$. Then, there are $y_1, \ldots, y_k \in G \cup H$ and $n_1, \ldots, n_k \in \omega$ such that $t_{n_1}(y_1) \wedge \ldots \wedge t_{n_k}(y_k) \leq x$. Since $G, H \in \mathcal{F}^\circ(A)$, we have $t_{n_i}(y_i) \in G \cup H$ for $1 \leq i \leq k$. Then, $x \vee t_{n_i}(y_i) \in (F \cap G) \cup (F \cap H)$. Hence, $x = x \vee \bigwedge_{1 \leq i \leq k} t_{n_i}(y_i) = \bigwedge_{1 \leq i \leq k}(x \vee t_{n_i}(y_i)) \in [(F \cap G) \cup (F \cap H))^\circ_A = (F \cap G) \vee (F \cap H)$. Therefore, $\langle \mathcal{F}^\circ(A), \subseteq \rangle$ is distributive. □

We let $Con(A)$ be the set of all congruence relations in a DH-algebra $\mathbb{A} = (A, \sim)$. The least and largest members of $Con(A)$ are denoted by Δ_A (identity) and ∇_A (totality), respectively. For every $\theta \in Con(A)$ and $x \in A$, we let $[x]_\theta$ be the equivalence class of x. We let $\mathbb{A}/\theta$ be the quotient algebra of $\mathbb{A}$ modulo θ. For every $F \in \mathcal{F}^\circ(A)$, we let $\theta_F \subseteq A \times A$ be defined by

$$\langle x, y \rangle \in \theta_F \text{ if and only if } x \wedge z = y \wedge z \text{ for some } z \in F.$$

Clearly, θ_F is an equivalence relation on A. Conversely, for each $\theta \in Con(A)$, we let $F_\theta = [1]_\theta$. The following lemma is derived from Lemma 3.2 [8], and here we offer details of proof.

Lemma 3. *For every DH-algebra $\mathbb{A} = (A, \sim)$ and $F \in \mathcal{F}^\circ(A)$, $\theta_F \in Con(A)$ and $[1]_{\theta_F} = F$.*

Proof. It suffices to show that θ_F is compatible with respect to operators in $\mathbb{A}$. The intuitionistic cases are shown regularly. We assume $\langle x, y \rangle \in \theta_F$. Then, $x \wedge z = y \wedge z$ for some $z \in F$. Clearly, $\neg{\sim}z \in F$. We have ${\sim}x \leq {\sim}x \vee {\sim}z = {\sim}(x \wedge z) = {\sim}(y \wedge z) = {\sim}y \vee {\sim}z$. Obviously, ${\sim}y \leq \neg{\sim}z \to {\sim}y$ and ${\sim}z \leq \neg{\sim}z \to {\sim}y$. Hence, ${\sim}x \leq \neg{\sim}z \to {\sim}y$, and so ${\sim}x \wedge \neg{\sim}z \leq {\sim}y$. Similarly, ${\sim}y \wedge \neg{\sim}z \leq {\sim}x$. Then, ${\sim}x \wedge \neg{\sim}z = {\sim}y \wedge \neg{\sim}z$. Hence, $\langle {\sim}x, {\sim}y \rangle \in \theta_F$. We assume $x \in F$. Then, $\langle x, 1 \rangle \in \theta_F$ and so $x \in [1]_{\theta_F}$. We assume $x \in [1]_{\theta_F}$. Then, $\langle x, 1 \rangle \in \theta_F$ and so $x \wedge z = 1 \wedge z = z$ for some $z \in F$. Then, $z \leq x$ and so $x \in F$. Hence, $[1]_{\theta_F} = F$. □

Lemma 4. *For every DH-algebra $\mathbb{A} = (A, \sim)$ and $\theta \in Con(A)$, $F_\theta \in \mathcal{F}^\circ(A)$.*

Proof. We let $x, y \in F_\theta$. Then, $\langle x, 1 \rangle \in \theta$ and $\langle y, 1 \rangle \in \theta$. Then, $\langle x \wedge y, 1 \rangle \in \theta$ and so $x \wedge y \in F_\theta$. We let $x \in F_\theta$ and $x \leq y$. Then, $\langle x, 1 \rangle \in \theta$ and $x = x \wedge y$. Then, $\langle x \wedge y, y \rangle \in \theta$ and so $\langle x, y \rangle \in \theta$. Hence, $\langle y, 1 \rangle \in \theta$ and so $y \in F_\theta$. We let $x \in F_\theta$. Then, $\langle x, 1 \rangle \in \theta$ and so $\langle \boxplus x, \boxplus 1 \rangle \in \theta$. Clearly, $\boxplus 1 = 1$ and so $\boxplus x \in F_\theta$. Then, $F_\theta \in \mathcal{F}^\circ(A)$. □

Theorem 1. *For every DH-algebra $\mathbb{A}$, $\langle Con(A), \subseteq \rangle$ is lattice-isomorphic to $\langle \mathcal{F}^\circ(A), \subseteq \rangle$.*

Proof. Function $f : \mathcal{F}^\circ(A) \to Con(A)$ is defined by setting $f(F) = \theta_F$ for each $F \in \mathcal{F}^\circ(A)$. By Lemma 3 (1), $\theta_F \in Con(A)$. Function $g : Con(A) \to \mathcal{F}^\circ(A)$ is defined by setting $g(\theta) = F_\theta$ for each $\theta \in Con(A)$. By Lemma 4, $F_\theta \in \mathcal{F}^\circ(A)$. We assume $F \in \mathcal{F}^\circ(A)$. By Lemma 3 (2), $F = [1]_{\theta_F}$ and so $F = F_{\theta_F}$. Hence, $g \circ f$ is the identity on $\mathcal{F}^\circ(A)$. We assume $\theta \in Con(A)$. We suppose $\langle x, y \rangle \in \theta$. Then, $x \leftrightarrow y \in F_\theta$. Clearly, $x \wedge (x \leftrightarrow y) = y \wedge (x \leftrightarrow y)$ and so $\langle x, y \rangle \in \theta_{F_\theta}$. We suppose $\langle x, y \rangle \in \theta_{F_\theta}$. Then, there exists $z \in F_\theta = [1]_\theta$ such that $x \wedge z = y \wedge z$. By $\langle z, 1 \rangle \in \theta$, we have $\langle x \wedge z, x \rangle \in \theta$ and $\langle y \wedge z, y \rangle \in \theta$. Then, $\langle x, y \rangle \in \theta$. Hence, $f \circ g$ is the identity on $Con(A)$. Clearly, f and g are order preserving. Hence, f is a lattice isomorphism. □

Corollary 1. *$\mathcal{DH}$ is congruence distributive.*

Proof. By Theorem 1 and Lemma 2. □

Algebra $\mathbb{A}$ is a subdirect product of family $\{\mathbb{A}_i : i \in I\}$ if there exists an injective homomorphism $f : A \to \prod_{i\in I} A_i$ such that $\pi_i \circ f : A \to A_i$ is surjective for every $i \in I$ and projection $\pi_i : \prod_{i\in I} A_i \to A_i$. Algebra $\mathbb{A}$ is subdirectly irreducible if for every subdirect embedding $f : A \to \prod_{i\in I} A_i$ there exists $i \in I$ such that $\pi_i \circ f$ is an isomorphism.

Theorem 2. *A DH-algebra $\mathbb{A} = (A, \sim)$ is subdirectly irreducible if and only if the set of all nontrivial regular filters $\mathcal{F}^\circ(A) \setminus \{\Delta_F\}$ has a least member.*

Proof. By the characterization of subdirectly irreducibles in universal algebra (cf., e.g., [17]), $\mathbb{A}$ is subdirectly irreducible if and only if $Con(A) \setminus \{\Delta_A\}$ has a least member. Then, the characterization follows from Lemma 1. □

For a class of DH-algebras $\mathcal{K}$, we let $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$ and $\mathsf{P}(\mathcal{K})$ denote the classes of homomorphic images, subalgebras and direct products of members of $\mathcal{K}$, respectively. The variety generated by $\mathcal{K}$ is exactly $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$. If $\mathcal{K}$ is a variety, we let $\Lambda(\mathcal{K})$ be the lattice of all subvarieties of $\mathcal{K}$ with respect to the meet $\cap$ and join $\oplus$. The bottom $\mathcal{T}$ in $\Lambda(\mathcal{K})$ is the variety of trivial algebras, which consists of algebraic structures with only one element. We note that $\mathcal{K}_1 \oplus \mathcal{K}_2$ is the smallest variety containing $\mathcal{K}_1 \cup \mathcal{K}_2$. The lattice of all subvarieties of $\mathcal{DH}$ corresponds to the set of all intuitionistic De Morgan logics.

Now, we offer more observations on lattice $\Lambda(\mathcal{DH})$ using the above results. Similar results for modal algebras are given by Blok [18]. We let $\mathcal{S} = \{\mathbb{A}_i : i \in I\}$ be a nonempty family of DH-algebras and U an ultrafilter on I. Then, $F = \{a \in \prod_{i\in I} A_i : \{i \in I \mid a(i) = 1\} \in U\}$ is a regular filter in the direct product $\prod_{i\in I} \mathbb{A}_i$. Aalgebra $\prod_{i\in I} \mathbb{A}_i/\theta_F$ is called an ultraproduct of $\mathcal{S}$. For a class of DH-algebras $\mathcal{K}$, we let $\mathsf{P_U}(\mathcal{K})$ be the class of all ultraproducts of nonempty families of algebras in $\mathcal{K}$. We let $\mathcal{K}_{\mathrm{SI}}$ be the class of all subdirectly irreducibles in $\mathcal{K}$.

Lemma 5. *We let $\mathcal{K}$ be a class of DH-algebras. Then, the following hold:*

(1) $\mathsf{V}(\mathcal{K})_{\mathrm{SI}} \subseteq \mathsf{HSP_U}(\mathcal{K})$.
(2) *if $\mathcal{K}$ is a finite set of finite DH-algebras; then,* $\mathsf{V}(\mathcal{K})_{\mathrm{SI}} \subseteq \mathsf{HS}(\mathcal{K})$.
(3) *if $\mathbb{A}$ and $\mathbb{B}$ are subdirectly irreducible; then,* $\mathsf{V}(\mathbb{A}) = \mathsf{V}(\mathbb{B})$ *if and only if $\mathbb{A}$ is isomorphic to $\mathbb{B}$.*
(4) *for all $\mathcal{J}, \mathcal{K} \in \Lambda(\mathcal{DH})$, $(\mathcal{J} \oplus \mathcal{K})_{\mathrm{SI}} = \mathcal{J}_{\mathrm{SI}} \cup \mathcal{K}_{\mathrm{SI}}$.*

Proof. By Corollary 1, $\mathsf{V}(\mathcal{K})$ is congruence distributive. By Jónsson Corollary 3.2 [19], $\mathsf{V}(\mathcal{K})_{\mathrm{SI}} \subseteq \mathsf{HSP_U}(\mathcal{K})$. By Corollary 3.4 [19], for a finite set of finite DH-algebras $\mathcal{K}$, $\mathsf{V}(\mathcal{K})_{\mathrm{SI}} \subseteq \mathsf{HS}(\mathcal{K})$. By Corollary 3.5 [19], we obtain (3). By the definition of $\oplus$ and (3), subdirectly irreducible algebras in $\mathcal{J} \oplus \mathcal{K}$ are exactly those in $\mathcal{J}$ or $\mathcal{K}$. Thus, (4) holds (cf., e.g., Corollary 3.5 [18]). □

Every variety is generated by its subdirectly irreducible members. We let $\mathcal{S}_{\mathrm{DH}}$ be the set of all finitely generated subdirectly irreducible DH-algebras up to isomorphism. As in [18], we define the preorder $\preceq$ on $\mathcal{S}_{\mathrm{DH}}$ as follows:

$$\mathbb{A} \preceq \mathbb{B} \text{ if and only if } \mathbb{A} \in \mathsf{HSP_U}(\mathbb{B}).$$

Subset $\mathcal{K} \subseteq \mathcal{S}_{\mathrm{DH}}$ is a downset in $\langle \mathcal{S}_{\mathrm{DH}}, \preceq \rangle$ if $\mathbb{A} \preceq \mathbb{B}$ and $\mathbb{B} \in \mathcal{K}$ implies $\mathbb{A} \in \mathcal{K}$. We let $\mathcal{O}(\mathcal{S}_{\mathrm{DH}})$ be the set of all downsets. Clearly, $\langle \mathcal{O}(\mathcal{S}_{\mathrm{DH}}), \cap, \cup \rangle$ is a distributive lattice.

Theorem 3. *Lattice $\Lambda(\mathcal{DH})$ is embedded into $\mathcal{O}(\mathcal{S}_{\mathrm{DH}})$.*

Proof. Map $f : \Lambda(\mathcal{DH}) \to \mathcal{O}(\mathcal{S}_{\mathrm{DH}})$ is defined by setting $f(\mathcal{K}) = \mathcal{K} \cap \mathcal{S}_{\mathrm{DH}}$ for every $\mathcal{K} \in \mathcal{DH}$. We assume $\mathcal{J} \neq \mathcal{K} \in \Lambda(\mathcal{DH})$. Then, $\mathcal{J} = \mathsf{V}(\mathcal{J} \cap \mathcal{S}_{\mathrm{DH}})$ and $\mathcal{K} = \mathsf{V}(\mathcal{K} \cap \mathcal{S}_{\mathrm{DH}})$. Then, $f(\mathcal{J}) \neq f(\mathcal{K})$. Hence, f is injective. Clearly, $f(\mathcal{J} \cap \mathcal{K}) = f(\mathcal{J}) \cap f(\mathcal{K})$. By Lemma 5 (4), $f(\mathcal{J} \oplus \mathcal{K}) = (\mathcal{J} \oplus \mathcal{K}) \cap \mathcal{S}_{\mathrm{DH}} = (\mathcal{J} \oplus \mathcal{K})_{\mathrm{SI}} \cap \mathcal{S}_{\mathrm{DH}} = (\mathcal{J}_{\mathrm{SI}} \cup \mathcal{K}_{\mathrm{SI}}) \cap \mathcal{S}_{\mathrm{DH}} = (\mathcal{J}_{\mathrm{SI}} \cap \mathcal{S}_{\mathrm{DH}}) \cup (\mathcal{K}_{\mathrm{SI}} \cap \mathcal{S}_{\mathrm{DH}}) = f(\mathcal{J}) \cup f(\mathcal{K})$. Hence, f is a lattice embedding. □

Corollary 2. *Lattice $\Lambda(\mathcal{DH})$ is distributive.*

We note that $\Lambda(\mathcal{DH})$ is a complete distributive lattice. Next, we can start the investigation of logics. In what follows, we first make some observations on the discrete dualities between De Morgan frames and DH-algebras.

3. De Morgan Frame and Discrete Duality

An intuitionistic frame ("I-frame" for short) is a pair $\mathbb{F} = (W, \leq)$ where $W \neq \varnothing$ and $\leq$ is a partial order on W. For interpreting $\sim$, an involution is added into I-frames and so De Morgan frames (cf., e.g., [14,15]) are available.

Definition 2. *A De Morgan frame ("DM-frame" for short) is a triple $\mathfrak{F} = (W, \leq, g)$ where $\leq$ is a partial order on W and $g : W \to W$ is a function such that the following conditions hold:*

(G1) *g is an involution, i.e., $g(g(w)) = w$ for all $w \in W$.*
(G2) *for all $w, u \in W$, if $w \leq u$, then $g(u) \leq g(w)$.*

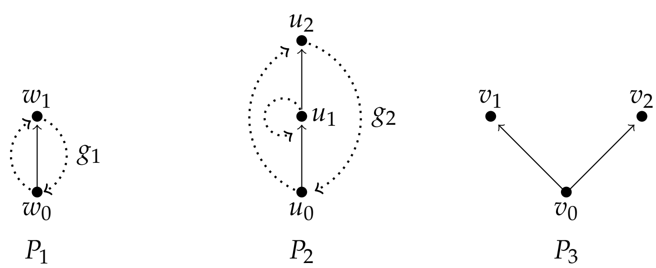
We write $w < u$ if $w \leq u$ and $w \neq u$. For every $w \in W$ and $X \subseteq W$, we let $w{\uparrow} = \{u \in W : w \leq u\}$ and $X{\uparrow} = \bigcup_{w \in X} w{\uparrow}$. A subset $X \subseteq W$ is an upset in $\mathfrak{F}$ if $X = X{\uparrow}$. We let $Up(W)$ be the sets of all upsets in $\mathfrak{F}$. Operations $\sim_g$ and $\to_R$ on the powerset $\mathcal{P}(W)$ are defined by setting

$$\sim_g X = \overline{g(X)} \text{ and } X \to_R Y = \{w \in W : w{\uparrow} \cap X \subseteq Y\}.$$

We use the set operations $\cap$, $\cup$ and $\overline{(.)}$ (complementation) on $\mathcal{P}(W)$.

We let $\mathfrak{F} = (W, \leq, g)$ be a DM-frame. By (G1), map g is bijective. Condition (G2) means that g is antitone. Some I-frames can be expanded to a DM-frame by adding an antitone involution but this does not hold in general.

Example 1. *We consider the following posets and try to expand them with an antitone involution:*



Here, thick arrows stand for the order and dotted arrows for the involution. There is a unique antitone involution on P_1, i.e., $g_1(w_0) = w_1$ and $g_1(w_1) = w_0$. We suppose $g_1(w_0) = w_0$. Then, $g_1(w_1) = w_1$, which contradicts $w_0 \leq w_1$. The unique antitone involution on P_2 is map g_2 where $g_2(u_0) = u_2$, $g_2(u_1) = u_1$ and $g_2(u_2) = u_0$. There is no antitone involution on P_3.

Lemma 6. *We let $\mathfrak{F} = (W, \leq, g)$ be a DM-frame. For all $w \in W$ and $X, Y \subseteq W$, the following hold:*

(1) *$w \leq u$ if and only if $g(u) \leq g(w)$.*
(2) *$w \in g(X)$ if and only if $g(w) \in X$.*
(3) *$g(g(X)) = X$, $g(\overline{X}) = \sim_g X$ and $g(\sim_g X) = \overline{X}$.*
(4) *$g(X \cup Y) = g(X) \cup g(Y)$ and $g(X \cap Y) = g(X) \cap g(Y)$.*
(5) *$g(g(w){\uparrow}) = w{\downarrow}$.*
(6) *$X \subseteq Y$ if and only if $g(X) \subseteq g(Y)$.*
(7) *$w \in g(X \to_R Y)$ if and only if $w{\downarrow} \cap g(X) \subseteq g(Y)$.*

Proof. By (G1) and (G2), we obain (1). For (2), we assume $w \in g(X)$. Then, $w = g(u)$ for some $u \in X$. Then, $g(w) = g(g(u)) = u \in X$. If $g(w) \in X$, then $w = g(g(w)) \in g(X)$.

Clearly, (3) and (4) follow from (2). For (5), we assume $u \in g(g(w)\uparrow)$. By (2), $g(w) \leq g(u)$. By (1), $u \in w\downarrow$. The other direction is similar. For (6), we assume $X \subseteq Y$ and $w \in g(X)$. By (2), $g(w) \in X$ and so $w \in g(Y)$. We assume $g(X) \subseteq g(Y)$ and $w \in X$. Then, $g(w) \in g(X)$ and so $g(w) \in g(Y)$. Then, $g(w) = g(u)$ for some $u \in Y$. Then, $w = g(g(w)) = g(g(u)) = u \in Y$. For (7), we assume $w \in g(X \to_R Y)$. By (2), $g(w) \in X \to_R Y$. Then, $g(w)\uparrow \cap X \subseteq Y$. By (4) and (6), $g(g(w)\uparrow) \cap g(X) \subseteq g(Y)$. By (5), $w\downarrow \cap g(X) \subseteq g(Y)$. The other direction is similar. $\square$

Lemma 7. *We let $\mathfrak{F} = (W, \leq, g)$ and $X, Y \in Up(W)$. Then, $X \to_R Y, \sim_g X \in Up(W)$.*

Proof. We assume $X, Y \in Up(W)$. Clearly, $X \to_R Y \in Up(W)$ (cf., e.g., [20]). We let $w \in \sim_g X$ and $w \leq u$. Then, $w \notin g(X)$ and $g(u) \leq g(w)$. By $w \leq u$ and (G2), $g(u) \leq g(w)$. Since $X \in Up(W)$, we have $g(u) \notin X$. By Lemma 6 (2), $u \in \overline{g(X)} = \sim_g X$. Hence, $\sim_g X \in Up(W)$. $\square$

Clearly, $\varnothing, W \in Up(W)$. We note that $Up(W)$ is closed under $\cap$ and $\cup$. By Lemma 7, $Up(W)$ is closed under $\sim_g$ and $\to_R$. The dual of a DM-frame $\mathfrak{F} = (W, \leq, g)$ is defined as $\mathfrak{F}^+ = (Up(W), \cap, \cup, \sim_g, \to_R, \varnothing, W)$.

Proposition 1. *For every DM-frame $\mathfrak{F} = (W, \leq, g)$, the dual $\mathfrak{F}^+$ is a DH-algebra.*

Proof. Clearly, $\sim_g \varnothing = \overline{g(\varnothing)} = W$ and $\sim_g W = \overline{g(W)} = \varnothing$. Obviously, $(Up(W), \cap, \cup, \to_R, \varnothing)$ is a Heyting algebra. Moreover, for all $X, Y \in Up(W)$, the following hold:

$$\begin{aligned}
\sim_g\sim_g(X) &= \overline{g(\sim_g(X))} = \overline{\overline{X}} = X \text{ (using Lemma 6 (3))},\\
\sim_g(X \cap Y) &= \overline{g(X \cap Y)} = \overline{g(X) \cap g(Y)} = \overline{g(X)} \cup \overline{g(Y)} = \sim_g X \cup \sim_g Y,\\
\sim_g(X \cup Y) &= \overline{g(X \cup Y)} = \overline{g(X) \cup g(Y)} = \overline{g(X)} \cap \overline{g(Y)} = \sim_g X \cap \sim_g Y.
\end{aligned}$$

Hence, $\mathfrak{F}^+$ is a DH-algebra. $\square$

We let $\mathbb{A} = (A, \sim)$ be a DH-algebra. The sets of all filters and ideals in $\mathbb{A}$ are denoted by $\mathcal{F}(A)$ and $\mathcal{I}(A)$, respectively. A filter (or ideal) $K \subseteq A$ is proper in $\mathbb{A}$ if $K \neq A$. A proper filter $F \subseteq A$ is prime if $x \vee y \in F$ implies $x \in A$ or $y \in A$. A proper ideal $F \subseteq A$ is prime if $x \wedge y \in F$ implies $x \in A$ or $y \in A$. We let $\mathcal{F}_{\mathrm{p}}(A)$ and $\mathcal{I}_{\mathrm{p}}(A)$ be sets of all prime filters and prime ideals in $\mathbb{A}$, respectively. A nonempty subset $K \subseteq A$ has the finite meet property if $x_1 \wedge \ldots \wedge x_n \neq 0$ for all $x_1, \ldots, x_n \in K$. The filter (resp. ideal) in A generated by a subset $K \subseteq A$ is denoted by $[K)$ (resp. $(K]$). If $K = \{x\}$, we write $[x)$ for $[\{x\})$, and $(x]$ for $(\{x\}]$.

Lemma 8. *We let $\mathbb{A} = (A, \sim)$ be a DH-algebra, $F \in \mathcal{F}(A)$ and $I \in \mathcal{I}(A)$. Then, the following hold:*

(1) *if $x \not\leq y$, then there exists $G \in \mathcal{F}_{\mathrm{p}}(A)$ such that $x \in G$ and $y \notin G$.*
(2) *$F \in \mathcal{F}_{\mathrm{p}}(A)$ if and only if $\overline{F} \in \mathcal{I}_{\mathrm{p}}(A)$.*
(3) *$F \in \mathcal{F}_{\mathrm{p}}(A)$ if and only if $\sim F \in \mathcal{I}_{\mathrm{p}}(A)$.*
(4) *if $F \in \mathcal{F}_{\mathrm{p}}(A)$ and $x \to y \notin F$, there exists $G \in \mathcal{F}_{\mathrm{p}}(A)$ such that $F \subseteq G$, $x \in G$ and $y \notin G$.*

Proof. For (1), we assume $x \not\leq y$. Then, $y \notin [x)$. We let $\mathcal{Z} = \{H \in \mathcal{F}(A) : x \in H \,\&\, y \notin H\}$. By Zorn's lemma, there exists a $\subseteq$-maximal element $G \in \mathcal{Z}$. Then, $G \in \mathcal{F}_{\mathrm{p}}(A)$, $x \in G$ and $y \notin G$. Item (2) is obtained by definition. For (3), assume $F \in \mathcal{F}_{\mathrm{p}}(A)$. We suppose $x, y \in F$. Then, $x \wedge y \in F$ and so $\sim x \vee \sim y = \sim(x \wedge y) \in \sim F$. We suppose $x \in F$ and $y \leq \sim x$ where $y \in A$. Then, $x = \sim\sim x \leq \sim y$. Then, $\sim y \in F$ and so $y = \sim\sim y \in \sim F$. We suppose $\sim x \wedge \sim y \in \sim F$. Then, $x \vee y = \sim(\sim x \wedge \sim y) \in F$ and so $x \in F$ or $y \in F$. Then, $\sim x \in \sim F$ or $\sim y \in \sim F$. Hence, $\sim F \in \mathcal{I}_{\mathrm{p}}(A)$. We assume $\sim F \in \mathcal{I}_{\mathrm{p}}(A)$. We suppose $x, y \in F$. Then, $\sim(x \wedge y) = \sim x \vee \sim y \in \sim F$ and so $x \wedge y \in F$. We suppose $x \in F$ and $x \leq y$. Then, $\sim y \leq \sim x \in \sim F$ and so $\sim y \in \sim F$. Then, $y \in F$. We suppose

$x \vee y \in F$. Then, $\sim x \wedge \sim y = \sim(x \vee y) \in \sim F$ and so $\sim x \in \sim F$ or $\sim y \in \sim F$. Then, $x \in F$ or $y \in F$. Hence, $F \in \mathcal{F}_{\mathrm{p}}(A)$. For (4), we assume $x \to y \notin F$. We let $K = F \cup \{x\}$ and $\mathcal{X} = \{G \in \mathcal{F}(A) : K \subseteq G \ \& \ y \notin G\}$. We suppose $y \in [K)$. Then, there exists $z \in F$ such that $x \wedge z \leq y$. By (Res), $z \leq x \to y$. Then, $x \to y \in F$ which contradicts the assumption. Hence, $y \notin [K)$ and so $[K) \in \mathcal{X}$. By Zorn's lemma, we let $G \in \mathcal{X}$ be a $\subseteq$-maximal element in $\mathfrak{X}$. Then, G is prime. Clearly, $F \subseteq G$, $x \in G$ and $y \notin G$. □

Definition 3. *We let $\mathbb{A} = (A, \sim)$ be a DH-algebra. The dual of $\mathbb{A}$ is defined as algebra $\mathbb{A}_+ = (\mathcal{F}_{\mathrm{p}}(A), \subseteq, g_A)$ where $g_A : \mathcal{F}_{\mathrm{p}}(A) \to \mathcal{F}_{\mathrm{p}}(A)$ is map $F \mapsto \overline{\sim F}$ for every $F \in \mathcal{F}_{\mathrm{p}}(A)$.*

Lemma 9. *For every $F \in \mathcal{F}_{\mathrm{p}}(A)$, $x \in g_A(F)$ if and only if $\sim x \notin F$.*

Proof. We assume $x \in g_A(F)$. Then, $x \notin \sim F$. Since $x = \sim\sim x$, we obtain $\sim x \notin F$. Conversely, we assume $\sim x \notin F$. Then, $x = \sim\sim x \notin \sim F$ and so $x \in g_A(F)$. □

Proposition 2. *For every DH-algebra $\mathbb{A} = (A, \sim)$, the dual $\mathbb{A}_+$ is a DM-frame.*

Proof. By Lemma 8, $g_A(F) \in \mathcal{F}_{\mathrm{p}}(A)$ for all $F \in \mathcal{F}_{\mathrm{p}}(A)$. We let $F \in \mathcal{F}_{\mathrm{p}}(A)$. By Lemma 9, $x \in g_A(g_A(F))$ if and only if $\sim x \notin g_A(F)$ if and only if $x \in F$. Hence, $g_A(g_A(F)) = F$ and so g_A is an involution. We assume $F \subseteq G \in \mathcal{F}_{\mathrm{p}}(A)$ and $x \in g_A(G)$. By Lemma 9, $\sim x \notin G$ and so $\sim x \notin F$. By Lemma 9, $x \in g_A(F)$. Then, $g_A(G) \subseteq g_A(F)$. Hence, $\mathbb{A}_+ = (\mathcal{F}_{\mathrm{p}}(A), \subseteq, g_A)$ is a DM-frame. □

We let $\mathfrak{F} = (W, \leq, g)$ and $\mathfrak{G} = (T, \leq, h)$ be DM-frames. Map $\pi : W \to T$ is an embedding from $\mathfrak{F}$ to $\mathfrak{G}$ if (i) π is injective; (ii) $\pi(g(w)) = h(\pi(w))$; (iii) $w \leq u$ if and only if $\pi(w) \leq \pi(u)$. We say $\mathfrak{F}$ is embedded into $\mathfrak{G}$ if there is an embedding from $\mathfrak{F}$ to $\mathfrak{G}$. A DH-algebra $\mathbb{A}$ is embedded into $\mathbb{B}$ if there is an injective DH-homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

Theorem 4. *We let $\mathfrak{F} = (W, \leq, g)$ be a DM-frame and $\mathbb{A} = (A, \sim)$ be a DH-algebra. Then, (1) $\mathfrak{F}$ is embedded into $(\mathfrak{F}^+)_+$; and (2) $\mathbb{A}$ is embedded into $(\mathbb{A}_+)^+$.*

Proof. (1) Map $\pi : W \to \mathcal{F}_{\mathrm{p}}(\mathfrak{F}^+)$ is defined by $\pi(w) = \{X \in Up(W) : w \in X\}$ for every $w \in W$. Clearly, $\pi(w) \in \mathcal{F}_{\mathrm{p}}(\mathfrak{F}^+)$. We assume $w \not\leq u$. Then, $w \in w{\uparrow}$ and $u \notin w{\uparrow}$. Then, $\pi(w) \neq \pi(u)$. Hence, π is injective. For every $F \in \mathcal{F}_{\mathrm{p}}(\mathfrak{F}^+)$, $g_{\mathfrak{F}^+}(F) = \overline{\sim_g F} = g(F)$. We assume $X \in \pi(g(w))$. Then, $g(w) \in X$ and so $w \in g(X)$. Then, $g(X) \in \pi(w)$ and so $X \in g(\pi(w)) = g_{\mathfrak{F}^+}(\pi(w))$. We assume $X \in g_{\mathfrak{F}^+}(\pi(w))$. Then, $X \in g(\pi(w))$ and so $g(X) \in \pi(w)$. Then, $w \in g(X)$ and so $X \in \pi(g(w))$. Hence, $\pi(g(w)) = g_{\mathfrak{F}^+}(\pi(w))$. We assume $w \leq u$. If $X \in \pi(w)$, then $w \in X$ and so $u \in X$, i.e., $X \in \pi(u)$. Hence, $\pi(w) \subseteq \pi(u)$. We assume $\pi(w) \subseteq \pi(u)$. Clearly, $w{\uparrow} \in \pi(w)$ and so $w{\uparrow} \in \pi(u)$. Then, $u \in w{\uparrow}$ and so $w \leq u$. Hence, π is an embedding.

(2) Map $\rho : A \to Up(\mathcal{F}_{\mathrm{p}}(A))$ is defined by setting $\rho(x) = \{F \in \mathcal{F}_{\mathrm{p}}(A) : x \in F\}$ for every $x \in A$. Clearly, $\rho(x) \in Up(\mathcal{F}_{\mathrm{p}}(A))$ for each $x \in A$. We assume $x \not\leq y$. By Lemma 18 (1), there exists $F \in \mathcal{F}_{\mathrm{p}}(A)$ such that $x \in F$ and $y \notin F$. Then, $F \in \rho(x)$ and $F \notin \rho(y)$. Then, $\rho(x) \neq \rho(y)$. Hence, ρ is injective. Clearly, $\rho(0) = \varnothing$ and $\rho(x \wedge y) = \rho(x) \cap \rho(y)$ and $\rho(x \vee y) = \rho(x) \cup \rho(y)$. We assume $F \in \rho(x \to y)$. We suppose $G \in F{\uparrow} \cap \rho(x)$. Then, $x \to y \in F \subseteq G$ and $x \in G$. By $x \wedge (x \to y) \leq y$ and $x \wedge (x \to y) \in G$, we obtain $y \in G$. Then, $G \in \rho(y)$. Then, $F \in \rho(x) \to_R \rho(y)$. Conversely, we assume $F \notin \rho(x \to y)$. Then, $x \to y \notin F$. By Lemma 18 (4), there exists $G \in \mathcal{F}_{\mathrm{p}}(A)$ such that $F \subseteq G$, $x \in G$ and $y \notin G$. Then, $F{\uparrow} \cap \rho(x) \not\subseteq \rho(y)$ and so $F \notin \rho(x) \to_R \rho(y)$. Hence, $\rho(x \to y) = \rho(x) \to_R \rho(y)$. We assume $F \in \rho(\sim x)$. Then, $\sim x \in F$. By Lemma 9, $x \notin g_A(F)$. Then, $g_A(F) \notin \rho(x)$ and so $F \notin g_A(\rho(x))$. Then, $\rho(\sim x) \subseteq \sim_{g_A}(\rho(x))$. Similarly, $\sim_{g_A}(\rho(x)) \subseteq \rho(\sim x)$. Hence, $\rho(\sim x) = \sim_{g_A}(\rho(x))$. Therefore, ρ is an injective DH-homomorphism. □

4. De Morgan Heyting Logics

We let $Var = \{p_i : i < \omega\}$ be a denumerable set of variables. The language of De Morgan Heyting logic is the intuitionistic propositional language with De Morgan negation $\sim$.

Definition 4. *The DH-formula algebra $\mathcal{L}_{\mathrm{DH}}$ is defined inductively as follows:*

$$\mathcal{L}_{\mathrm{DH}} \ni \varphi ::= p \mid \bot \mid \sim\varphi \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \to \varphi_2), \text{ where } p \in Var.$$

Formulas in $Var \cup \{\bot\}$ are called atomic. We let $\neg\varphi := \varphi \to \bot$ (intuitionistic negation), $\top := \neg\bot$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$ (equivalence).

The complexity of a DH-formula φ, denoted by $c(\varphi)$, is defined as the number of occurrences of unary or binary operators in φ. We let $\mathrm{Sf}(\varphi)$ be the set of all subformulas of φ. For a set of formulas Σ, we let $\mathrm{Sf}(\Sigma) = \bigcup_{\varphi\in\Sigma} \mathrm{Sf}(\varphi)$. We let $var(\varphi)$ be the set of all variables appearing in φ. We let $\varphi(p_1, \ldots, p_n)$ be the DH-formula φ such that $var(\varphi) \subseteq \{p_1, \ldots, p_n\}$. A substitution is a homomorphism $s : \mathcal{L}_{\mathrm{DH}} \to \mathcal{L}_{\mathrm{DH}}$ on the DH-formula algebra. We let φ^s be the DH-formula by applying the substitution s to φ. We let $\varphi(\psi_1/p_1, \ldots, \psi_n/p_n)$ be the DH-formula obtained from φ by substituting ψ_i for p_i (with $1 \leq i \leq n$) in φ. We let $\varphi(\psi/\chi)$ be the DH-formula obtained by replacing one or more occurrences of $\chi \in \mathrm{Sf}(\varphi)$ with ψ.

Definition 5. *A De Morgan Heyting logic ("DH-logic" for short) is a set of DH-formulas L satisfying the following conditions:*

(Int)	*every instance of intuitionistic propositional logic belongs to L.*
(M1)	$\sim\bot \leftrightarrow \top \in L$.
(M2)	$\sim\sim p \leftrightarrow p \in L$.
(M3)	$\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q \in L$.
(M4)	$\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q \in L$.
(MP)	*if* $\varphi \in L$ *and* $\varphi \to \psi \in L$, *then* $\psi \in L$.
(CP)	*if* $\varphi \to \psi \in L$, *then* $\sim\psi \to \sim\varphi \in L$.
(Sub)	*if* $\varphi \in L$, *then* $\varphi^s \in L$ *for every substitution* s.

A DH formula φ is a theorem of a DH-logic L (notation: $\vdash_L \varphi$) if $\varphi \in L$. A DH formula φ is L-derivable from a set of DH formulas Γ (notation: $\Gamma \vdash_L \varphi$) if there exists a finite subset $\Delta \subseteq \Gamma$ with $\vdash_L \bigwedge \Delta \to \varphi$. We note that $\bigwedge \Delta$ is the conjunction of all DH formulas in Δ. In particular, we let $\bigwedge \varnothing = \top$. A set of DH formulas Γ is L-consistent if $\Gamma \nvdash_L \bot$. A DH logic L_1 is a sublogic of L_2 (or L_2 is an extension of L_1) if $L_1 \subseteq L_2$.

Lemma 10. *For every DH-logic L, the following hold:*

(1) $\varphi, \Gamma \vdash_L \psi$ *if and only if* $\Gamma \vdash_L \varphi \to \psi$.
(2) $\vdash_L \varphi \to \psi$ *if and only if* $\vdash_L \sim\psi \to \sim\varphi$.
(3) *if* $\vdash_L \psi \leftrightarrow \chi$, *then* $\vdash_L \varphi \leftrightarrow \varphi(\psi/\chi)$.
(4) $\vdash_L \sim(\varphi \to \psi) \to (\varphi \to \sim\psi)$.
(5) $\vdash_L \varphi \to (\psi \to \chi)$ *if and only if* $\vdash_L \varphi \wedge \psi \to \chi$.

Proof. Clearly, (1) holds by the definition. By (CP) and (M2), we obtain (2). For (3), we assume $\vdash_L \psi \leftrightarrow \chi$. The case $\varphi = \psi$ is trivial. We suppose $\psi \in \mathrm{Sf}(\varphi) \setminus \{\varphi\}$. The proof proceeds by induction on $c(\varphi)$. We suppose $\varphi = \sim\delta$. By induction hypothesis, $\vdash_L \delta \leftrightarrow \delta(\psi/\chi)$. By (CP), $\vdash_L \sim\delta \leftrightarrow \sim\delta(\psi/\chi)$. We suppose $\varphi = \varphi_1 \odot \varphi_2$ for $\odot \in \{\wedge, \vee, \to\}$. We obtain $\vdash_L \varphi \leftrightarrow \varphi(\psi/\chi)$ by induction hypothesis and (Int). For (4), by (Int), $\vdash_L \psi \to \sim\varphi \vee (\varphi \to \psi)$. By (CP), $\vdash_L \sim(\sim\varphi \vee (\varphi \to \psi)) \to \sim\psi$. By (M4), (M2) and (3), $\vdash_L \varphi \wedge \sim(\varphi \to \psi) \to \sim\psi$. By (Int), $\vdash_L \sim(\varphi \to \psi) \to (\varphi \to \sim\psi)$. Item (5) holds by (Int). $\square$

We let $\{L_i : i \in I\}$ be a family of DH-logics. Intersection $\bigcap_{i \in I} L_i$ is a DH-logic. We let Γ be a set of DH formulas and L a DH-logic. The DH-logic generated by Γ over L is defined as $L \oplus \Gamma = \bigcap\{L' : L'$ is a DH-logic and $L \cup \Gamma \subseteq L'\}$. If $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, we write $L \oplus \varphi_1 \oplus \ldots \oplus \varphi_n$ for $L \oplus \Gamma$. The minimal DH-logic is denoted by J. We note that J is axiomatized by axioms (Int), (M1), all instances of (M2)– M4) and rules (MP) and (CP) in Definition 5. For every DH-logic L, the set of all extensions of L is denoted by $\mathsf{Ext}(L)$. Clearly, $\langle \mathsf{Ext}(L), \cap, \oplus \rangle$ forms a complete lattice.

Definition 6. *We let $\mathbb{A} = (A, \sim)$ be a DH-algebra. A valuation in $\mathbb{A}$ is a homomorphism $\sigma : \mathscr{F} \to A$. A DH formula φ is valid in $\mathbb{A}$ (notation: $\mathbb{A} \models \varphi$) if $\sigma(\varphi) = 1$ for every valuation σ in $\mathbb{A}$. We let $\mathcal{K}$ be a class of DH-algebras. A DH formula φ is valid in $\mathcal{K}$ (notation: $\mathcal{K} \models \varphi$) if $\mathbb{A} \models \varphi$ for every $\mathbb{A} \in \mathcal{K}$. The logic of $\mathcal{K}$ is defined as the set of DH formulas $\mathsf{Log}(\mathcal{K}) = \{\varphi \in \mathscr{F} : \mathcal{K} \models \varphi\}$. For a set of DH formulas Σ, we let $\mathsf{Alg}(\Sigma)$ be the variety of DH-algebras validating all DH formulas in Σ.*

For every DH-logic L, relation $\theta_L = \{\langle \varphi, \psi \rangle : \vdash_L \varphi \leftrightarrow \psi\}$ is a congruence on DH formula algebra $\mathscr{F}$. Quotient algebra $\mathbb{A}^L = (\mathscr{F}/\theta_L, \wedge, \vee, \to, \sim, 0)$ is called the Tarski–Lindenbaum algebra for L. Clearly, $\mathbb{A}^L \in \mathsf{Alg}(L)$. We let $[\varphi]_L$ be the equivalence class of φ module θ_L. Clearly, $[\varphi]_L \leq [\psi]_L$ if and only if $\vdash_L \varphi \to \psi$.

Lemma 11. *For every DH-logic L, $\vdash_L \varphi$ if and only if $\mathsf{Alg}(L) \models \varphi$.*

Proof. All axioms of J are valid and all rules of J preserve validity in $\mathcal{DH}$. Then, $L \subseteq \mathsf{Log}(\mathsf{Alg}(L))$. We assume $\nvdash_L \varphi$. Then, $[\varphi]_L \neq [\top]_L$. We let σ be the valuation in $\mathbb{A}^L$ such that $\sigma(p) = [p]_L$ for all $p \in Var$. Then, $\sigma(\chi) = [\chi]_L$ for all $\chi \in \mathcal{F}$. Then, $\sigma(\varphi) \neq [\top]_L$. Hence, $\mathbb{A}^L \not\models \varphi$. By $\mathbb{A}^L \in \mathsf{Alg}(L)$, we have $\mathsf{Alg}(L) \not\models \varphi$. $\square$

A model is a tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a DM frame and V is a valuation in $\mathfrak{F}^+$. A DH formula φ is true at w in a model $\mathfrak{M}$ (notation: $\mathfrak{M}, w \models \varphi$) if $w \in V(\varphi)$. We write $w \models \varphi$ if no confusion arises from the context. A DH formula φ is true in $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) if $V(\varphi) = W$. A DH formula φ is valid in a DM frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \varphi$) if $\mathfrak{F}^+ \models \varphi$. For a set of DH-formulas Γ, we let $\mathfrak{F} \models \Gamma$ denote that $\mathfrak{F} \models \varphi$ for all $\varphi \in \Gamma$. We let $\mathsf{Fr}(\Gamma) = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$. We note that $\mathsf{Fr}(\mathsf{J})$ is the class of all DM-frames. For a class of DM-frames $\mathcal{C}$, we let $\mathcal{C}^+ = \{\mathfrak{F}^+ : \mathfrak{F} \in \mathcal{C}\}$. DH formula φ is valid in $\mathcal{C}$ (notation: $\mathcal{C} \models \varphi$) if $\mathcal{C}^+ \models \varphi$. We let $\mathsf{Th}(\mathcal{C}) = \{\varphi \in \mathscr{F} : \mathcal{C} \models \varphi\}$ be the DH theory of $\mathcal{C}$. DH-logic L is Kripke complete if $L = \mathsf{Th}(\mathsf{Fr}(L))$.

Theorem 5. *For every DH formula φ, $\vdash_{\mathsf{J}} \varphi$ if and only if $\mathsf{Fr}(\mathsf{J}) \models \varphi$. Hence, $\mathsf{J} = \mathsf{Log}(\mathcal{DH})$.*

Proof. Clearly, $\mathfrak{F} \models \varphi$ for all $\varphi \in \mathsf{Int}$ and DM frame $\mathfrak{F}$. We assume $\nvdash_{\mathsf{J}} \varphi$. By Lemma 11, $\mathbb{A} \not\models \varphi$ for some DH-algebra $\mathbb{A}$. By Theorem 4 (2), $(\mathbb{A}_+)^+ \not\models \varphi$. Hence, $\mathsf{Fr}(\mathsf{J}) \models \varphi$. $\square$

For proving the Kripke completeness of some DH-logics, it is useful to define the canonical model for a DH-logic. We let L be a DH-logic. A set of DHcformulas Γ is an L theory if Γ is closed under $\vdash_L$, i.e., if $\Gamma \vdash_L \varphi$, then $\varphi \in \Gamma$. We let $\mathcal{T}(L)$ be the set of all L-theories. A consistent L-theory Γ is prime if $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$. We let $\mathcal{T}_{\mathrm{p}}(L)$ be the set of all prime L theories. A set of DH formulas Σ is a counter L theory if $\varphi \vdash_L \psi_1 \vee \ldots \vee \psi_n$ for some $\psi_1, \ldots, \psi_n \in \Sigma$ implies $\varphi \in \Sigma$. We let $\mathcal{T}^c(L)$ be the set of all counter-L theories. A consistent counter-L theory Σ is prime if $\varphi \wedge \psi \in \Sigma$ implies $\varphi \in \Sigma$ or $\psi \in \Sigma$. We let $\mathcal{T}^c_{\mathrm{p}}(L)$ be the set of all prime counter L-theories. For a set of DH-formulas Θ, the L theory and the counter-L theory generated by Θ are denoted by $[\Theta)_L$ and $(\Theta]_L$, respectively. If $\Theta = \{\varphi\}$, we write $[\varphi)_L$ and $(\varphi]_L$ for $[\{\varphi\})_L$ and $(\{\varphi\}]_L$, respectively.

Fact 2. *We let $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$ and $\Sigma \in \mathcal{T}^c_{\mathrm{p}}(L)$. The following hold for all $\varphi, \psi \in \mathcal{F}$:*

(1) $\varphi \wedge \psi \in \Gamma$ *if and only if* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.

(2) *$\varphi \vee \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.*
(3) *$\varphi \wedge \psi \in \Sigma$ if and only if $\varphi \in \Sigma$ or $\psi \in \Sigma$.*
(4) *$\varphi \vee \psi \in \Sigma$ if and only if $\varphi \in \Sigma$ and $\psi \in \Sigma$.*
(5) *$\sim\sim\varphi \in \Gamma$ if and only if $\varphi \in \Gamma$.*
(6) *$\sim\sim\varphi \in \Delta$ if and only if $\varphi \in \Delta$.*

Proof. These biconditions are obtained by the definition of prime L theory and counter-L theory. Details of proof are omitted. □

Lemma 12. *We let $\Gamma \in \mathcal{T}(L)$ and $\Sigma \in \mathcal{T}^c(L)$. Then, the following hold:*

(1) *if $\Xi \nvdash_L \psi$, then there exists $\Delta \in \mathcal{T}_{\mathrm{p}}(L)$ such that $\Xi \subseteq \Delta$ and $\psi \notin \Delta$.*
(2) *$\Sigma \in \mathcal{T}^c_{\mathrm{p}}(L)$ if and only if $\overline{\Sigma} \in \mathcal{T}_{\mathrm{p}}(L)$.*
(3) *$\varphi \in (\sim\Gamma]_L$ if and only if $\sim\varphi \in \Gamma$.*
(4) *$\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$ if and only if $(\sim\Gamma]_L \in \mathcal{T}^c_{\mathrm{p}}(L)$.*

Proof. For (1), we assume $\Xi \nvdash_L \psi$. Then, $[\Xi)_L$ is a consistent L theory and $\psi \notin [\Xi)_L$. We let $\mathcal{X} = \{\Theta \in \mathcal{T}(L) : \Xi \subseteq \Theta \;\&\; \psi \notin \Theta\}$. Then, $[\Xi)_L \in \mathcal{X}$. By Zorn's lemma, there exists a $\subseteq$-maximal element $\Delta \in \mathcal{X}$. Clearly, Δ is prime. For (2), we assume $\Sigma \in \mathcal{T}^c_{\mathrm{p}}(L)$. Then, $\perp \in \Sigma$ and so $\perp \notin \overline{\Sigma}$. We suppose $\varphi_1 \wedge \ldots \wedge \varphi_n \vdash_L \psi$ and $\varphi_1, \ldots, \varphi_n \in \overline{\Sigma}$. Then, $\varphi_1 \wedge \ldots \wedge \varphi_n \notin \Sigma$ and so $\psi \in \overline{\Sigma}$. We suppose $\varphi \vee \psi \in \overline{\Sigma}$. Then, $\varphi \in \overline{\Sigma}$ or $\psi \in \overline{\Sigma}$. Hence, $\overline{\Sigma} \in \mathcal{T}_{\mathrm{p}}(L)$. The other direction is shown similarly. For (3), we assume $\varphi \in (\sim\Gamma]_L$. Then, there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\varphi \vdash_L \sim\psi_1 \vee \ldots \vee \sim\psi_n$. Then, $\psi_1 \wedge \ldots \wedge \psi_n \vdash_L \sim\varphi$. Hence, $\sim\varphi \in \Gamma$. We assume $\sim\varphi \in \Gamma$. Then, $\sim\sim\varphi \in \sim\Gamma$. By $\varphi \vdash_L \sim\sim\varphi$, we have $\varphi \in (\sim\Gamma]_L$. For (4), we assume $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$. We suppose $\sim\perp \in (\sim\Gamma]_L$. Then, $\sim\sim\perp \in \Gamma$ and so $\perp \in \Gamma$ which contradicts the assumption. Hence, $\sim\perp \notin (\sim\Gamma]_L$. We suppose $\varphi \vdash_L \psi_1 \vee \psi_n$ and $\psi_1, \ldots, \psi_n \in (\sim\Gamma]_L$. By (3), $\sim\psi_1, \ldots, \sim\psi_n \in \Gamma$ and so $\sim(\psi_1 \vee \ldots \vee \psi_n) = \sim\psi_1 \wedge \ldots \wedge \sim\psi_n \in \Gamma$. Clearly, $\sim(\psi_1 \vee \ldots \vee \psi_n) \vdash_L \sim\varphi$. Then, $\sim\varphi \in \Gamma$. By (3), $\varphi \in (\sim\Gamma]_L$. We suppose $\chi_1 \wedge \chi_2 \in (\sim\Gamma]_L$. By (3), $\sim\chi_1 \vee \sim\chi_2 = \sim(\chi_1 \wedge \chi_2) \in \Gamma$. Then, $\sim\chi_1 \in \Gamma$ or $\sim\chi_2 \in \Gamma$. By (3), $\chi_1 \in (\sim\Gamma]_L$ or $\chi_2 \in (\sim\Gamma]_L$. Hence, $(\sim\Gamma]_L \in \mathcal{T}^c_{\mathrm{p}}(L)$. We assume $(\sim\Gamma]_L \in \mathcal{T}^c_{\mathrm{p}}(L)$. We suppose $\chi_1 \vee \chi_2 \in \Gamma$. Then $\sim(\chi_1 \vee \chi_2) \in \sim\Gamma$. By $\sim\chi_1 \wedge \sim\chi_2 \vdash_L \sim(\chi_1 \vee \chi_2)$, we have $\sim\chi_1 \wedge \sim\chi_2 \in (\sim\Gamma]_L$. Then, $\sim\chi_1 \in (\sim\Gamma]_L$ or $\sim\chi_2 \in (\sim\Gamma]_L$. By (3), $\sim\sim\chi_1 \in \Gamma$ or $\sim\sim\chi_2 \in \Gamma$. Since $\Gamma \in \mathcal{T}(L)$, we have $\chi_1 \in \Gamma$ or $\chi_2 \in \Gamma$. Hence, $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$. □

Definition 7. *We let L be a DH-logic. The canonical frame for L is defined as $\mathfrak{F}^L = (\mathcal{T}_{\mathrm{p}}(L), \subseteq, g^L)$, where g^L is the map given by setting $g^L(\Gamma) = \overline{(\sim\Gamma]_L}$ for every $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$. We let V^L be the valuation in $\mathfrak{F}^L$ such that $V^L(p) = \{\Gamma \in \mathcal{T}_{\mathrm{p}}(L) : p \in \Gamma\}$ for each $p \in Var$. We call $\mathfrak{M}^L = (\mathfrak{F}^L, V^L)$ the canonical model for L. A DH-logic L is called canonical if $\mathfrak{F}^L \models L$.*

We let L be a DH-logic. By Lemma 12 (2) and (4), $g^L(\Gamma) \in \mathcal{T}_{\mathrm{p}}(L)$ for every $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$. It follows that g^L is a function. The following lemma proves that g^L is an antitone involution.

Lemma 13. *For all $\Gamma, \Delta \in \mathcal{T}_{\mathrm{p}}(L)$, the following hold:*

(1) *$\varphi \in g^L(\Gamma)$ if and only if $\sim\varphi \notin \Gamma$.*
(2) *$g^L(g^L(\Gamma)) = \Gamma$.*
(3) *if $\Gamma \subseteq \Delta$, then $g^L(\Delta) \subseteq g^L(\Gamma)$.*

Proof. Clearly, (1) follows from Lemma 12 (3). For (2), by (1), $\varphi \in g^L(g^L(\Gamma))$ if and only if $\sim\varphi \notin g^L(\Gamma)$ if and only if $\varphi \in \Gamma$. For (3), we assume $\Gamma \subseteq \Delta$. Then, $(\sim\Gamma]_L \subseteq (\sim\Delta]_L$ and so $g^L(\Delta) \subseteq g^L(\Gamma)$. □

Lemma 14. *For every DH-logic L, $\mathfrak{M}^L, \Gamma \models \varphi$ if and only if $\varphi \in \Gamma$.*

Proof. The proof proceeds by induction on $c(\varphi)$. Case $\varphi \in Var \cup \{\perp\}$ is trivial. We assume $\varphi = \sim\psi$. We suppose $\mathfrak{M}^L, \Gamma \models \sim\psi$. Then, $\mathfrak{M}^L, g^L(\Gamma) \not\models \psi$. By induction hypothesis,

$\psi \notin g^L(\Gamma)$. By Lemma 13 (1), $\sim\psi \in \Gamma$. We suppose $\sim\psi \in \Gamma$. By Lemma 13 (1), $\psi \notin g^L(\Gamma)$. By induction hypothesis, $\mathfrak{M}^L, g^L(\Gamma) \not\models \psi$. Hence, $\mathfrak{M}^L, \Gamma \models \sim\psi$. Cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \varphi_1 \vee \varphi_2$ are shown easily by induction hypothesis. We assume $\varphi = \varphi_1 \to \varphi_2$. We suppose $\varphi_1 \to \varphi_2 \in \Gamma$, $\Gamma \subseteq \Delta$ and $\mathfrak{M}^L, \Delta \models \varphi_1$. By induction hypothesis, $\varphi_1 \in \Delta$. By $\varphi_1, \varphi_1 \to \varphi_2 \vdash_L \varphi_2$, we obtain $\varphi_2 \in \Delta$. By induction hypothesis, $\mathfrak{M}^L, \Delta \models \varphi_2$. Hence, $\mathfrak{M}^L, \Gamma \models \varphi_1 \to \varphi_2$. We suppose $\varphi_1 \to \varphi_2 \notin \Gamma$. Then, $\Gamma \nvdash_L \varphi_1 \to \varphi_2$ and so $\varphi_1, \Gamma \nvdash_L \varphi_2$. By Lemma 12 (1), there exists $\Delta \in \mathcal{T}_{\mathrm{p}}(L)$ such that $\varphi_1 \in \Delta$, $\Gamma \subseteq \Delta$ and $\varphi_2 \notin \Delta$. By induction hypothesis, $\mathfrak{M}^L, \Delta \models \varphi_1$ and $\mathfrak{M}^L, \Delta \not\models \varphi_2$. Hence, $\mathfrak{M}^L, \Gamma \not\models \varphi_1 \to \varphi_2$. □

Theorem 6. *Every canonical DH-logic is Kripke complete.*

Proof. We let L be a DH-logic and $\mathfrak{F}^L \models L$. We assume $\nvdash_L \varphi$. Then, $\top \nvdash_L \varphi$. By Lemma 12 (1), there exists $\Gamma \in \mathcal{T}_{\mathrm{p}}(L)$ with $\varphi \notin \Gamma$. By Lemma 14, $\mathfrak{M}^L, \Gamma \not\models \varphi$. Then, $\mathfrak{F}^L \not\models \varphi$. □

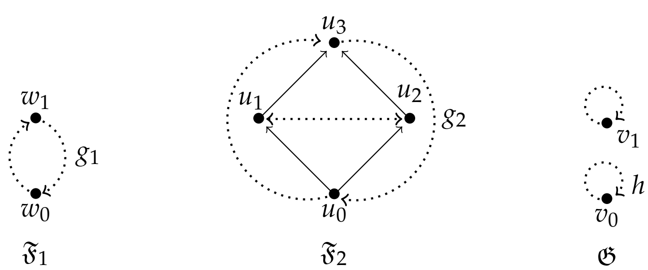
We consider the following DH-logics: $\mathsf{B} = \mathsf{J} \oplus p \vee \neg p$, $\mathsf{D} = \mathsf{J} \oplus \neg p \to \sim p$ and $\mathsf{C} = \mathsf{J} \oplus p \vee \sim p$. These characteristic DH-formulas have their corresponding frame conditions.

Lemma 15. *For every DM-frame $\mathfrak{F} = (W, \leq, g)$, the following hold:*

(1) $\mathfrak{F} \models p \vee \neg p$ *if and only if* $w\uparrow = \{w\}$ *for all* $w \in W$.
(2) $\mathfrak{F} \models p \vee \sim p$ *if and only if* $g(w) = w$ *for all* $w \in W$.
(3) $\mathfrak{F} \models \neg p \to \sim p$ *if and only if* $w\uparrow \cap g(w)\uparrow \neq \varnothing$ *for all* $w \in W$.

Proof. Item (1) holds in intuitionistic logic (cf., e.g., [20]). For (2), we assume $g(w) = w$ for all $w \in W$. We let V be a valuation in $\mathfrak{F}$ and $w \in W$. We suppose $w \not\models p$. Then, $g(w) \models \sim p$. Hence, $\mathfrak{F} \models p \vee \sim p$. We assume $g(w) \neq w$ for some $w \in W$. We suppose $g(w) \nleq w$. We let V_1 be a valuation in $\mathfrak{F}$ with $V_1(p) = g(w)\uparrow$. Then, $w \not\models p$, $g(w) \models p$ and so $w \not\models \sim p$. Hence, $w \not\models p \vee \sim p$. We suppose $w \nleq g(w)$. We let V_2 be a valuation in $\mathfrak{F}$ with $V_2(p) = w\uparrow$. Then, $w \models p$ and $g(w) \not\models p$. Then, $g(w) \not\models \sim p$ and so $g(w) \not\models p \vee \sim p$. Hence, $\mathfrak{F} \not\models p \vee \sim p$. For (3), we assume $w\uparrow \cap g(w)\uparrow \neq \varnothing$ for all $w \in W$. We let V be a valuation in $\mathfrak{F}$ and $w \in W$. We suppose $w \models \neg p$. By assumption, we let $w \leq u$ and $g(w) \leq u$. Then, $u \not\models p$. By $g(w) \leq u$, we obtain $g(w) \not\models p$ and so $w \models \sim p$. Hence, $w \models \neg p \to \sim p$. Conversely, we assume $\mathfrak{F} \models \neg p \to \sim p$. We let $w \in W$ and U be a valuation in $\mathfrak{F}$ such that $U(p) = g(w)\uparrow$. Then, $g(w) \models p$ and so $w \not\models \sim p$. Then, $w \not\models \neg p$ and so $w \leq u$ and $u \models p$. Clearly, $g(w) \leq u$. Hence, $w\uparrow \cap g(w)\uparrow \neq \varnothing$. □

Example 2. *We consider DM frames $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{G}$ where the domain of points, partial order and involution are given in the following diagrams:*



By Lemma 15, $\mathfrak{F}_1 \in \mathsf{Fr}(\mathsf{B}) \setminus \mathsf{Fr}(\mathsf{C})$ and $\mathfrak{G} \in \mathsf{Fr}(\mathsf{C})$. Clearly, $\mathfrak{F}_1 \notin \mathsf{Fr}(\mathsf{D})$ and $\mathfrak{F}_2 \notin \mathsf{Fr}(\mathsf{B})$. We show $\mathfrak{F}_2 \in \mathsf{Fr}(\mathsf{D}) \setminus \mathsf{Fr}(\mathsf{C})$. Clearly, $\mathfrak{F}_2 \notin \mathsf{Fr}(\mathsf{C})$. We have $u_3 \in u_0\uparrow \cap g_2(u_0)\uparrow$ and $u_3 \in u_3\uparrow \cap g_2(u_3)\uparrow$. By $u_1 \leq u_3$ and $g(u_1) = u_2 \leq u_3$, we obtain $u_3 \in u_1\uparrow \cap g_2(u_1)\uparrow$. Similarly, $u_3 \in u_2\uparrow \cap g_2(u_2)\uparrow$. Thus, $\mathfrak{F}_2 \in \mathsf{Fr}(\mathsf{D})$.

A DH-logic L_1 is a proper sublogic of L_2 (or L_2 is a proper extension of L_1) (notation: $L_1 \subset L_2$) if $L_1 \subseteq L_2$ and $L_1 \neq L_2$. We say that L_1 is incomparable with L_2 (notation:

$L_1 \parallel L_2$) if $L_1 \not\subseteq L_2$ and $L_2 \not\subseteq L_1$. We use the interval $[L_1, L_2]$ for the set of DH-logics $\{L : L_1 \subseteq L \subseteq L_2\}$. Similarly, we let $(L_1, L_2) = \{L : L_1 \subset L \subset L_2\}$.

Lemma 16. $\mathsf{C} = \mathsf{J} \oplus {\sim}p \leftrightarrow \neg p$.

Proof. We let $L = \mathsf{J} \oplus {\sim}p \leftrightarrow \neg p$. Clearly, $\vdash_{\mathsf{C}} \top \to p \vee {\sim}p$. By (CP), $\vdash_{\mathsf{C}} {\sim}(p \vee {\sim}p) \to {\sim}{\sim}\bot$. By (M2) and (M4), $\vdash_{\mathsf{C}} {\sim}p \wedge p \to \bot$. Then, $\vdash_{\mathsf{C}} {\sim}p \to \neg p$. Combined with $\vdash_{\mathsf{C}} p \vee {\sim}p$, we obtain $\vdash_{\mathsf{C}} p \vee \neg p$. By (Int), we obtain $\vdash_{\mathsf{C}} \varphi \vee \psi \leftrightarrow (\neg\varphi \to \psi)$ since the excluded middle law $p \vee \neg p$ holds in C. By $\vdash_{\mathsf{C}} p \vee {\sim}p$, we obtain $\vdash_{\mathsf{C}} \neg p \to {\sim}p$. Hence, $\vdash_{\mathsf{C}} {\sim}p \leftrightarrow \neg p$. Now, by $\vdash_J \neg(p \wedge \neg p)$ and Lemma 10 (3) and (Sub), $\vdash_L {\sim}(p \wedge {\sim}p)$. By (M2) and (M3), $\vdash_L p \vee {\sim}p$. $\square$

Proposition 3. *The DH-logics* $\mathsf{B}, \mathsf{D} \in (\mathsf{J}, \mathsf{C})$ *and* $\mathsf{B} \parallel \mathsf{D}$ *(Figure 1).*

Proof. By the proof of Lemma 16, $\mathsf{B}, \mathsf{D} \in [\mathsf{J}, \mathsf{C}]$. Clearly, $\mathsf{Fr}(\mathsf{J}) \subset \mathsf{Fr}(\mathsf{B})$ and $\mathsf{Fr}(\mathsf{J}) \subset \mathsf{Fr}(\mathsf{D})$. Hence, $\mathsf{J} \subset \mathsf{B}$ and $\mathsf{J} \subset \mathsf{D}$. By DM frames in Example 2, we have $\mathsf{B} \subset \mathsf{C}$ and $\mathsf{D} \subset \mathsf{C}$. Hence, $\mathsf{B}, \mathsf{D} \in (\mathsf{J}, \mathsf{C})$. Moreover, $\mathfrak{F}_1 \notin \mathsf{Fr}(\mathsf{D})$ and $\mathfrak{F}_2 \notin \mathsf{Fr}(\mathsf{B})$. Hence, $\mathsf{B} \parallel \mathsf{D}$. $\square$

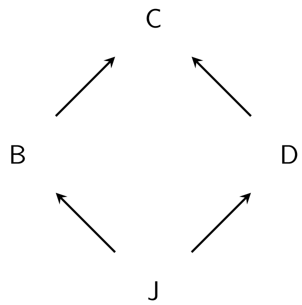


Figure 1. The relation between some DH-logics.

Theorem 7. *The DH-logics* B, D *and* C *are Kripke complete.*

Proof. By Theorem 6, it suffices to show that canonical frames for these DH-logics belong to their DM frames, respectively. For B, by Lemma 15 (1), it suffices to show $\Gamma{\uparrow} = \{\Gamma\}$ for every Γ in $\mathfrak{F}^{\mathsf{B}}$. We assume $\Gamma \subseteq \Delta$. We suppose $\varphi \in \Delta \setminus \Gamma$. By $\vdash_{\mathsf{B}} \varphi \vee \neg\varphi$, we have $\varphi \vee \neg\varphi \in \Gamma$. Then, $\neg\varphi \in \Gamma$ and so $\neg\varphi \in \Delta$. Then, $\varphi \wedge \neg\varphi \in \Delta$ and so $\bot \in \Delta$, which contradicts the consistency of Δ. Hence, $\Delta \subseteq \Gamma$ and so $\Delta = \Gamma$. For D, by Lemma 15 (1), it suffices to show $\Gamma{\uparrow} \cap g^{\mathsf{D}}(\Gamma){\uparrow} \neq \varnothing$ for every Γ in $\mathfrak{F}^{\mathsf{D}}$. We consider the D theory $\Sigma = [\Gamma \cup g^{\mathsf{D}}(\Gamma))_{\mathsf{D}}$. We note that Γ and $g^{\mathsf{D}}(\Gamma)$ are closed under taking $\wedge$. We suppose $\bot \in \Sigma$. Then, there exist $\varphi \in \Gamma$ and $\psi \in g^{\mathsf{D}}(\Gamma)$ such that $\vdash_{\mathsf{D}} \varphi \wedge \psi \to \bot$. Then, $\vdash_{\mathsf{D}} \varphi \to \neg\psi$. By $\vdash_{\mathsf{D}} \neg\psi \to {\sim}\psi$, we have $\vdash_{\mathsf{D}} \varphi \to {\sim}\psi$. By $\psi \in g^{\mathsf{D}}(\Gamma)$ and Lemma 13 (1), ${\sim}\psi \notin \Gamma$. Then, $\varphi \notin \Gamma$, which contradicts $\varphi \in \Gamma$. Hence, $\bot \notin \Sigma$. Then, there exists $G \in \mathcal{T}_{\mathrm{p}}(\mathsf{D})$ with $\Sigma \subseteq G$. Then, $G \in \Gamma{\uparrow} \cap g^{\mathsf{D}}(\Gamma){\uparrow}$. For C, by Lemma 15 (2), it suffices to show $g^{\mathsf{C}}(\Gamma) = \Gamma$ for every Γ in $\mathfrak{F}^{\mathsf{C}}$. Clearly, $g^{\mathsf{C}}(\Gamma) = \overline{({\sim}\Gamma]}_{\mathsf{C}}$. We assume $\varphi \in g^{\mathsf{C}}(\Gamma)$. By Lemma 13 (1), ${\sim}\varphi \notin \Gamma$. Since $\varphi \vee {\sim}\varphi \in \Gamma$, we have $\varphi \in \Gamma$. We assume $\varphi \notin g^{\mathsf{C}}(\Gamma)$. Then, $\varphi \in ({\sim}\Gamma]_{\mathsf{C}}$. By Lemma 12 (3), ${\sim}\varphi \in \Gamma$. By Lemma 16, $\neg\varphi \in \Gamma$. We suppose $\varphi \in \Gamma$. Then, $\varphi \wedge \neg\varphi \in \Gamma$. Clearly, $\varphi \wedge \neg\varphi \vdash_{\mathsf{C}} \bot$. Then, $\bot \in \Gamma$ which contradicts $\Gamma \in \mathcal{T}_{\mathrm{p}}(\mathsf{C})$. Hence, $\varphi \notin \Gamma$. It follows that $g^{\mathsf{C}}(\Gamma) = \Gamma$. $\square$

A DH-logic L is finitely approximable if for every $\varphi \notin L$ there exists a finite DM frame $\mathfrak{F}$ such that $\mathfrak{F} \models L$ and $\mathfrak{F} \not\models \varphi$. Thus, a DH-logic L is finitely approximable if and only if $L = \mathsf{Th}(\mathsf{Fr}_{<\omega}(L))$ where $\mathsf{Fr}_{<\omega}(L)$ is the set of all finite members in $\mathsf{Fr}(L)$. Now, we extend the Lemmon-filtration method to show the finite approximability of some DH-logics.

A set of DH formulas Σ is DH-closed if $\Sigma = \{\sim^n \varphi : \varphi \in \mathrm{Sf}(\Sigma)\ \&\ n \geq 0\}$. For every set of DH-formulas Θ, we let Θ^c be the minimal DH-closed set of formulas containing Θ. We note that Θ^c is obtained from Θ by taking subformulas and operation $\sim$.

Definition 8. *We let Σ be a DH-closed set of DH formulas and $\mathfrak{M} = (W, \leq, g, V)$ a DM model. The equivalence relation $\approx_\Sigma$ on W is defined as follows:*

$$w \approx_\Sigma u \text{ if and only if } \forall \varphi \in \Sigma(w \in V(\varphi) \iff u \in V(\varphi)).$$

We let $[w]_\Sigma = \{u \in W : w \approx_\Sigma u\}$ for each $w \in W$. We let $W_\Sigma = \{[w]_\Sigma : w \in W\}$. The filtration of $\mathfrak{M}$ through Σ is defined as structure $\mathfrak{M}_\Sigma = (W_\Sigma, \leq_\Sigma, g_\Sigma, V_\Sigma)$ where

(C1) $g_\Sigma([w]_\Sigma) = [g(w)]_\Sigma$ *for each* $w \in W$.
(C2) $[w]_\Sigma \leq_\Sigma [u]_\Sigma$ *if and only if* $\forall \varphi \in \Sigma(w \in V(\varphi) \Rightarrow u \in V(\varphi))$.
(C3) $V_\Sigma(p) = \{[w]_\Sigma \in W_\Sigma : w \in V(p)\}$ *for each* $p \in Var$.

We let $\mathfrak{F}_\Sigma = (W_\Sigma, \leq_\Sigma, g_\Sigma)$ which is the underlying frame of $\mathfrak{M}_\Sigma$.

We note that g_Σ is a function on W_Σ. We assume $u \in [w]_\Sigma$, i.e., $w \approx_\Sigma u$. We suppose $\varphi \in \Sigma$ and $g(w) \in V(\varphi)$. Then, $w \notin V(\sim\varphi)$. Since Σ is closed under $\sim$, we have $\sim\varphi \in \Sigma$. By assumption, $u \notin V(\sim\varphi)$. Then, $g(u) \in V(\varphi)$. Similarly, $g(u) \in V(\varphi)$ implies $g(w) \in V(\varphi)$. Then, $g(w) \approx_\Sigma g(u)$ and so $[g(w)]_\Sigma = [g(u)]_\Sigma$. Hence, g_Σ is a function. Moreover, if $w \leq u$, then clearly $[w]_\Sigma \leq_\Sigma [u]_\Sigma$.

Lemma 17. *Filtration $\mathfrak{M}_\Sigma = (W_\Sigma, \leq_\Sigma, g_\Sigma, V_\Sigma)$ is a DM model.*

Proof. Clearly, $\leq_\Sigma$ is a partial order on W_Σ. For every $w \in W$, $g_\Sigma(g_\Sigma([w]_\Sigma)) = [g(g(w))]_\Sigma = [w]_\Sigma$. Hence, g_Σ is an involution. We assume $[w]_\Sigma \leq_\Sigma [u]_\Sigma$. We suppose $\varphi \in \Sigma$ and $g(u) \in V(\varphi)$. Then, $u \notin V(\sim\varphi)$. Clearly, $\sim\varphi \in \Sigma$. By assumption, $w \notin V(\sim\varphi)$ and so $g(w) \in V(\varphi)$. Then, $[g(u)]_\Sigma \leq_\Sigma [g(w)]_\Sigma$. Hence, g_Σ is antitone. We assume $[w]_\Sigma \in V_\Sigma(p)$ and $[w]_\Sigma \leq_\Sigma [u]_\Sigma$. Then, $w \in V(p)$. By (C2), $u \in V(p)$, and so $[u]_\Sigma \in V_\Sigma(p)$. Hence, V_Σ is a valuation in $\mathfrak{F}_\Sigma$. □

We say that Σ is finitely based in $\mathfrak{M}$ if there exists a finite subset $\Delta \subseteq \Sigma$ such that for every $\varphi \in \Sigma$ there exists $\psi \in \Delta$ with $V(\varphi) = V(\psi)$. Such subset Δ is called a finite base for Σ in $\mathfrak{M}$. We note that, in this case, $|W_\Sigma| \leq 2^{|\Delta|}$, and so $\mathfrak{M}_\Sigma$ is finite.

Lemma 18. *We let Σ be a DH-closed set of DH formulas and $\mathfrak{M} = (W, \leq, g, V)$ a DM model. For every $\varphi \in \Sigma$ and $w \in W$, $w \in V(\varphi)$ if and only if $[w]_\Sigma \in V_\Sigma(\varphi)$.*

Proof. The proof proceeds by induction on $c(\varphi)$. The case of atomic formulas is trivial. We assume $\varphi = \sim\psi$. We suppose $w \in V(\sim\psi)$. Then, $g(w) \notin V(\psi)$. By induction hypothesis, $[g(w)]_\Sigma \notin V_\Sigma(\psi)$ and so $g_\Sigma([w]_\Sigma) \notin V_\Sigma(\psi)$. Then, $[w]_\Sigma \in V_\Sigma(\sim\psi)$. The other direction is shown similarly. We let $\varphi = \varphi_1 \to \varphi_2$. We assume $w \in V(\varphi_1 \to \varphi_2)$. We suppose $[w]_\Sigma \leq [u]_\Sigma$ and $[u]_\Sigma \in V_\Sigma(\varphi_1)$. By induction hypothesis, $u \in V(\varphi_1)$. By $[w]_\Sigma \leq [u]_\Sigma$, $w \in V(\varphi_1 \to \varphi_2)$ and (C2), we obtain $u \in V(\varphi_1 \to \varphi_2)$. Then, $u \in V(\varphi_2)$. By induction hypothesis, $[u]_\Sigma \in V_\Sigma(\varphi_2)$. Hence, $[w]_\Sigma \in V_\Sigma(\varphi_1 \to \varphi_2)$. We assume $[w]_\Sigma \in V_\Sigma(\varphi_1 \to \varphi_2)$. We suppose $w \leq u$ and $u \in V(\varphi_1)$. Then, $[w]_\Sigma \leq_\Sigma [u]_\Sigma$. By induction hypothesis, $[u]_\Sigma \in V_\Sigma(\varphi_1)$. Then, $[u]_\Sigma \in V_\Sigma(\varphi_2)$. By induction hypothesis, $u \in V(\varphi_2)$. Hence, $w \in V(\varphi_1 \to \varphi_2)$. The case $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ is shown easily by induction hypothesis. □

Theorem 8. *The DH-logics* J, B *and* C *are finitely approximable.*

Proof. (1) We sssume $\varphi \notin \mathsf{J}$. Since J is Kripke complete, there is a DM model $\mathbb{M} = (W, \leq, g, V)$ such that $w \notin V(\varphi)$ for some $w \in W$. We let $\Sigma = \{\sim^n \psi : \psi \in \mathrm{Sf}(\varphi)\ \&\ n \geq 0\}$. By Lemma 18, $[w]_\Sigma \notin V_\Sigma(\varphi)$. Hence, $\mathfrak{F}_\Sigma \not\models \varphi$. Clearly, $\Delta = \mathrm{Sf}(\varphi) \cup \{\sim\psi : \psi \in \mathrm{Sf}(\varphi)\}$ is a finite base for Σ in $\mathfrak{M}$. Then, $\mathfrak{F}_\Sigma$ is a finite DM-frame. Clearly, $\mathfrak{F}_\Sigma \models \mathsf{J}$. Hence, J is finitely approximable.

(2) We assume $\varphi \notin \mathsf{B}$. Since B is Kripke complete, there exists a DM frame $\mathfrak{F} = (W, \leq, g)$ such that $\mathfrak{F} \models \mathsf{B}$ and $\mathfrak{F} \not\models \varphi$. We let $\mathfrak{M} = (\mathfrak{F}, V)$ be a DM model and $w \notin V(\varphi)$. We consider $\Theta = \mathsf{Sf}(\varphi) \cup \{\neg\psi : \psi \in \mathsf{Sf}(\varphi)\}$ and $\Sigma = \Theta^c$. Then, $\Theta \cup \{\sim\chi : \chi \in \Theta\}$ is a finite base for Σ in $\mathfrak{M}$. By Lemma 18, $[w]_\Sigma \notin V_\Sigma(\varphi)$. Hence, $\mathfrak{F}_\Sigma \not\models \varphi$. Now, we show $\mathfrak{F}_\Sigma \models \mathsf{B}$. By Lemma 15, it suffices to show $[u]_\Sigma{\uparrow} = \{[u]_\Sigma\}$. We suppose $[u]_\Sigma \leq_\Sigma [v]_\Sigma$. We suppose $\xi \in \Sigma$ and $u \notin V(\xi)$. By $\mathfrak{F} \models \mathsf{B}$, we obtain $\mathfrak{F} \models \xi \vee \neg\xi$. Then, $u \in V(\neg\xi)$. Since $\neg\xi \in \Sigma$, by $[u]_\Sigma \leq_\Sigma [v]_\Sigma$, we obtain $v \in V(\neg\xi)$. By $v \leq v$, we obtain $v \notin V(\xi)$. Hence, $[v]_\Sigma \leq_\Sigma [u]$. It follows that $[u]_\Sigma = [v]_\Sigma$. Then, $\mathfrak{F}_\Sigma \models \mathsf{B}$. Hence, B is finitely approximable.

(3) We assume $\varphi \notin \mathsf{C}$. Since C is Kripke complete, there is a DM frame $\mathfrak{F} = (W, \leq, g)$ such that $\mathfrak{F} \models \mathsf{C}$ and $\mathfrak{F} \not\models \varphi$. We let $\mathfrak{M} = (\mathfrak{F}, V)$ be a DM model and $w \notin V(\varphi)$. We let $\Sigma = \{\sim^n\psi : \psi \in \mathsf{Sf}(\varphi) \;\&\; n \geq 0\}$. By Lemma 18, $[w]_\Sigma \notin V_\Sigma(\varphi)$. Hence, $\mathfrak{F}_\Sigma \not\models \varphi$. We let $\psi \in \Sigma$ and $w \notin V(\psi)$. Clearly, $\psi \vee \sim\psi \in \mathsf{C}$. Then, $w \in V(\sim\psi)$. Then, $g(w) \notin V(\psi)$. Then, $[g(w)]_\Sigma \leq_\Sigma [w]_\Sigma$. We suppose $g(w) \notin V(\psi)$. Then, $g(w) \in V(\sim\psi)$ and so $w \notin V(\psi)$. Then, $[w]_\Sigma \leq_\Sigma [g(w)]_\Sigma$. Hence, $[g(w)]_\Sigma = [w]_\Sigma$. Then, $\mathfrak{F}_\Sigma \models \mathsf{C}$ and so C is finitely approximable. $\square$

Corollary 3. *The DH-logics* J, B *and* C *are decidable.*

5. Conservativity and the Lattice Ext(J)

We let $\mathcal{L}_\mathrm{I}$ be the set of intuitionistic formulas (''I-formulas" for short) built from Var by using $\bot, \wedge, \vee$ and $\to$. We let $\mathbb{F} = (W, \leq)$ be an I-frame. We say that (i) a point $w \in W$ is a root of $\mathbb{F}$ if $w{\uparrow} = W$; and (ii) $\mathbb{F}$ is rooted if $\mathbb{F}$ has a root. An intuitionistic model (''I-model" for short) is a triple $\mathbb{M} = (W, \leq, V)$ where $(W, \leq)$ is an I-frame and $V : Var \to Up(W)$. We write $\mathbb{M}, w \models \varphi$ if $w \in V(\varphi)$. An I-formula φ is valid at w in $\mathbb{F}$ (notation: $\mathbb{F}, w \models \varphi$) if $\mathbb{F}, V, w \models \varphi$ for every valuation V in $\mathbb{F}$. An I-formula φ is valid in $\mathbb{F}$ (notation: $\mathbb{F} \models \varphi$) if $\mathbb{F}, w \models \varphi$ for every $w \in W$. We let $\mathsf{IF}(\Sigma)$ be the class of all I-frames validating all formulas in Σ. We let $\mathsf{IF}_\rho(\Sigma)$ be the set of all rooted members in $\mathsf{IF}(\Sigma)$. The I-theory $\mathsf{Th}(\mathcal{K})$ of a class of I-frames $\mathcal{K}$ is the set of all I-formulas valid in every I-frame in $\mathcal{K}$. The intuitionistic logic is defined as $\mathsf{Int} = \{\varphi \in \mathcal{L}_\mathrm{I} : \mathbb{F} \models \varphi$ for every I-frame $\mathbb{F}\}$.

Definition 9. *A superintuitionistic logic is a set of I-formulas S such that* $\mathsf{Int} \subseteq S$ *and S is closed under (MP) and (Sub). We let S be a superintuitionistic logic and Σ a set of formulas. The superintuitionistic logic generated by Σ over S is denoted by $S \oplus \Sigma$. We write* $\mathsf{Ext}(S)$ *for the lattice of s.i. logics extending S. We say that S is Kripke complete if* $S = \mathsf{Th}(\mathsf{IF}(S))$. *The DH expansion of S, denoted by S^δ, is defined as the smallest DH-logic containing S.*

We let $\mathbb{F} = (W, \leq)$ be an I-frame and $\varnothing \neq X \subseteq W$. The subframe of $\mathbb{F}$ generated by X is defined as $\mathbb{F}_X = (X{\uparrow}, \leq^X)$ where $\leq^X = \leq \cap X^2$. We let $\mathbb{M} = (\mathbb{F}, V)$ be an I-model. The submodel of $\mathbb{M}$ generated by X is $\mathbb{M}_X = (\mathbb{F}_X, V_X)$, where $V_X(p) = V(p) \cap X$ for each $p \in Var$. If $X = \{w\}$, we write $\mathbb{F}_w$ for $\mathbb{F}_{\{w\}}$, and $\mathbb{M}_w$ for $\mathbb{M}_{\{w\}}$. For every $u \in X$, the following hold:

$$\mathbb{M}, u \models \varphi \Leftrightarrow \mathbb{M}_X, u \models \varphi. \tag{1}$$

$$\mathbb{F}, u \models \varphi \Leftrightarrow \mathbb{F}_X, u \models \varphi. \tag{2}$$

$$\mathbb{F} \models \varphi \Rightarrow \mathbb{F}_X \models \varphi. \tag{3}$$

These preservation results (1)–(3) can be found in, e.g., [20].

Lemma 19. *A superintuitionistic logic S is Kripke complete if and only if* $S = \mathsf{Th}(\mathsf{IF}_\rho(S))$.

Proof. The right-to-left direction holds, obviously. We assume S is Kripke complete. Clearly, $S \subseteq \mathsf{Th}(\mathsf{IF}_\rho(S))$. We suppose $\varphi \notin S$. Then, $\mathbb{F} \not\models \varphi$ for some I-frame $\mathbb{F} = (W, \leq)$.

Then, $\mathbb{F}, w \not\models \varphi$ for some $w \in W$. Then, $\mathbb{F}_w \not\models \varphi$. Clearly, $\mathbb{F}_w \in \mathsf{IF}_\rho(S)$. Hence, $\varphi \notin \mathsf{Th}(\mathsf{IF}_\rho(S))$. $\Box$

DH-logic L is a conservative extension of a superintuitionistic logic S (or S is the I-fragment of L) if $S = L \cap \mathcal{L}_\mathrm{I}$. Next, we prove DH-expansion S^δ of a Kripke complete superintuitionistic logic S which preserves expansion is a conservative extension of S.

Definition 10. *We let $W = \{w_i : i < \lambda\}$ ($\lambda > 0$) and $\mathbb{F} = (W, \leq)$ be an I-frame with root $w_0 \in W$. The expansion of $\mathbb{F}$ is defined as the structure $\mathbb{F}^* = (W \cup W^*, \leq^*, g)$ where*

(1) $W^* = \{w_i^* : 0 < i < \lambda\}$;
(2) $\leq^* = \leq \cup \{\langle w_i^*, w_j^* \rangle : w_i \leq w_j \;\&\; 0 < j \leq i < \lambda\} \cup \{\langle w_i^*, w_j \rangle : 0 < i < \lambda \;\&\; j < \lambda\}$;
(3) *function $g : W \cup W^* \to W \cup W^*$ is defined as follows:*

$$g(w_i) = \begin{cases} w_0, & \text{if } i = 0. \\ w_i^*, & \text{if } i > 0. \end{cases}$$
$$g(w_i^*) = w_i.$$

For an I-model $\mathbb{M} = (\mathbb{F}, V)$, we let $\mathbb{M}^* = (\mathbb{F}^*, V)$ be the expansion of $\mathbb{M}$.

Expansion $\mathbb{F}^*$ is obtained from the rooted I-frame $\mathbb{F}$ by putting it over the order dual of $\mathbb{F}$ such that the root of $\mathbb{F}$ coincides the top of its order dual. The resulting involution g links each point in $\mathbb{F}$ with its mirror point in the order dual.

Lemma 20. *If $\mathbb{F}$ is a rooted I-frame, then the expansion $\mathbb{F}^*$ is a DM-frame.*

Proof. We let $W = \{w_i : i < \kappa\}$ and $\mathbb{F} = (W, \leq)$ be an I-frame with root w_0. Clearly, $\leq^*$ is a partial order. If $i < \kappa$, then $g(g(w_i)) = w_i$. If $0 < i < \kappa$, then $g(g(w_i^*)) = w_i^*$. Hence, g is an involution on $W \cup W^*$. If $w_i \leq w_j$, then $g(w_j) \leq^* g(w_i)$. If $w_i^* \leq^* w_j^*$, then $w_j \leq w_i$ and so $g(w_i^*) \leq^* g(w_j^*)$. We suppose $w_i^* \leq^* w_j$. Then, $g(w_j^*) = w_j$. If $j = 0$, then $g(w_0) = w_0$ and so $w_0 \leq^* w_j$. If $j > 0$, then $g(w_j) = w_j^*$ and so $w_j^* \leq^* w_0 \leq^* w_j$. Hence, g is antitone. $\Box$

Example 3. *We let $W = \{w_i : i < 7\}$ and $\mathbb{F} = (W, \leq)$ be the I-frame where $\leq = \{\langle w_i, w_i \rangle, \langle w_i, w_j \rangle : i \leq j < 7\}$. The expansion of $\mathbb{F}$ is the DM frame given in Figure 2.*

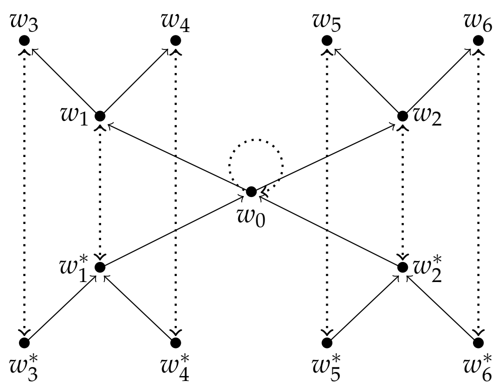


Figure 2. Expansion of the I-frame $\mathbb{F}$.

Lemma 21. *We let $\mathbb{F} = (W, \leq)$ be a rooted I-frame and $\mathbb{M} = (\mathbb{F}, V)$ an I-model. For every I-formula $\varphi \in \mathcal{L}_\mathrm{I}$ and $w \in W$, the following hold:*

(1) $\mathbb{M}, w \models \varphi$ *if and only if* $\mathbb{M}^*, w \models \varphi$.
(2) $\mathbb{F}, w \models \varphi$ *if and only if* $\mathbb{F}^*, w \models \varphi$.

Proof. Item (1) is shown regularly by induction on $c(\varphi)$. We note that the truth of an I-formula at w in $\mathbb{M}$ and $\mathbb{M}^*$ is determined only by $w\uparrow$. Moreover, (2) follows from (1). $\square$

A superintuitionistic logic S *preserves expansion* if for every roooted I-frame $\mathbb{F}$, $\mathbb{F} \models S$ implies $\mathbb{F}^* \models S^\delta$. Then, we obtain the following theorem:

Theorem 9. *We let S be a Kripke complete superintuitionistic logic. If S preserves expansion, then DH expansion S^δ is a conservative extension of S.*

Proof. We assume S preserves expansion. By Lemma 19, $S = \mathsf{Th}(\mathsf{Fr}_\rho(S))$. Clearly, $S \subseteq S^\delta \cap \mathcal{L}_\mathrm{I}$. We suppose $\varphi \in \mathcal{L}_\mathrm{I} \setminus S$. Then, there exists a rooted I-frame $\mathbb{F} = (W, \leq)$ such that $\mathbb{F} \models S$ and $\mathbb{F} \not\models \varphi$. By Lemma 21, $\mathbb{F}^* \not\models \varphi$. By $\mathbb{F} \models S$ and assumpton, $\mathbb{F}^* \models S^\delta$. Then, $\varphi \notin S^\delta$. Hence, S^δ is a conservative extension of S. $\square$

Corollary 4. J *is a conservative extension of* Int.

Proof. By Lemma 20, the intuitionistic propositional logic Int preserves expansion. By Theorem 9, J is a conservative extension of Int. $\square$

Let us consider DH-logic $\mathsf{C} = \mathsf{J} \oplus p \vee {\sim}p = \mathsf{J} \oplus {\sim}p \leftrightarrow \neg p$. The classical propositional logic is defined as $\mathsf{K} = \mathsf{Int} \oplus p \vee \neg p$.

Lemma 22. *For every I-formula $\varphi \in \mathcal{L}_\mathrm{I}$, $\varphi \in \mathsf{C}$ if and only if $\varphi \in \mathsf{K}$.*

Proof. We assume $\varphi \in \mathsf{K}$. Clearly, $\mathsf{K}^\delta = \mathsf{B}$ and so $\mathsf{K} \subseteq \mathsf{B} \subseteq \mathsf{C}$. Hence, $\varphi \in \mathsf{C}$. We assume $\varphi \notin \mathsf{K}$. Then, there exists substitution $s : \mathit{Var} \to \mathcal{L}_\mathrm{I}$ such that $\varphi^s \leftrightarrow \bot \in \mathsf{K}$ and so $\neg\varphi^s \in \mathsf{K}$. Then, $\neg\varphi^s \in \mathsf{C}$. We suppose $\varphi \in \mathsf{C}$. By (Sub), $\varphi^s \in \mathsf{C}$ which contradicts $\neg\varphi^s \in \mathsf{C}$. Hence, $\varphi \notin \mathsf{C}$. $\square$

Lemma 23. *For every DH-formula φ, there exists a unique I-formula $\varphi^\sharp$ such that $\vdash_\mathsf{C} \varphi \leftrightarrow \varphi^\sharp$.*

Proof. We let φ be a DH formula. By Lemma 10 (3), we replace each subformula ${\sim}\psi \in \mathsf{Sf}(\varphi)$ with $\neg\psi$, and so obtain a unique I-formula $\varphi^\sharp$ such that $\vdash_\mathsf{C} \varphi \leftrightarrow \varphi^\sharp$. $\square$

DH-logic L is consistent if $\bot \notin L$. We note that $\mathcal{L}_\mathrm{DH}$ is the only inconsistent DH-logic. Next, we show that every consistent DH-logic belongs to $[\mathsf{J}, \mathsf{C}]$ and hence C is the second largest DH-logic in the lattice $\mathsf{Ext}(\mathsf{J})$.

Theorem 10. *If L is a consistent DH-logic, then $L \in [\mathsf{J}, \mathsf{C}]$.*

Proof. We let L be consistent DH-logics. We suppose $\mathsf{C} \subsetneq L$. Then, there exists ψ such that $\varphi \in L$ and $\varphi \notin \mathsf{C}$. By Lemma 23, we have $\varphi^\sharp \notin \mathsf{C}$. By Lemma 22, we obtain $\varphi^\sharp \notin \mathsf{K}$. Then, there exists substitution $s : \mathit{Var} \to \mathcal{L}_\mathrm{I}$ such that $(\varphi^\sharp)^s \leftrightarrow \bot \in \mathsf{K}$. Then, $\neg(\varphi^\sharp)^s \in \mathsf{K}$. Then, $\neg(\varphi^\sharp)^s \in \mathsf{C}$. By Lemma 23, $\neg\varphi^s \in \mathsf{C}$. Then, $\neg\varphi^s \in L$, which contradicts $\varphi \in L$. $\square$

An immediate consequence of Theorem 10 is that C is the unique DH-logic which is post complete, i.e., $\subseteq$-maximal consistent. Let us continue the study on the link between C and J. It is well known (Glivenko's theorem [20]) that K is embedded into intuitionistic logic Int via the double negation translation:

(GT) for every $\varphi \in \mathcal{L}_\mathrm{I}$, $\varphi \in \mathsf{K}$ if and only if $\neg\neg\varphi \in \mathsf{Int}$.

Now, we can extend Glivenko's theorem to DH-logics C and J. By Lemma 23, there exists a function $G^\sharp : \mathcal{L}_\mathrm{DH} \to \mathcal{L}_\mathrm{I}$ where $G^\sharp(\varphi) = \neg\neg\varphi^\sharp$ for each $\varphi \in \mathcal{L}_\mathrm{DH}$.

Theorem 11 (Embedding)**.** *For every $\varphi \in \mathcal{L}_\mathrm{DH}$, $\varphi \in \mathsf{C}$ if and only if $G^\sharp(\varphi) \in \mathsf{J}$.*

Proof. We asume $\neg\neg\varphi^\sharp \in \mathsf{J}$. Then $\neg\neg\varphi^\sharp \in \mathsf{C}$. Clearly, $\neg\neg\psi \leftrightarrow \psi \in \mathsf{C}$ for every $\psi \in \mathcal{L}_{\mathrm{DH}}$. Hence, $\varphi^\sharp \in \mathsf{C}$. By Lemma 23, $\varphi \in \mathsf{C}$. We assume $\varphi \in \mathsf{C}$. By Lemma 23, $\varphi^\sharp \in \mathsf{C}$. Since $\varphi^\sharp \in \mathcal{L}_{\mathrm{I}}$, by Lemma 22, $\varphi^\sharp \in \mathsf{K}$. By (GT), $\neg\neg\varphi^\sharp \in \mathsf{Int}$. By Corollary 4, $\neg\neg\varphi^\sharp \in \mathsf{J}$. □

6. A Gentzen Sequent Calculus for B

In this section, we study the proof theory of the DH-logic $\mathsf{B} = \mathsf{J} \oplus p \vee \neg p$. This logic can be viewed as a logic merging classical and De Morgan logics. It is Kripke complete but not post complete. We note that DH-algebra $(B, \sim)$ is an algebra for B if and only if $x \vee \neg x = 1$. Such a DH-algebra is clearly a Boolean algebra with a De Morgan negation, and we call it a De Morgan Boolean algebra (''DB-algebra" for short).

Lemma 24. *We let $\mathbb{B} = (B, \sim)$ be a DB-algebra. Then, $\sim\neg x = \neg\sim x$ for all $x \in B$.*

Proof. We let $x \in B$. Then, $\sim(x \vee \neg x) = \sim 1 = 0$. Then, $\sim x \wedge \sim\neg x = 0$. By (Res), $\sim\neg x \leq \neg\sim x$. We assume $y \leq \neg\sim x$. By (Res), $y \wedge \sim x = 0$. Then, $\sim(y \wedge \sim x) = \sim y \vee \sim\sim x = \sim y \vee x = 1$. Then, $\neg x = \neg x \wedge (\sim y \vee x) = (\neg x \wedge \sim y) \vee (\neg x \wedge x) = \neg x \wedge \sim y \leq \sim y$. Then, $y = \sim\sim y \leq \sim\neg x$. Hence, $\neg\sim x \leq \sim\neg x$. It follows that $\sim\neg x = \neg\sim x$. □

Corollary 5. $\vdash_{\mathsf{B}} \sim\neg\varphi \leftrightarrow \neg\sim\varphi$*.*

Now, we establish a Gentzen sequent calculus for DH-logic B. The basics of Gentzen sequent calculus for classical and intuitionistic logics can be found in, e.g., [21]. For the simplicity of formulation, we consider the set of DH formulas built from variables in *Var* using the connectives $\bot, \neg, \wedge, \vee$ and $\sim$. Thus, we use abbreviation $\varphi \to \psi := \neg\varphi \vee \psi$. A sequent is expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are a finite (possibly empty) multiset of formulas. We use s, t, etc., for sequents. Sequent rule (R) is expression

$$\frac{s_1 \ldots s_n}{s_0}(R),$$

where $s_1, \ldots, s_n$ are the premisses and s_0 is the conclusion of (R).

Definition 11. *Sequent calculus* GB *consists of the following initial sequents and rules:*

(1) *Initial sequents:*

$$(\mathrm{Id}_1)\ p, \Gamma \Rightarrow \Delta, p \quad (\mathrm{Id}_2) \sim p, \Gamma \Rightarrow \Delta, \sim p \quad (\bot)\ \bot, \Gamma \Rightarrow \Delta \quad (\sim\bot)\ \Gamma \Rightarrow \Delta, \sim\bot$$

.

(2) *Logical rules:*

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}(\neg\Rightarrow) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}(\Rightarrow\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\varphi}{\sim\neg\varphi, \Gamma \Rightarrow \Delta}(\sim\neg\Rightarrow) \quad \frac{\sim\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\neg\varphi}(\Rightarrow\sim\neg)$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}(\wedge\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}(\Rightarrow\wedge)$$

$$\frac{\sim\varphi, \Gamma \Rightarrow \Delta \quad \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}(\sim\wedge\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \sim\varphi, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \wedge \psi)}(\Rightarrow\sim\wedge)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}(\vee\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}(\Rightarrow\vee)$$

$$\frac{\sim\varphi, \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \vee \psi), \Gamma \Rightarrow \Delta}(\sim\vee\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \sim\varphi \quad \Gamma \Rightarrow \Delta, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \vee \psi)}(\Rightarrow\sim\vee)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\sim\sim\varphi, \Gamma \Rightarrow \Delta}(\sim\sim\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \sim\sim\varphi}(\Rightarrow\sim\sim)$$

The formula with connectives in the below sequent of a logical rule is called principal. A derivation in GB *is a finite tree of sequents in which each node is either an initial sequent or derived from child node(s) by a rule. We use $\mathcal{D}, \mathcal{E}$, etc., for derivations. The height of derivation $\mathcal{D}$, denoted by $|\mathcal{D}|$, is the maximal length of branches in $\mathcal{D}$. A single node derivation has height* 0*. Sequent s is derivable in* GB *(notation:* $\mathsf{GB} \vdash s$*) if there exists a derivation in* GB *with root node s. For every $n \geq 0$, we use $\mathsf{GB} \vdash_n s$ for s that has a derivation in* GB *with a height of at most n. Sequent rule (R) with premisses $s_1, \ldots, s_n$ and conclusion s_0 is admissible in* GB *if $\mathsf{GB} \vdash s_0$ whenever $\mathsf{GB} \vdash s_i$ for each $1 \leq i \leq n$. Sequent rule (R) with premisses $s_1, \ldots, s_n$ and conclusion s_0 is height-preserving admissible in* GB *if for every $k \geq 0$, $\mathsf{GB} \vdash_k s_0$ whenever $\mathsf{GB} \vdash_k s_i$ for each $1 \leq i \leq n$. Prefix* GB *is omitted if no confusion arises from the context.*

Lemma 25. *For every formula φ, $\mathsf{GB} \vdash \varphi, \Gamma \Rightarrow \Delta, \varphi$.*

Proof. The proof proceeds by induction on $c(\varphi)$. Case $\varphi \in Var \cup \{\bot\}$ is trivial. Cases $\varphi = \varphi_1 \odot \varphi_2$ for $\odot \in \{\wedge, \vee\}$ are easily shown by induction hypothesis. We suppose $\varphi = \neg\psi$. We obtain $\vdash \neg\psi, \Gamma \Rightarrow \Delta, \neg\psi$ by induction hypothesis and rules $(\neg\Rightarrow)$ and $(\Rightarrow\neg)$. We let $\varphi = {\sim}\psi$. The proof proceeds by subinduction on $c(\psi)$. Case $\psi \in Var \cup \{\bot\}$ is trivial. We suppose $\psi = \neg\chi$. By induction hypothesis, $\vdash {\sim}\chi, \Gamma \Rightarrow \Delta, {\sim}\chi$. By $({\sim}\neg\Rightarrow)$ and $(\Rightarrow{\sim}\neg)$, $\vdash {\sim}\neg\chi, \Gamma \Rightarrow \Delta, {\sim}\neg\chi$. We suppose $\psi = {\sim}\chi$. By induction hypothesis, $\vdash \chi, \Gamma \Rightarrow \Delta, \chi$. By $({\sim}{\sim}\Rightarrow)$ and $(\Rightarrow{\sim}{\sim})$, $\vdash {\sim}{\sim}\chi, \Gamma \Rightarrow \Delta, {\sim}{\sim}\chi$. We suppose $\psi = \psi_1 \wedge \psi_2$. By induction hypothesis, $\vdash {\sim}\psi_i, \Gamma \Rightarrow \Delta, {\sim}\psi_i$ for $i = 1, 2$. By $({\sim}\wedge\Rightarrow)$ and $(\Rightarrow{\sim}\wedge)$, $\vdash {\sim}(\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta, {\sim}(\psi_1 \wedge \psi_2)$. Case $\psi = \psi_1 \vee \psi_2$ is shown similarly. □

Lemma 26. *The following rules of weakening are height-preserving admissible in* GB*:*

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}(\mathrm{Wk}\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}(\Rightarrow\mathrm{Wk})$$

Proof. We assume $\vdash_k \Gamma \Rightarrow \Delta$. We show $\vdash_k \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \varphi$ simultaneously by induction on $k \geq 0$. We suppose $k = 0$. Then, $\Gamma \Rightarrow \Delta$ is an initial sequent and so both $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are initial sequents. We let $k > 0$ and $\Gamma \Rightarrow \Delta$ be derived by a rule (R). We obtain $\vdash_k \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \varphi$ by induction hypothesis and the rule (R). For example, we let (R) be $({\sim}\neg\Rightarrow)$. We let the premise and conclusion of (R) be $\vdash_{k-1} \Gamma' \Rightarrow \Delta, {\sim}\chi$ and $\vdash_k {\sim}\neg\chi, \Gamma' \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \varphi, \Gamma' \Rightarrow \Delta, {\sim}\chi$ and $\vdash_{k-1} \Gamma' \Rightarrow \Delta, {\sim}\chi, \varphi$. By $({\sim}\neg\Rightarrow)$, $\vdash_k \varphi, {\sim}\neg\chi, \Gamma' \Rightarrow \Delta$ and $\vdash_k {\sim}\neg\chi, \Gamma' \Rightarrow \Delta, \varphi$. □

Lemma 27. *For every $k \geq 0$, the following hold in* GB*:*

(1) *if $\vdash_k \neg\varphi, \Gamma \Rightarrow \Delta$, then $\vdash_k \Gamma \Rightarrow \Delta, \varphi$.*
(2) *if $\vdash_k \Gamma \Rightarrow \Delta, \neg\varphi$, then $\vdash_k \varphi, \Gamma \Rightarrow \Delta$.*
(3) *if $\vdash_k {\sim}\neg\varphi, \Gamma \Rightarrow \Delta$, then $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\varphi$.*
(4) *if $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\neg\varphi$, then $\vdash_k {\sim}\varphi, \Gamma \Rightarrow \Delta$.*
(5) *if $\vdash_k \varphi \wedge \psi, \Gamma \Rightarrow \Delta$, then $\vdash_k \varphi, \psi, \Gamma \Rightarrow \Delta$.*
(6) *if $\vdash_k \Gamma \Rightarrow \Delta, \varphi \wedge \psi$, then $\vdash_k \Gamma \Rightarrow \Delta, \varphi$ and $\vdash_k \Gamma \Rightarrow \Delta, \psi$.*
(7) *if $\vdash_k {\sim}(\varphi \wedge \psi), \Gamma \Rightarrow \Delta$, then $\vdash_k {\sim}\varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k {\sim}\psi, \Gamma \Rightarrow \Delta$.*
(8) *if $\vdash_k \Gamma \Rightarrow \Delta, {\sim}(\varphi \wedge \psi)$, then $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\varphi, {\sim}\psi$.*
(9) *if $\vdash_k \varphi \vee \psi, \Gamma \Rightarrow \Delta$, then $\vdash_k \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \psi, \Gamma \Rightarrow \Delta$.*
(10) *if $\vdash_k \Gamma \Rightarrow \Delta, \varphi \vee \psi$, then $\vdash_k \Gamma \Rightarrow \Delta, \varphi, \psi$.*
(11) *if $\vdash_k {\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta$, then $\vdash_k {\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta$.*
(12) *if $\vdash_k \Gamma \Rightarrow \Delta, {\sim}(\varphi \vee \psi)$, then $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\varphi$ and $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\psi$.*
(13) *if $\vdash_k {\sim}{\sim}\varphi, \Gamma \Rightarrow \Delta$, then $\vdash_k \varphi, \Gamma \Rightarrow \Delta$.*
(14) *if $\vdash_k \Gamma \Rightarrow \Delta, {\sim}{\sim}\varphi$, then $\vdash_k \Gamma \Rightarrow \Delta, \varphi$.*

Proof. The proof proceeds by induction on $k \geq 0$. Here, we prove only (3), and other items are shown similarly. Case $k = 0$ is trivial. We let $k > 0$ and ${\sim}\neg\varphi, \Gamma \Rightarrow \Delta$ be derived by rule (R). If ${\sim}\neg\varphi$ is principal in (R), then $\vdash_{k-1} \Gamma \Rightarrow \Delta, {\sim}\varphi$ and so $\vdash_k \Gamma \Rightarrow \Delta, {\sim}\varphi$. We suppose

$\sim\neg\varphi$ is not principal in (R). We obtain $\vdash_k \Gamma \Rightarrow \Delta, \sim\varphi$ by induction hypothesis and rule (R). For example, we let (R) be $(\sim\wedge\Rightarrow)$ with premisses $\vdash_{k-1} \sim\psi_1, \sim\neg\varphi, \Gamma' \Rightarrow \Delta$ and $\vdash_{k-1} \sim\psi_2, \sim\neg\varphi, \Gamma' \Rightarrow \Delta$, and conclusion $\vdash_k \sim(\psi_1 \wedge \psi_2), \sim\neg\varphi, \Gamma' \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \sim\psi_1, \Gamma' \Rightarrow \Delta, \sim\varphi$ and $\vdash_{k-1} \sim\psi_2, \Gamma' \Rightarrow \Delta, \sim\varphi$. By $(\sim\wedge\Rightarrow)$, $\vdash_k \sim(\psi_1 \wedge \psi_2), \Gamma' \Rightarrow \Delta, \sim\varphi$. Other cases are shown similarly. □

Lemma 28. *The following rules of contraction are height-preserving admissible in* GB*:*

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}(\mathrm{Ctr}{\Rightarrow}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}({\Rightarrow}\mathrm{Ctr})$$

Proof. We assume $\vdash_k \varphi, \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \varphi, \varphi$. We show $\vdash_k \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \varphi$ simultaneously by induction on $k \geq 0$. Case $k = 0$ is trivial. We let $k > 0$ and the premisses be obtained by rule (R). If φ is not principal in (R), we obtain $\vdash_k \varphi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \varphi$ by induction hypothesis and rule (R). We suppose φ is principal in (R). The proof proceeds by induction on $c(\varphi)$. We have the following cases:

(1) $\varphi = \neg\psi$. Then, $\vdash_{k-1} \neg\psi, \Gamma \Rightarrow \Delta, \psi$ and $\vdash_{k-1} \psi, \Gamma \Rightarrow \Delta, \neg\psi$. By Lemma 27 (1) and (2), $\vdash_{k-1} \Gamma \Rightarrow \Delta, \psi, \psi$ and $\vdash_{k-1} \psi, \psi, \Gamma \Rightarrow \Delta$. By induction hypothesis, $\vdash_k \Gamma \Rightarrow \Delta, \psi$ and $\vdash_k \psi, \Gamma \Rightarrow \Delta$. By $(\neg\Rightarrow)$ and $(\Rightarrow\neg)$, $\vdash_k \neg\psi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \neg\psi$.

(2) $\varphi = \varphi_1 \wedge \varphi_2$. Then, (i) $\vdash_{k-1} \varphi_1, \varphi_2, \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta$; (ii) $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2, \varphi_1$; and (iii) $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2, \varphi_2$. By (i) and Lemma 27 (5), $\vdash_{k-1} \varphi_1, \varphi_2, \varphi_1, \varphi_2, \Gamma \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \varphi_1, \varphi_2, \Gamma \Rightarrow \Delta$. By $(\wedge\Rightarrow)$, $\vdash_k \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta$. By (ii), (iii) and Lemma 27 (6), $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1, \varphi_1$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_2, \varphi_2$. By induction hypothesis, $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_2$. By $(\Rightarrow\wedge)$, $\vdash_k \Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2$.

(3) $\varphi = \varphi_1 \vee \varphi_2$. Then, (i) $\vdash_{k-1} \varphi_1, \varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta$; (ii) $\vdash_{k-1} \varphi_2, \varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta$; and (iii) $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2, \varphi_1, \varphi_2$. By (i), (ii) and Lemma 27 (9), $\vdash_{k-1} \varphi_1, \varphi_1, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \varphi_2, \varphi_2, \Gamma \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \varphi_1, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \varphi_2, \Gamma \Rightarrow \Delta$. By $(\vee\Rightarrow)$, $\vdash_k \varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta$. By (iii) and Lemma 27 (10), $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1, \varphi_2, \varphi_1, \varphi_2$. By induction hypothesis, $\vdash_{k-1} \Gamma \Rightarrow \Delta, \varphi_1, \varphi_2$. By $(\Rightarrow\vee)$, $\vdash_k \Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2$.

(4) $\varphi = \sim\psi$. The proof proceeds by subinduction on $c(\psi)$. We suppose $\psi = \neg\chi$. Then, $\vdash_{k-1} \sim\neg\chi, \Gamma \Rightarrow \Delta, \sim\chi$ and $\vdash_{k-1} \sim\chi, \Gamma \Rightarrow \Delta, \sim\neg\chi$. By Lemma 27 (3) and (4), $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim\chi, \sim\chi$ and $\vdash_{k-1} \sim\chi, \sim\chi, \Gamma \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim\chi$ and $\vdash_{k-1} \sim\chi, \Gamma \Rightarrow \Delta$. By $(\sim\neg\Rightarrow)$ and $(\Rightarrow\sim\neg)$, $\vdash_k \sim\neg\chi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \sim\neg\chi$. We suppose $\psi = \psi_1 \wedge \psi_2$. Then (i) $\vdash_{k-1} \sim\psi_1, \sim(\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta$; (ii) $\vdash_{k-1} \sim\psi_2, \sim(\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta$; and (iii) $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim(\psi_1 \wedge \psi_2), \sim\psi_1, \sim\psi_2$. By (i), (ii) and Lemma 27 (3) and (4), $\vdash_{k-1} \sim\psi_1, \sim\psi_1, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \sim\psi_2, \sim\psi_2, \Gamma \Rightarrow \Delta$. By induction hypothesis, $\vdash_{k-1} \sim\psi_1, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \sim\psi_2, \Gamma \Rightarrow \Delta$. By $(\sim\wedge\Rightarrow)$, $\vdash_k \sim(\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta$. By (iii) and Lemma 27 (8), $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim\psi_1, \sim\psi_2, \sim\psi_1, \sim\psi_2$. By induction hypothesis, $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim\psi_1, \sim\psi_2$. By $(\Rightarrow\sim\wedge)$, $\vdash_k \Gamma \Rightarrow \Delta, \sim(\psi_1 \wedge \psi_2)$. Case $\psi = \psi_1 \vee \psi_2$ is shown similarly. We suppose $\psi = \sim\chi$. Then, $\vdash_{k-1} \chi, \sim\sim\chi, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta, \sim\sim\chi, \chi$. By Lemma 27 (13) and (14), $\vdash_{k-1} \chi, \chi, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta, \chi, \chi$. By induction hypothesis, $\vdash_{k-1} \chi, \Gamma \Rightarrow \Delta$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta, \chi$. By $(\sim\sim\Rightarrow)$ and $(\Rightarrow\sim\sim)$, $\vdash_k \sim\sim\chi, \Gamma \Rightarrow \Delta$ and $\vdash_k \Gamma \Rightarrow \Delta, \sim\sim\chi$. □

Lemma 29. *The following hold in* GB*:*

(1) *if* $\vdash \Gamma \Rightarrow \Delta, \bot$, *then* $\vdash \Gamma \Rightarrow \Delta, \Sigma$.
(2) *if* $\vdash \sim\bot, \Gamma \Rightarrow \Delta$, *then* $\vdash \Sigma, \Gamma \Rightarrow \Delta$.

Proof. For (1), we assume $\vdash_k \Gamma \Rightarrow \Delta, \bot$. We prove $\vdash \Gamma \Rightarrow \Delta, \Sigma$ by induction on $k \geq 0$. Case $k = 0$ is trivial. We let $k > 0$ and $\Gamma \Rightarrow \Delta, \bot$ be derived by rule (R). We obtain $\vdash \Gamma \Rightarrow \Delta, \Sigma$ by induction hypothesis and rule (R). For example, we let (R) be $(\sim\neg\Rightarrow)$ with premise $\vdash_{k-1} \Gamma' \Rightarrow \Delta, \bot, \sim\varphi$ and conclusion $\vdash_k \sim\neg\varphi, \Gamma' \Rightarrow \Delta, \bot$. By induction hypothesis, $\vdash_{k-1} \Gamma' \Rightarrow \Delta, \Sigma, \sim\varphi$. By $(\sim\neg\Rightarrow)$, $\vdash \sim\neg\varphi, \Gamma' \Rightarrow \Delta, \Sigma$. Other cases are shown similarly. We note that the proof of (2) is similar. □

Theorem 12. *The following rule (Cut) is admissible in* GB*:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}(\mathrm{Cut})$$

Proof. We assume $\vdash_m \Gamma \Rightarrow \Delta, \alpha$ and $\vdash_n \alpha, \Sigma \Rightarrow \Theta$. We prove $\vdash \Gamma, \Sigma \Rightarrow \Delta, \Theta$ by induction on the cut height $m + n$ and subinduction on $c(\alpha)$.

We assume $m = 0$ or $n = 0$. We suppose $m = 0$. Then, $\Gamma \Rightarrow \Delta, \alpha$ is an initial sequent. If $\Gamma \Rightarrow \Delta$ is an initial sequent, so is the conclusion of (Cut). We suppose $\alpha \in \Gamma$. Then, we obtain $\vdash \Gamma, \Sigma \Rightarrow \Delta, \Theta$ from the right premiss of (Cut) by (Wk$\Rightarrow$). We suppose $\alpha = {\sim}\bot$. By the right premiss of (Cut) and Lemma 29 (2), we obtain $\vdash \Gamma, \Sigma \Rightarrow \Delta, \Theta$. We suppose $n = 0$. Then, $\alpha, \Sigma \Rightarrow \Theta$ is an initial sequent. If $\Sigma \Rightarrow \Theta$ is an initial sequent, so is the conclusion of (Cut). If $\alpha \in \Theta$, then we obtain $\vdash \Gamma, \Sigma \Rightarrow \Delta, \Theta$ from the left premiss of (Cut) by ($\Rightarrow$Wk). We suppose $\alpha = \bot$. By the left premiss of (Cut) and Lemma 29 (1), we obtain $\vdash \Gamma, \Sigma \Rightarrow \Delta, \Theta$.

We assume $m > 0$ and $n > 0$. We let the left and right premisses of (Cut) be obtained by rules (R_1) and (R_2), respectively. We have the following cases:

1. α is not principal in (R_1). We apply (Cut) to the premiss(es) of (R_1) and $\vdash_n \alpha, \Sigma \Rightarrow \Theta$, and then apply (R_1). For example, we let (R_1) be $({\sim}{\wedge}{\Rightarrow})$ and the derivation ends with

$$\frac{{\sim}\varphi, \Gamma' \Rightarrow \Delta, \alpha \quad {\sim}\psi, \Gamma' \Rightarrow \Delta, \alpha}{{\sim}(\varphi \wedge \psi), \Gamma' \Rightarrow \Delta, \alpha}({\sim}{\wedge}{\Rightarrow})$$

By induction hypothesis, we have the following derivation:

$$\frac{\dfrac{{\sim}\varphi, \Gamma' \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Theta}{{\sim}\varphi, \Gamma', \Sigma \Rightarrow \Delta, \Theta}(\mathrm{Cut}) \quad \dfrac{{\sim}\psi, \Gamma' \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Theta}{{\sim}\psi, \Gamma', \Sigma \Rightarrow \Delta, \Theta}(\mathrm{Cut})}{{\sim}(\varphi \wedge \psi), \Gamma', \Sigma \Rightarrow \Delta, \Theta}({\sim}{\wedge}{\Rightarrow})$$

Other cases are shown similarly.

2. α is principal only in (R_1). We apply (Cut) to $\vdash_m \Gamma \Rightarrow \Delta, \alpha$ and the premis(es) of (R_2), and then apply (R_2). For example, we let (R_2) be $({\sim}{\wedge}{\Rightarrow})$ and the derivation ends with

$$\frac{{\sim}\varphi, \alpha, \Sigma' \Rightarrow \Theta \quad {\sim}\psi, \alpha, \Sigma' \Rightarrow \Theta}{{\sim}(\varphi \wedge \psi), \alpha, \Sigma' \Rightarrow \Theta}({\sim}{\wedge}{\Rightarrow})$$

By induction hypothesis, we have the following derivation:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad {\sim}\varphi, \alpha, \Sigma' \Rightarrow \Theta}{{\sim}\varphi, \Gamma, \Sigma' \Rightarrow \Delta, \Theta}(\mathrm{Cut}) \quad \dfrac{\Gamma \Rightarrow \Delta, \alpha \quad {\sim}\psi, \alpha, \Sigma' \Rightarrow \Theta}{{\sim}\psi, \Gamma, \Sigma' \Rightarrow \Delta, \Theta}(\mathrm{Cut})}{{\sim}(\varphi \wedge \psi), \Gamma, \Sigma' \Rightarrow \Delta, \Theta}({\sim}{\wedge}{\Rightarrow})$$

Other cases are shown similarly.

3. α is principal in both (R_1) and (R_2). The proof proceeds by induction on $c(\alpha)$. We have the following cases:

   (a) $\alpha = \alpha_1 \wedge \alpha_2$ and the derivations end with

$$\frac{\Gamma \Rightarrow \Delta, \alpha_1 \quad \Gamma \Rightarrow \Delta, \alpha_2}{\Gamma \Rightarrow \Delta, \alpha_1 \wedge \alpha_2}({\Rightarrow}{\wedge}) \qquad \frac{\alpha_1, \alpha_2, \Sigma \Rightarrow \Theta}{\alpha_1 \wedge \alpha_2, \Sigma \Rightarrow \Theta}({\wedge}{\Rightarrow})$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha_2 \quad \dfrac{\Gamma \Rightarrow \Delta, \alpha_1 \quad \alpha_1, \alpha_2, \Sigma \Rightarrow \Theta}{\alpha_2, \Gamma, \Sigma \Rightarrow \Delta, \Theta}(\mathrm{Cut})}{\Gamma, \Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}(\mathrm{Cut})}{\Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}({\Rightarrow}\mathrm{Ctr})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}(\mathrm{Ctr}{\Rightarrow})$$

(b) $\alpha = \alpha_1 \vee \alpha_2$ and the derivations end with

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha_1, \alpha_2}{\Gamma \Rightarrow \Delta, \alpha_1 \vee \alpha_2}(\Rightarrow\vee) \quad \dfrac{\alpha_1, \Sigma \Rightarrow \Theta \quad \alpha_2, \Sigma \Rightarrow \Theta}{\alpha_1 \vee \alpha_2, \Sigma \Rightarrow \Theta}(\vee\Rightarrow)$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha_1, \alpha_2 \quad \alpha_1, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \alpha_2}\text{(Cut)} \qquad \alpha_2, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Theta, \Theta}\text{(Cut)}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \Theta}(\Rightarrow\text{Ctr})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}(\text{Ctr}\Rightarrow)$$

(c) $\alpha = \neg\beta$ and the derivations end with

$$\dfrac{\beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\beta}(\Rightarrow\neg) \quad \dfrac{\Sigma \Rightarrow \Theta, \beta}{\neg\beta, \Sigma \Rightarrow \Theta}(\neg\Rightarrow)$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\Sigma \Rightarrow \Theta, \beta \quad \beta, \Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\text{(Cut)}$$

(d) $\alpha = {\sim}\neg\beta$ and the derivations end with

$$\dfrac{{\sim}\beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\neg\beta}(\Rightarrow{\sim}\neg) \quad \dfrac{\Sigma \Rightarrow \Theta, {\sim}\beta}{{\sim}\neg\beta, \Sigma \Rightarrow \Theta}({\sim}\neg\Rightarrow)$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\Sigma \Rightarrow \Theta, {\sim}\beta \quad {\sim}\beta, \Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\text{(Cut)}$$

(e) $\alpha = {\sim}(\alpha_1 \wedge \alpha_2)$ and the derivations end with

$$\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha_1, {\sim}\alpha_2}{\Gamma \Rightarrow \Delta, {\sim}(\alpha_1 \wedge \alpha_2)}(\Rightarrow{\sim}\wedge) \quad \dfrac{{\sim}\alpha_1, \Sigma \Rightarrow \Theta \quad {\sim}\alpha_2, \Sigma \Rightarrow \Theta}{{\sim}(\alpha_1 \wedge \alpha_2), \Sigma \Rightarrow \Theta}({\sim}\wedge\Rightarrow)$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha_1, {\sim}\alpha_2 \quad {\sim}\alpha_1, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, {\sim}\alpha_2}\text{(Cut)} \qquad {\sim}\alpha_2, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Theta, \Theta}\text{(Cut)}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \Theta}(\Rightarrow\text{Ctr})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}(\text{Ctr}\Rightarrow)$$

(f) $\alpha = {\sim}(\alpha_1 \vee \alpha_2)$ and the derivations end with

$$\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha_1 \quad \Gamma \Rightarrow \Delta, {\sim}\alpha_2}{\Gamma \Rightarrow \Delta, {\sim}(\alpha_1 \vee \alpha_2)}(\Rightarrow{\sim}\vee) \quad \dfrac{{\sim}\alpha_1, {\sim}\alpha_2, \Sigma \Rightarrow \Theta}{{\sim}(\alpha_1 \vee \alpha_2), \Sigma \Rightarrow \Theta}({\sim}\vee\Rightarrow)$$

By induction hypothesis, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha_2 \qquad \dfrac{\Gamma \Rightarrow \Delta, {\sim}\alpha_1 \quad {\sim}\alpha_1, {\sim}\alpha_2, \Sigma \Rightarrow \Theta}{{\sim}\alpha_2, \Gamma, \Sigma \Rightarrow \Delta, \Theta}\text{(Cut)}}{\Gamma, \Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}\text{(Cut)}}{\Gamma, \Sigma \Rightarrow \Delta, \Delta, \Theta}(\Rightarrow\text{Ctr})}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}(\text{Ctr}\Rightarrow)$$

(g) $\alpha = {\sim}{\sim}\beta$ and the derivations end with

$$\frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\beta}({\Rightarrow}{\sim}{\sim}) \quad \frac{\beta, \Sigma \Rightarrow \Theta}{{\sim}{\sim}\beta, \Sigma \Rightarrow \Theta}({\sim}{\sim}{\Rightarrow})$$

By induction hypothesis, we have the following derivation:

$$\frac{\Sigma \Rightarrow \Theta, \beta \quad \beta, \Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,(\mathrm{Cut})$$

This completes the proof. □

For every finite multiset $\Gamma = \varphi_1, \ldots, \varphi_n$, we let ${\sim}\Gamma = {\sim}\varphi_1, \ldots, {\sim}\varphi_n$. Then, we have the following result on the admissibility of the contraposition rule.

Lemma 30. *The following rule of contraposition is admissible in* GB*:*

$$\frac{\Gamma \Rightarrow \Delta}{{\sim}\Delta \Rightarrow {\sim}\Gamma}(\mathrm{CPs})$$

Proof. We assume $\vdash_k \Gamma \Rightarrow \Delta$. We prove $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma$ by induction on $k \geq 0$. We assume $k = 0$. If $\Gamma \Rightarrow \Delta$ is an instance of (Id_1), then ${\sim}\Delta \Rightarrow {\sim}\Gamma$ is an instance of (Id_2). We suppose $\Gamma = {\sim}p, \Gamma'$ and $\Delta = \Delta', {\sim}p$. Clearly, $\vdash p, {\sim}\Delta' \Rightarrow {\sim}\Gamma', p$. By $({\sim}{\sim}{\Rightarrow})$ and $({\Rightarrow}{\sim}{\sim})$, we obtain $\vdash {\sim}{\sim}p, {\sim}\Delta' \Rightarrow {\sim}\Gamma', {\sim}{\sim}p$. If $\bot \in \Gamma$, then ${\sim}\Delta \Rightarrow {\sim}\Gamma$ is an instance of $({\sim}\bot)$. We suppose ${\sim}\bot \in \Delta$. Then, ${\sim}{\sim}\bot \in {\sim}\Delta$. By $(\bot)$ and $({\sim}{\sim}{\Rightarrow})$, we obtain $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma$. Now, we assume $k > 0$ and $\Gamma \Rightarrow \Delta$ is derived by rule (R).

1. (R) is $(\neg{\Rightarrow})$. We let the premise and conclusion of (R) be $\vdash_{k-1} \Gamma' \Rightarrow \Delta, \varphi$ and $\vdash_k \neg\varphi, \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\varphi, {\sim}\Delta \Rightarrow {\sim}\Gamma'$. By $({\Rightarrow}{\sim}\neg)$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}\neg\varphi$.
2. (R) is $({\Rightarrow}\neg)$. We let the premiss and conclusion of (R) be $\vdash_{k-1} \varphi, \Gamma \Rightarrow \Delta'$ and $\vdash_k \Gamma \Rightarrow \Delta', \neg\varphi$, respectively. By induction hypothesis, $\vdash {\sim}\Delta' \Rightarrow {\sim}\Gamma, {\sim}\varphi$. By $({\sim}\neg{\Rightarrow})$, $\vdash {\sim}\neg\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$.
3. (R) is $({\sim}\neg{\Rightarrow})$. We let the premise and conclusion of (R) be $\vdash_{k-1} \Gamma' \Rightarrow \Delta, {\sim}\varphi$ and $\vdash_k {\sim}\neg\varphi, \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta, {\sim}{\sim}\varphi \Rightarrow {\sim}\Gamma'$. Clearly, $\vdash \varphi \Rightarrow {\sim}{\sim}\varphi$. By (Cut), $\vdash {\sim}\Delta, \varphi \Rightarrow {\sim}\Gamma'$. By $({\Rightarrow}\neg)$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \neg\varphi$. By $({\Rightarrow}{\sim}{\sim})$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}\neg\varphi$.
4. (R) is $({\Rightarrow}{\sim}\neg)$. We let the premise and conclusion of (R) be $\vdash_{k-1} {\sim}\varphi, \Gamma \Rightarrow \Delta'$ and $\vdash_k \Gamma \Rightarrow \Delta', {\sim}\neg\varphi$, respectively. By induction hypothesis, $\vdash {\sim}\Delta' \Rightarrow {\sim}\Gamma, {\sim}{\sim}\varphi$. Clearly, $\vdash {\sim}{\sim}\varphi \Rightarrow \varphi$. By (Cut), $\vdash {\sim}\Delta' \Rightarrow {\sim}\Gamma, \varphi$. By $(\neg{\Rightarrow})$, $\vdash \neg\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}{\sim}{\Rightarrow})$, $\vdash {\sim}{\sim}\neg\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$.
5. (R) is $(\wedge{\Rightarrow})$. We let the premise and conclusion of (R) be $\vdash_{k-1} \varphi, \psi, \Gamma' \Rightarrow \Delta$ and $\vdash_k \varphi \wedge \psi, \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}\varphi, {\sim}\psi$. By $({\Rightarrow}{\sim}\wedge)$, we obtain $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}(\varphi \wedge \psi)$.
6. (R) is $({\Rightarrow}\wedge)$. We let the premises of (R) be $\vdash_{k-1} \Gamma \Rightarrow \Delta', \varphi$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta', \psi$, and the conclusion of (R) be $\vdash_k \Gamma \Rightarrow \Delta', \varphi \wedge \psi$. By induction hypothesis, $\vdash {\sim}\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$ and $\vdash {\sim}\psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}\wedge{\Rightarrow})$, $\vdash {\sim}(\varphi \wedge \psi), {\sim}\Delta' \Rightarrow {\sim}\Gamma$.
7. (R) is $({\sim}\wedge{\Rightarrow})$. We let the premises of (R) be $\vdash_{k-1} {\sim}\varphi, \Gamma' \Rightarrow \Delta$ and $\vdash_{k-1} {\sim}\psi, \Gamma' \Rightarrow \Delta$, and the conclusion of (R) be $\vdash_k {\sim}(\varphi \wedge \psi), \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}\varphi$ and $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}\psi$. By $\vdash {\sim}{\sim}\varphi \Rightarrow \varphi$ and $\vdash {\sim}{\sim}\psi \Rightarrow \psi$ and (Cut), $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \varphi$ and $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \psi$. By $({\Rightarrow}\wedge)$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \varphi \wedge \psi$. By $({\Rightarrow}{\sim}{\sim})$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}(\varphi \wedge \psi)$.
8. (R) is $({\Rightarrow}{\sim}\wedge)$. We let the premise and conclusion of (R) be $\vdash_{k-1} \Gamma \Rightarrow \Delta', {\sim}\varphi, {\sim}\psi$ and $\vdash_k \Gamma \Rightarrow \Delta', {\sim}(\varphi \wedge \psi)$, respectively. By induction hypothesis, $\vdash {\sim}{\sim}\varphi, {\sim}{\sim}\psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $\vdash \varphi \Rightarrow {\sim}{\sim}\varphi$ and $\vdash \psi \Rightarrow {\sim}{\sim}\psi$ and (Cut), $\vdash \varphi, \psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $(\wedge{\Rightarrow})$, $\vdash \varphi \wedge \psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}{\sim}{\Rightarrow})$, $\vdash {\sim}{\sim}(\varphi \wedge \psi), {\sim}\Delta' \Rightarrow {\sim}\Gamma$.

9. (R) is $(\vee\Rightarrow)$. We let the premises of (R) be $\vdash_{k-1} \varphi, \Gamma' \Rightarrow \Delta$ and $\vdash_{k-1} \psi, \Gamma' \Rightarrow \Delta$, and the conclusion of (R) be $\vdash_k \varphi \vee \psi, \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}\varphi$ and $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}\psi$. By $(\Rightarrow{\sim}\vee)$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}(\varphi \vee \psi)$.
10. (R) is $(\Rightarrow\vee)$. We let the premises and conclusion of (R) be $\vdash_{k-1} \Gamma \Rightarrow \Delta', \varphi, \psi$ and $\vdash_k \Gamma \Rightarrow \Delta', \varphi \vee \psi$. By induction hypothesis, $\vdash {\sim}\varphi, {\sim}\psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}\vee\Rightarrow)$, $\vdash {\sim}(\varphi \vee \psi), {\sim}\Delta' \Rightarrow {\sim}\Gamma$.
11. (R) is $({\sim}\vee\Rightarrow)$. We let the premise and conclusion of (R) be $\vdash_{k-1} {\sim}\varphi, {\sim}\psi, \Gamma' \Rightarrow \Delta$ and $\vdash_k {\sim}(\varphi \vee \psi), \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}\varphi, {\sim}{\sim}\psi$. By $\vdash {\sim}{\sim}\varphi \Rightarrow \varphi$ and $\vdash {\sim}{\sim}\psi \Rightarrow \psi$ and (Cut), $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \varphi, \psi$. By $(\Rightarrow\vee)$, we obtain $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', \varphi \vee \psi$. By $(\Rightarrow{\sim}{\sim})$, we obtain $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}(\varphi \vee \psi)$.
12. (R) is $(\Rightarrow{\sim}\vee)$. We let the premises of (R) be $\vdash_{k-1} \Gamma \Rightarrow \Delta', {\sim}\varphi$ and $\vdash_{k-1} \Gamma \Rightarrow \Delta', {\sim}\psi$, and the conclusion of (R) be $\vdash_k \Gamma \Rightarrow \Delta', {\sim}(\varphi \vee \psi)$. By induction hypothesis, $\vdash {\sim}{\sim}\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$ and $\vdash {\sim}{\sim}\psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $\vdash \varphi \Rightarrow {\sim}{\sim}\varphi$ and $\vdash \psi \Rightarrow {\sim}{\sim}\psi$ and (Cut), $\vdash \varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$ and $\vdash \psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $(\vee\Rightarrow)$, $\vdash \varphi \vee \psi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}{\sim}\Rightarrow)$, $\vdash {\sim}{\sim}(\varphi \vee \psi), {\sim}\Delta' \Rightarrow {\sim}\Gamma$.
13. (R) is $({\sim}{\sim}\Rightarrow)$. We let the premise and conclusion of (R) be $\vdash_{k-1} \varphi, \Gamma' \Rightarrow \Delta$ and $\vdash_k {\sim}{\sim}\varphi, \Gamma' \Rightarrow \Delta$, respectively. By induction hypothesis, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}\varphi$. By $(\Rightarrow{\sim}{\sim})$, $\vdash {\sim}\Delta \Rightarrow {\sim}\Gamma', {\sim}{\sim}{\sim}\varphi$.
14. (R) is $(\Rightarrow{\sim}{\sim})$. We let the premise and conclusion of (R) be $\vdash_{k-1} \Gamma \Rightarrow \Delta', \varphi$ and $\vdash_k \Gamma \Rightarrow \Delta', {\sim}{\sim}\varphi$, respectively. By induction hypothesis, $\vdash {\sim}\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$. By $({\sim}{\sim}\Rightarrow)$, $\vdash {\sim}{\sim}{\sim}\varphi, {\sim}\Delta' \Rightarrow {\sim}\Gamma$.

This completes the proof. □

A DM frame $\mathfrak{F} = (W, \leq, g)$ is called a DB frame if $w{\uparrow} = \{w\}$ for every $w \in W$. Sequent $\Gamma \Rightarrow \Delta$ is valid in a DB frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \Gamma \Rightarrow \Delta$) if $\mathfrak{F} \models \bigwedge \Gamma \to \bigvee \Delta$. Here, $\bigwedge \Gamma$ is the conjunction of formulas in Γ, and $\bigvee \Delta$ is the disjunction of formulas in Δ. In particular, we let $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$. We write $\mathsf{Fr}(\mathsf{B}) \models \Gamma \Rightarrow \Delta$ if $\mathfrak{F} \models \Gamma \Rightarrow \Delta$ for all DB-frames $\mathfrak{F}$.

Lemma 31. $\mathsf{GB} \vdash\ \Rightarrow \varphi \to \psi$ *if and only if* $\mathsf{GB} \vdash \varphi \Rightarrow \psi$.

Proof. We assume $\vdash\ \Rightarrow \varphi \to \psi$. We have the following derivation:

$$\dfrac{\dfrac{\Rightarrow \neg\varphi \vee \psi \qquad \dfrac{\neg\varphi \Rightarrow \neg\varphi, \psi \quad \psi \Rightarrow \neg\varphi, \psi}{\neg\varphi \vee \psi \Rightarrow \neg\varphi, \psi}(\vee\Rightarrow)}{\Rightarrow \neg\varphi, \psi}(\mathrm{Cut}) \qquad \dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}(\neg\Rightarrow)}{\varphi \Rightarrow \psi}(\mathrm{Cut})$$

We assume $\vdash \varphi \Rightarrow \psi$. By $(\Rightarrow\neg)$, $\vdash\ \Rightarrow \neg\varphi, \psi$. By $(\Rightarrow\vee)$, $\vdash\ \Rightarrow \neg\varphi \vee \psi$. □

Lemma 32. *If* $\vdash_{\mathsf{B}} \varphi$*, then* $\mathsf{GB} \vdash\ \Rightarrow \varphi$.

Proof. We assume $\vdash_{\mathsf{B}} \varphi$. We note that, if φ is an axiom of B, then $\mathsf{GB} \vdash\ \Rightarrow \varphi$. For example, we have the following derivation of the sequent $\Rightarrow {\sim}(\psi \wedge \chi) \to {\sim}\psi \vee {\sim}\chi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{{\sim}\psi \Rightarrow {\sim}\psi, {\sim}\chi \quad {\sim}\chi \Rightarrow {\sim}\psi, {\sim}\chi}{{\sim}(\psi \wedge \chi) \Rightarrow {\sim}\psi, {\sim}\chi}({\sim}\wedge\Rightarrow)}{\Rightarrow \neg{\sim}(\psi \wedge \chi), {\sim}\psi, {\sim}\chi}(\Rightarrow\neg)}{\Rightarrow \neg{\sim}(\psi \wedge \chi), {\sim}\psi \vee {\sim}\chi}(\Rightarrow\vee)}{\Rightarrow \neg{\sim}(\psi \wedge \chi) \vee ({\sim}\psi \vee {\sim}\chi)}(\Rightarrow\vee)$$

We suppose φ is obtained from $\psi \to \varphi$ and ψ by (MP). By induction hypothesis, we have $\mathsf{GB} \vdash\ \Rightarrow \psi \to \varphi$ and $\mathsf{GB} \vdash\ \Rightarrow \psi$. By Lemma 31, $\mathsf{GB} \vdash \psi \Rightarrow \varphi$. By (Cut), $\mathsf{GB} \vdash\ \Rightarrow \varphi$. We suppose $\varphi = {\sim}\psi \to {\sim}\chi$ is obtained from $\chi \to \psi$ by (CP). By induction hypothesis,

$\mathsf{GB} \vdash\ \Rightarrow \chi \to \psi$. By Lemma 31, $\mathsf{GB} \vdash \chi \Rightarrow \psi$. By (CPs), $\mathsf{GB} \vdash {\sim}\psi \Rightarrow {\sim}\chi$. By Lemma 31, $\mathsf{GB} \vdash\ \Rightarrow {\sim}\psi \to {\sim}\chi$. It follows that $\mathsf{GB} \vdash\ \Rightarrow \varphi$. □

Lemma 33. $\mathsf{GB} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{GB} \vdash\ \Rightarrow \bigwedge \Gamma \to \bigvee \Delta$.

Theorem 13. *For every sequent* $\Gamma \Rightarrow \Delta$, $\mathsf{GB} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{Fr}(\mathsf{B}) \models \Gamma \Rightarrow \Delta$.

Proof. We assume $\mathsf{GB} \vdash_k \Gamma \Rightarrow \Delta$. We prove $\mathsf{Fr}(\mathsf{B}) \models \Gamma \Rightarrow \Delta$ by induction on $k \geq 0$. Case $k = 0$ is trivial. All rules in GB preserve validity in $\mathsf{Fr}(\mathsf{B})$. We note that, by Corollary 5, $\mathsf{Fr}(\mathsf{B}) \models {\sim}\neg\varphi \leftrightarrow \neg{\sim}\varphi$, and so the rules $({\sim}\neg\Rightarrow)$ and $(\Rightarrow{\sim}\neg)$ preserve validity in $\mathsf{Fr}(\mathsf{B})$. We assume $\mathsf{Fr}(\mathsf{B}) \models \Gamma \Rightarrow \Delta$. Then, $\mathsf{Fr}(\mathsf{B}) \models \bigwedge \Gamma \to \bigvee \Delta$. By the Kripke completeness of B, we have $\vdash_{\mathsf{B}} \bigwedge \Gamma \to \bigvee \Delta$. By Lemma 32, $\mathsf{GB} \vdash\ \Rightarrow \bigwedge \Gamma \to \bigvee \Delta$. By Lemma 33, $\mathsf{GB} \vdash \Gamma \Rightarrow \Delta$. □

Finally, we return to sequent calculus GB. It is a structural rule-free and cut-free system for DH-logic B. We let s be sequent $\Gamma \Rightarrow \Delta$. The weight of s is defined as the sum $wt(s) = \sum\{c(\varphi) : \varphi \in \Gamma \cup \Delta\}$. Clearly, in each logical rule in GB, the weight of each premise is strictly less than that of the conclusion. The proof search space of a given sequent in GB must be finite. This means that GB is terminating. It yields a proof search procedure and thus the decidability of derivability in GB.

Fact 3. *The derivability of a sequent in* GB *is decidable.*

7. Concluding Remarks

De Morgan Heyting logics as combinations of intuitionistic and De Morgan logics are developed in this paper. We show some properties of variety $\mathcal{DH}$. DM frames are introduced and dualities between DM frames and DH-algebras are developed. The Kripke completeness, finite approximability and the conservativity in lattice $\mathsf{Ext}(\mathsf{J})$ are investigated. A cut-free terminating Gentzen sequent calculus is given for the logic of De Morgan Boolean algebras. A glimpse on these results offers hints at some interesting problems for further study.

- First, the finite approximability and finite model property under the Kripke semantics should be analyzed deeply. It is still unknown whether DH-logic $\mathsf{C} = \mathsf{J} \oplus \neg p \to {\sim}p$ is finite approximable or not. Moreover, the finite approximability in lattice $\mathsf{Ext}(\mathsf{J})$ needs further explanation. For example, it is not known whether all extensions of the Gödelian DH-logic $\mathsf{J} \oplus (p \to q) \vee (q \to p)$ are finitely approximable or not.
- Second, the modal companions of DH-logics have not yet been studied. The intuitionistic part of J should have, e.g., the standard modal logic S4 as its counterpart (cf., e.g., [22]), and the De Morgan negation should be translated into a (negative) modal logic. Thus, the basic modal companion of DH-logics should be the product of S4 and a modal logic which represents the De Morgan part.
- Third, if we enrich the cut-free Gentzen sequent calculus for classical propositional logic with rules corresponding to axiom $\neg p \leftrightarrow {\sim}p$, we obtain a cut-free and terminating sequent calculus for DH-logic $\mathsf{C} = \mathsf{J} \oplus p \vee {\sim}p$. However, cut-free Gentzen sequent calculi for J and other DH-logics are not known yet.

Furthermore, the properties like disjunction property, tabularity, pretabularity, interpolation in sublattices of $\mathsf{Ext}(\mathsf{J})$ are interesting problems.

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