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Generalized Halanay Inequalities and Asymptotic Behavior of Nonautonomous Neural Networks with Infinite Delays

Dehao Ruan and Yao Lu *

School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665, China; mathhope@sina.com

* Correspondence: luyaomath@gpnu.edu.cn

Abstract: This paper focuses on the asymptotic behavior of nonautonomous neural networks with delays. We establish criteria for analyzing the asymptotic behavior of nonautonomous recurrent neural networks with delays by means of constructing some new generalized Halanay inequalities. We do not require to constructi any complicated Lyapunov function and our results improve some existing works. Lastly, we provide some illustrative examples to demonstrate the effectiveness of the obtained results.

Keywords: generalized Halanay inequalities; dissipativity; asymptotic behavior; recurrent neural networks

MSC: 34A40; 35B40; 92B20

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1. Introduction

Recently, neural networks (NNs) have garnered significant attention and have found extensive applications across various domains, including image restoration [1], pattern recognition [2] and associative memory [3]. In practical applications, time delays are an unavoidable factor stemming from the finite switching speed of amplifiers. It's well-established that time delays can potentially induce oscillations and instability in systems. Consequently, the asymptotic behavior of NNs with delays has been a focal point of research for numerous authors.

The study of asymptotic behavior such as dissipativity [4–7] attracting sets [8], stabilization [9–11] and stability offers potent tools for addressing the problem of controlling dynamics systems. In the asymptotic behavior analysis, one powerful tool is Lyapunov function or functional. Wang and Zhu [12] used a novel Lyapunov–Krasovskill functional to consider the stability of discrete-time semi-Markov jump linear systems with time delay. Fan et al. [13] using multiple Lyapunov-Krasovskii functionals to investigate the stability of switched stochastic nonlinear systems. Xu et al. [14] used the improved Lyapunov Razumikhin method to consider exponential stability of stochastic nonlinear delay systems. Zhu and Zhu [15] constructed the Lyapunov-Krasovskii functional to the stability of stochastic Highly Nonlinear Systems.

Especially, Cao and Zhou [16], Cao [17], Mohanmad and Gopalsamy [18], Sun et al. [19], Zeng et al. [7], Zeng et al. [20], Zhang et al. [21], Zhang et al. [22], Zhao and Cao [23], Zheng and Zhang [24], and Zhou and Zhang [25] used the Lypunov functional to investigate the stability of delayed cellular NNs with constant coefficient, respectively. Jiang and Cao [26], Jiang and Teng [27,28], Long et al. [29], Rehim et al. [30], Song and Zhao [6], Yu et al. [31], Zhang et al. [32], Zhang et al. [33] investigated the stability of recurrent NNs with variable coefficient by constructing Lyapunov function or functional, respectively. Through the construction of Lyapunov functions or functionals, one can find some interesting results. Nevertheless, constructing an appropriate Lyapunov function or functional can be a challenging task, particularly in the context of nonautonomous NNs with unbounded delays [34].

On the other hand, Halanay inequalities can be also used to consider the asymptotic behavior of NNs [5,29,34–38]. It should be noted that only [5,35,38] considered the unbounded coefficient functions, and unbounded delay functions. Hien et al. [35] considered the generalized exponential stability of one-dimensional Halalay inequalities and gave application to nonautonomous NNs. Later, Lu et al. [38] studied the global generalized exponential stability of nonautonomous NNs by multi-dimensional generalized Halanay inequalities which extended the results in [35]. However, when the coefficient functions are constants and the delay functions are infinite the works in [35,38] do not work. Hien et al. [5] considered the global dissipativity of nonautonomous NNs with delays. Howerer, their delay functions are required to be proportional.

Inspired by the preceding discussion, in this paper, we propose some generalized Halanay inequalities to investigate the asymptotic behaviour of neural networks with unbounded variable coefficients and infinite delay, and our assumptions are less restrictive than most of existing works. Our results not only enhance but also extend the results initially presented in [5,35,38].

The structure of this paper unfolds as follows. Section 2 provides an introduction to some preliminaries, definitions and model descriptions. Section 3 investigates the asymptotic behavior of NNs with delays by means of constructing some generalized Halanay inequalities. Section 4 offers some examples and simulations to exemplify the practical utility of our theoretical results. Finally, this paper concludes in the Section 5.

Notations: let $N_n = \{1, 2, \ldots, n\}$ and A^T denotes the transpose of matrix A. $\mathbb{R}^n$ is the n-dimensional Euclidean space equipped with the norm $\|q\| = \max\limits_{i \in N_n}\{|q_i|\}$ for $q = (q_1, q_2, \ldots, q_n)^T \in \mathbb{R}^n$. For $t_0 \geq 0$, $BC((-\infty, t_0], \mathbb{R}^n)$ stands for the space of all bounded and continuous functions $\psi : (-\infty, t_0] \to \mathbb{R}^n$ equipped with the norm $\|\psi\|_\infty := \sup\limits_{\theta \leq t_0} \|\psi(\theta)\|$. For any sets D and E, define $D - E := \{x | x \in D,\ x \notin E\}$. $b_+ = \max\{0, b\}$.

2. Preliminaries and Model Description

This paper investigates the following NNs with delays

$$\begin{cases} \frac{dq_i(t)}{dt} = -\alpha_i(t)q_i(t) + \sum\limits_{j=1}^{n} [\beta_{ij}(t) f_j(q_j(t)) + \gamma_{ij}(t) g_j(q_j(t - \tau_{ij}(t)))] + h_i(t), \quad t \geq t_0, \\ q_i(t) = \psi_i(t), \quad t \in (-\infty, t_0], \quad i \in N_n, \end{cases} \tag{1}$$

where $q_i(t)$ is the neuron state variable of the neural network, $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ is the initial value, $q(t, \psi) = (q_1(t), \ldots, q_n(t))^T \in BC((-\infty, t_0]$ denotes the solution (1) with initial value ψ, sometimes we write $q(t)$ for short. $\alpha_i(t)$ stands for self-feedback coefficient, $\beta_{ij}(t)$ and $\gamma_{ij}(t)$ stand for neuron connect weight. $\tau_{ij}(t) \geq 0$ represents the transmission delay. $h_i(t)$ is the external bias, f_j and g_j stand for the activation functions. If the initial value of $q_j(t)$ defined on $[\min\limits_{t \geq t_0}\{t - \tau_{ij}(t)\}, t_0]$, define $q_j(t) = q_j(\min\limits_{t \geq t_0}\{t - \tau_{ij}(t)\})$ for $t < \min\limits_{t \geq t_0}\{t - \tau_{ij}(t)\}$, then (1) is clearly defined.

Now, we introduce four definitions of asymptotic behavior.

Definition 1 ([5])**.** *A compact set $\Omega \subset \mathbb{R}^n$ is called to be a global attracting set of (1), provided $\limsup\limits_{t \to +\infty} d(q(t, \psi), \Omega) = 0$, where $d(q, \Omega) := \inf\limits_{x \in \Omega} \|q - x\|$ represents the distance between q and Ω.*

Definition 2 ([5])**.** *A compact set $\Omega \subset \mathbb{R}^n$ is called to be a global generalized exponential attracting set of (1), provided there exists a $\rho(\psi) \geq 0$ satisfies that*

$$d(q(t, \psi), \Omega) \leq \rho(\psi) e^{-\lambda(t)}, \quad t \geq t_0, \tag{2}$$

where $\lambda(t) \geq 0$ is a nondecreasing function satisfies that $\lim_{t\to+\infty} \lambda(t) = +\infty$.

Remark 1. *Substituting $\lambda(t)$ with $\lambda(t-t_0)$, $\lambda\ln(t-t_0+1)$, and $\lambda\ln(\ln(t-t_0+e))$, $(\lambda>0)$, respectively, results in Ω becoming a global exponential, polynomial as well as logarithmic attracting set of system (1), correspondingly.*

Definition 3 ([5])**.** *System (1) is called to be globally dissipative, provided there is a bounded set $\mathcal{B} \subset \mathbb{R}^n$ satisfies that for any bounded set $\Psi \subset \mathbb{R}^n$, there exists a time $t_B = t_B(\Psi)$ satisfies that for any initial value $\psi \in \Psi$, $q(t) = q(t,\psi) \in \mathcal{B}$ for $t \geq t_B(\Psi)$. Then $\mathcal{B}$ is called an absorbing set of (1).*

Remark 2. *If Ω is a global generalized exponential attracting set of (1), this implies (1) is globally dissipative. For any bounded set $\Psi \subset \mathbb{R}^n$, there exists an absorbing set of (1) such that $\mathcal{B}_\varepsilon = \{x \in \mathbb{R}^n : d(x,\Omega) \leq \varepsilon\}$.*

Definition 4 ([38])**.** *System (1) is called to be globally generalized exponential stable, provided for any two solutions $q^{(1)}(t) = (q_1^{(1)}(t),\ldots,q_n^{(1)}(t))^T$ and $q^{(2)}(t) = (q_1^{(2)}(t),\ldots,q_n^{(2)}(t))^T$, each having distinct initial values $\psi^{(1)},\psi^{(2)} \in BC((-\infty,t_0],\mathbb{R}^n)$, there exists a non-negative function $\varrho(\psi^{(1)}-\psi^{(2)})$ and a non-decreasing function $\lambda(t) \geq 0$ with property $\lim_{t\to+\infty}\lambda(t) = +\infty$ such that*

$$|q^{(1)}(t) - q^{(2)}(t)| \leq \varrho(\psi^{(1)}-\psi^{(2)})e^{-\lambda(t)}, \quad t \geq t_0,$$

where $\lambda(t)$ represents the decay rate.

3. Main Results

In this section, the asymptotic behavior of (1) is discussed by means of generalized Halanay inequalities.

Theorem 1. *Let the following conditions hold*

(**C.1**) *For $i,j \in N_n$ and $t \geq t_0$, $\alpha_i(t) \geq 0$, $\beta_{ij}(t)$, $\gamma_{ij}(t)$, $h_i(t)$ are all integrable functions.*

(**C.2**) *For $j \in N_n$ and $q_1,q_2 \in \mathbb{R}$, there exist constants F_j, G_j such that*

$$|f_j(q_1) - f_j(q_2)| \leq F_j|q_1-q_2|, \quad |g_j(q_1) - g_j(q_2)| \leq G_j|q_1-q_2|.$$

(**C.3**) *For each $i \in N_n$, there exist positive constants $\eta_1,\eta_2,\ldots,\eta_n$, $(\max\{\eta_1,\eta_2,\ldots,\eta_n\} = 1)$ and non-negative constants μ_i such that*

$$\eta_i\alpha_i(t) - \sum_{j=1}^{n}(|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j)\eta_j \geq 0, \quad t \geq t_0,$$

and

$$\sup_{\{t|t\geq t_0\}-D}\left\{\frac{\sum_{j=1}^{n}(|\beta_{ij}(t)f_j(0)| + |\gamma_{ij}(t)g_j(0)|) + |h_i(t)|}{\eta_i\alpha_i(t) - \sum_{j=1}^{n}(|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j)\eta_j}\right\} := \mu_i,$$

where

$$D = \left\{t|\eta_i\alpha_i(t) - \sum_{j=1}^{n}(|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j)\eta_j = \sum_{j=1}^{n}(|\beta_{ij}(t)f_j(0)| + |\gamma_{ij}(t)g_j(0)|) + |h_i(t)| = 0\right\},$$

$$\sum_{j=1}^{n}(|\beta_{ij}(t)f_j(0)| + |\gamma_{ij}(t)g_j(0)|) + |h_i(t)| := \mu_i(t).$$

Then systems (1) is globally dissipative and $\max_{k\in N_n}\left\{\frac{\sup_{t\le t_0}|\psi_k(\theta)|}{\eta_k}, \mu_k\right\}$ *is an absorbing set of (1).*

Remark 3. *Conditions* (**C.1**)–(**C.3**) *imply the local Lipschitz condition and local linear growth condition. So the existence and uniqueness of solution can be guaranteed.*

Proof. Assume $q(t) = (q_1(t),\dots,q_n(t))^T$ is the solution of (1) with initial value $\psi = (\psi_1,\dots,\psi_n)^T$. Let

$$z(t) = (z_1(t),\dots,z_n(t))^T = (\eta_1^{-1}q_1(t),\dots,\eta_n^{-1}q_n(t))^T, \tag{3}$$

then

$$\begin{cases} \frac{dz_i(t)}{dt} = -\alpha_i(t)z_i(t) + \eta_i^{-1}\sum_{j=1}^{n}[\beta_{ij}(t)f_j(q_j(t)) + \gamma_{ij}(t)g_j(q_j(t-\tau_{ij}(t)))] + \eta_i^{-1}h_i(t), \quad t\ge t_0, \\ z_i(t) = \eta_i^{-1}\psi_i(t), \quad t\in(-\infty,t_0]. \end{cases} \tag{4}$$

For each $i\in N_n$ and $t\ge t_0$, from (**C.2**), (3) and (4), we have

$$\begin{aligned} D^+|z_i(t)| \le& -\alpha_i(t)|z_i(t)| + \eta_i^{-1}\sum_{j=1}^{n}|\beta_{ij}(t)|(F_j\eta_j|z_j(t)| + |f_j(0)|) \\ &+ \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{ij}(t)|(\sup_{t-\tau_{ij}(t)\le s\le t}(G_j\eta_j|z_j(s)| + |g_j(0)|) + \eta_i^{-1}|h_i(t)| \\ =& -\alpha_i(t)|z_i(t)| + \eta_i^{-1}\sum_{j=1}^{n}|\beta_{ij}(t)|F_j\eta_j|z_j(t)| + \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{ij}(t)|G_j\eta_j\sup_{t-\tau_{ij}(t)\le s\le t}|z_j(s)| \\ &+ \eta_i^{-1}\left[\sum_{j=1}^{n}(|\beta_{ij}(t)f_j(0)| + |\gamma_{ij}(t)g_j(0)|) + |h_i(t)|\right], \end{aligned} \tag{5}$$

where D^+ is the upper-right Dini derivative. Define $M := \max_{k\in N_n}\left\{\sup_{t\le t_0}\frac{|\psi_k(\theta)|}{\eta_k}, \mu_k\right\}$. It is clear that $|z_i(t)| \le M$ for $t\le t_0$ and $i\in N_n$. Suppose there exist $i_1\in N_n$, $\epsilon_1>0$ and $t_1>t_0$ such that $|z_{i_1}(t_1)| = M+\epsilon_1$, and $|z_j(t)|\le M+\epsilon_1$ for $t\le t_1$ and $j\in N_n$. Then we get $D^+|z_{i_1}(t)|\Big|_{t=t_1} > 0$. In contrast

$$\begin{aligned} D^+|z_{i_1}(t)|\Big|_{t=t_1} \le& -\alpha_{i_1}(t_1)|z_{i_1}(t_1)| + \eta_{i_1}^{-1}\sum_{j=1}^{n}|\beta_{i_1j}(t_1)|F_j\eta_j|z_j(t_1)| \\ &+ \eta_{i_1}^{-1}\sum_{j=1}^{n}|\gamma_{i_1j}(t_1)|G_j\eta_j\sup_{t_1-\tau_{i_1j}(t_1)\le s\le t_1}|z_j(s)| + \eta_i^{-1}\mu_{i_1}(t_1) \\ \le& -\alpha_{i_1}(t_1)(M+\epsilon_1) + \eta_{i_1}^{-1}\sum_{j=1}^{n}|\beta_{i_1j}(t_1)|F_j\eta_j(M+\epsilon_1) + \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{i_1j}(t_1)|G_j\eta_j(M+\epsilon_1) \\ &+ \mu_{i_1}\left[\alpha_{i_1}(t_1) - \eta_i^{-1}\sum_{j=1}^{n}|\beta_{i_1j}(t_1)|F_j\eta_j - \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{i_1j}(t_1)|G_j\eta_j\right] \\ =& -\left[\alpha_{i_1}(t_1) - \eta_{i_1}^{-1}\sum_{j=1}^{n}|\beta_{i_1j}(t_1)|F_j\eta_j - \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{i_1j}(t_1)|G_j\eta_j\right](M-\mu_{i_1}+\epsilon_1) \le 0. \end{aligned}$$

This signifies a contradiction, implying that

$$|z_i(t)| \le \max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}, \mu_k\right\}, \quad t \ge t_0, \quad i \in N_n.$$

So we get

$$|q_i(t)| \le \max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}, \mu_k\right\}\eta_i, \quad t \ge t_0, \quad i \in N_n.$$

Then

$$\|q(t)\| \le \max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}, \mu_k\right\}, \quad t \ge t_0.$$

This completes the proof. □

Remark 4. *Condition* $(\max\{\eta_1, \eta_2, \dots, \eta_n\} = 1)$ *can be omitted, but in order to see our main results clearly, so we reserve it.*

Theorem 2. *Assume* (**C.1**)–(**C.3**) *and the following conditions hold:*
(**C.4**) *For* $i, j \in N_n$, *there exist constants* $\alpha_i > 0$ *and* $\alpha(t)$ *such that*

$$0 \le \alpha_i\alpha(t) \le \alpha_i(t) \text{ for } t \ge t_0, \quad \lim_{t\to+\infty}\int_{t_0}^{t}\alpha(s)ds \to +\infty, \quad \sup_{t\ge t_0}\left\{\int_{t-\tau_{ij}(t)}^{t}\alpha^*(s)ds\right\} := \tau_{ij} < +\infty,$$

where

$$\alpha^*(t) := \begin{cases} \alpha(t), & t \ge t_0, \\ 0, & t < t_0. \end{cases}$$

(**C.5**) *For* $i, j \in N_n$,

$$\sup_{\{t|t\ge t_0\}-\{t|\alpha_i(t)=|\beta_{ij}(t)|F_j=0\}}\left\{\frac{|\beta_{ij}(t)|F_j}{\alpha_i(t)}\right\} := \rho_{ij}^{(1)}, \quad \sup_{\{t|t\ge t_0\}-\{t|\alpha_i(t)=|\gamma_{ij}(t)|G_j=0\}}\left\{\frac{|\gamma_{ij}(t)|G_j}{\alpha_i(t)}\right\} := \rho_{ij}^{(2)},$$

and

$$-\eta_i + \sum_{j=1}^{n}(\rho_{ij}^{(1)} + \rho_{ij}^{(2)})\eta_j < 0,$$

where, $\eta_1, \eta_2, \dots, \eta_n$ *were introduced in Theorem 1.*

Then we have the following assertions:

(**1**) *For* $i \in N_n$,

$$|q_i(t)| \le \left[\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\} - \max_{k\in N_n}\{\mu_k\}\right)_+ e^{-\lambda^*\int_{t_0}^{t}\alpha(s)ds} + \max_{k\in N_n}\{\mu_k\}\right]\eta_i, \quad t \ge t_0.$$

where λ^* *represents the smallest solution to the following equations*

$$\frac{\lambda}{\alpha_i} + \frac{\sum\limits_{i=1}^{n}(\rho_{ij}^{(1)} + \rho_{ij}^{(2)}e^{\lambda\tau_{ij}})\eta_j}{\eta_i} - 1 = 0, \quad i \in N_n.$$

(**2**) *The set*

$$\Omega := \left\{u \in \mathbb{R}^n : \|u\| \le \max_{k\in N_n}\{\mu_k\}\right\}$$

is a global generalized exponential attracting set of (1).

(**3**) *System (1) is globally dissipative.*

Proof. When $\max_{k\in N_n}\left\{\frac{\sup_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}\le\max_{k\in N_n}\{\mu_k\}$, the proof is deduce from Theorem (1). Now, suppose $\max_{k\in N_n}\left\{\frac{\sup_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}>\max_{k\in N_n}\{\mu_k\}$ and define

$$K_i(\lambda):=\frac{\lambda}{\alpha_i}+\eta_i^{-1}\sum_{j=1}^{n}(\rho_{ij}^{(1)}+\rho_{ij}^{(2)}e^{\lambda\tau_{ij}})\eta_j-1,\quad \lambda\in[0,+\infty).$$

Note that, for each $i\in N_n$, $K_i(\lambda)$ is continuous on $[0,+\infty)$, $K_i(0)=\eta_i^{-1}\sum_{j=1}^{n}(\rho_{ij}^{(1)}+\rho_{ij}^{(2)})\eta_j-1<0$,

$$K_i'(\lambda)=\frac{1}{\alpha_i}+\eta_i^{-1}\sum_{j=1}^{n}\tau_{ij}\eta_j\rho_{ij}^{(2)}e^{\lambda\tau_{ij}}>0,$$

and $\lim_{\lambda\to+\infty}K_i(\lambda)=+\infty$. So for $i\in N_n$, equation $K_i(\lambda)=0$ has an unique solution $\lambda_i\in(0,+\infty)$. Define $\lambda^*:=\min_{k\in N_n}\{\lambda_k\}$, then

$$\frac{\lambda^*}{\alpha_i}+\eta_i^{-1}\sum_{j=1}^{n}(\rho_{ij}^{(1)}+\rho_{ij}^{(2)}e^{\lambda^*\tau_{ij}})\eta_j-1\le 0,\quad i\in N_n. \tag{6}$$

Multiply both sides of (6) by $\alpha_i(t)$, we get

$$\frac{\lambda^*\alpha_i(t)}{\alpha_i}+\eta_i^{-1}\sum_{j=1}^{n}(\rho_{ij}^{(1)}+\rho_{ij}^{(2)}e^{\lambda^*\tau_{ij}})\eta_j\alpha_i(t)-\alpha_i(t)\le 0,\quad t\ge t_0,\quad i\in N_n. \tag{7}$$

From (**C.4**), (**C.5**) and (7), we have

$$\eta_i^{-1}\sum_{j=1}^{n}(\rho_{ij}^{(1)}+\rho_{ij}^{(2)}e^{\lambda^*\tau_{ij}})\eta_j\alpha_i(t)-\alpha_i(t)\le-\lambda^*\alpha(t),\quad t\ge t_0,\quad i\in N_n \tag{8}$$

and

$$\eta_i^{-1}\sum_{j=1}^{n}(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_je^{\lambda^*\tau_{ij}})\eta_j-\alpha_i(t)\le-\lambda^*\alpha(t),\quad t\ge t_0,\quad i\in N_n. \tag{9}$$

For $t\in\mathbb{R}$, define

$$v(t):=\left(\max_{k\in N_n}\left\{\frac{\sup_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}-\max_{k\in N_n}\{\mu_k\}\right)e^{-\lambda^*\int_{t_0}^{t}\alpha^*(s)ds}+\max_{k\in N_n}\{\mu_k\}. \tag{10}$$

Then

$$\begin{aligned}(v(s)-\max_{k\in N_n}\{\mu_k\})&=(v(t)-\max_{k\in N_n}\{\mu_k\})e^{\lambda^*\int_s^t\alpha^*(u)du}\\&\le(v(t)-\max_{k\in N_n}\{\mu_k\})e^{\lambda^*\int_{t-\tau_{ij}(t)}^{t}\alpha^*(u)du},\quad i,j\in N_n,\quad s\in[t-\tau_{ij}(t),t].\end{aligned}$$

Hence

$$\sup_{t-\tau_{ij}(t)\le s\le t}\{v(s)-\max_{k\in N_n}\{\mu_k\}\}\le(v(t)-\max_{k\in N_n}\{\mu_k\})e^{\lambda^*\int_{t-\tau_{ij}(t)}^{t}\alpha^*(u)du},\quad t\ge t_0,\quad i,j\in N_n. \tag{11}$$

By (11) and the definition of τ_{ij}, we get

$$\sup_{t-\tau_{ij}(t)\le s\le t}\{v(s)-\max_{k\in N_n}\{\mu_k\}\}\le(v(t)-\max_{k\in N_n}\{\mu_k\})e^{\lambda^*\tau_{ij}},\quad t\ge t_0,\quad i,j\in N_n. \tag{12}$$

Thus, for $t\ge t_0$ and $i\in N_n$, from (**C.3**)–(**C.5**), (8)–(10) and (12), we get

$$\begin{aligned}
\frac{dv(t)}{dt}&=-\lambda^*\alpha(t)\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}-\max_{k\in N_n}\{\mu_k\}\right)e^{-\lambda^*\int_{t_0}^t\alpha^*(s)ds}\\
&=-\lambda^*\alpha(t)(v(t)-\max_{k\in N_n}\{\mu_k\})\\
&\ge-\left[1-\eta_i^{-1}\sum_{j=1}^n\left(\rho_{ij}^{(1)}+\rho_{ij}^{(2)}e^{\lambda^*\tau_{ij}}\right)\eta_j\right]\alpha_i(t)(v(t)-\max_{k\in N_n}\{\mu_k\})\\
&\ge-\alpha_i(t)v(t)+\eta_i^{-1}\sum_{j=1}^n\rho_{ij}^{(1)}\alpha_i(t)\eta_jv(t)+\eta_i^{-1}\sum_{j=1}^n\rho_{ij}^{(2)}\alpha_i(t)(t)e^{\lambda^*\tau_{ij}}\eta_jv(t)\\
&\quad+\left[\alpha_i(t)-\eta_i^{-1}\sum_{j=1}^n\left(|\beta_{ij}(t)|F_j\eta_j+|\gamma_{ij}(t)|G_j\eta_j\right)\right]\max_{k\in N_n}\{\mu_k\}\\
&\ge-\alpha_i(t)v(t)+\eta_i^{-1}\sum_{j=1}^n|\beta_{ij}(t)|F_j\eta_jv(t)+\eta_i^{-1}\sum_{j=1}^n|\gamma_{ij}(t)|G_j\eta_j\sup_{t-\tau_{ij}(t)\le s\le t}v(s)+\eta_i^{-1}\mu_i(t).
\end{aligned}\tag{13}$$

At last, we show when $\max_{k\in N_n}\left\{\frac{\sup_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}>\max_{k\in N_n}\{\mu_k\}$, $|z_i(t)|\le v(t)$ for $t\ge t_0$ and $i\in N_n$ by reduction to absurdity. Clearly, $|z_i(t)|\le v(t)$ for $t\in(-\infty,t_0]$. Suppose there exist $i_2\in N_n$, $\varepsilon_2>0$ and $t_2>t_0$ such that $|z_{i_2}(t_2)|=v(t_2)+\epsilon_2$, and $|z_j(t)|\le v(t)+\epsilon_2$ for $t\in(-\infty,t_2]$ and $j\in N_n$, then we get $D^+\left(|z_{i_2}(t)|-\frac{dv(t)}{dt}\right)\Big|_{t=t_2}>0$. In contrast, from (5) and (13), we get

$$\begin{aligned}
D^+\left(|z_{i_2}(t)|-\frac{dv(t)}{dt}\right)\Big|_{t=t_2}&\le-\alpha_{i_2}(t_2)(|z_{i_2}(t_2)|-v(t_2))+\eta_{i_2}^{-1}\sum_{j=1}^n|\beta_{i_2j}(t_2)|F_j\eta_j(|z_j(t_2)|-v(t_2))\\
&\quad+\eta_{i_2}^{-1}\sum_{j=1}^n|\gamma_{i_2j}(t_2)|G_j\eta_j\sup_{t_2-\tau_{i_2j}(t_2)\le s\le t_2}(|z_j(s)|-v(s))\\
&=-\left[\alpha_{i_2}(t_2)-\eta_{i_2}^{-1}\sum_{j=1}^n|\beta_{i_2j}(t_2)|F_j\eta_j-\eta_i^{-1}\sum_{j=1}^n|\gamma_{i_2j}(t_2)|G_j\eta_j\right]\epsilon_2\le 0.
\end{aligned}$$

This is a contradiction, the proof is completed. So for each $i\in N_n$, we get

$$|q_i(t)|\le\left[\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}-\max_{k\in N_n}\{\mu_k\}\right)_+e^{-\lambda^*\int_{t_0}^t\alpha(s)ds}+\max_{k\in N_n}\{\mu_k\}\right]\eta_i,\quad t\ge t_0.$$

and

$$\|q(t)\|\le\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}-\max_{k\in N_n}\{\mu_k\}\right)_+e^{-\lambda^*\int_{t_0}^t\alpha(s)ds}+\max_{k\in N_n}\{\mu_k\},\quad t\ge t_0. \tag{14}$$

Now, we proof the assertion (2). Define

$$\rho\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}\right):=\begin{cases}\max\limits_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}-\max\limits_{k\in N_n}\{\mu_k\}, & \max\limits_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}\ge\max\limits_{k\in N_n}\{\mu_k\},\\ 0, & \max\limits_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}<\max\limits_{k\in N_n}\{\mu_k\}.\end{cases}$$

By (14), we get

$$d(q(t),\Omega)\le\rho\left(\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}\right)e^{-\lambda^*\int_{t_0}^t\alpha(s)ds},\quad t\ge t_0.$$

This means that $\Omega=\left\{u\in\mathbb{R}^n:\|u\|\le\max\limits_{k\in N_n}\{\mu_k\}\right\}$ is the global generalized exponential attracting set of (1). Now, we prove the assertion (3). Obvious, the ball $B(0,\max\limits_{k\in N_n}\{\mu_k\}+\varepsilon):=\left\{u\in\mathbb{R}^n:\|u\|\le\max\limits_{k\in N_n}\{\mu_k\}+\varepsilon\right\}$ is an absorbing set of (1) for any $\varepsilon>0$. This completes the proof. □

Remark 5. *Hien et al. [5] investigated the dissipativity of the specific instance of the system (1), namely, the delay functions are proportional. Under condition* $(\mathbf{C.2})$ *and the following conditions*

$(\mathbf{C.4}')$ *For* $i,j\in N_n$*, there exist constants* $\alpha_i>0$ *and* $\alpha(t)$ *such that*

$$0<\alpha_i\alpha(t)\le\alpha_i(t)\ for\ t\ge0,\quad\lim_{t\to+\infty}\int_0^t\alpha(s)ds\to+\infty,\quad\sup_{t\ge0}\left\{\int_{q_{ij}t}^t\alpha(s)ds\right\}<+\infty.$$

$(\mathbf{C.5}')$ *For* $i,j\in N_n$*, there exist constants* $\hat{\beta}_{ij}$*,* $\hat{\gamma}_{ij}$ *and* $\hat{h}_i$ *such that*

$$\frac{|\beta_{ij}(t)|}{\alpha_i(t)}\le\hat{\beta}_{ij},\quad\frac{|\gamma_{ij}(t)|}{\alpha_i(t)}\le\hat{\gamma}_{ij},\quad\frac{|h_i(t)|}{\alpha_i(t)}\le\hat{h}_i,\quad t\ge t_0.$$

and for each $i\in N_n$*, there exist positive constants* $\eta_1,\eta_2,\dots,\eta_n$*,* $(\max\{\eta_1,\eta_2,\dots,\eta_n\}=1)$ *such that*

$$-\eta_i+\sum_{j=1}^n(F_j\hat{\beta}_{ij}+G_j\hat{\gamma}_{ij})\eta_j<0.$$

They got the following results

$$|q_i(t)|\le\left[\left(\frac{\|\psi\|_\infty}{\min\{\eta_1,\dots,\eta_n\}}-\frac{\hat{\gamma}}{\hat{m}}\right)_+e^{\lambda^*\int_{t_0}^t\alpha(s)ds}+\frac{\hat{\gamma}}{\hat{m}}\right)\Bigg]\eta_i,$$

and the global generalized exponential attracting set is

$$\Omega_1=\left\{q\in\mathbb{R}^n:\|q\|\le\frac{\hat{\gamma}}{\hat{m}}\right\},$$

where $\hat{\gamma}=\max\limits_{k\in N_n}\left\{\hat{h}_k+\sum\limits_{j=1}^n(\hat{b}_{kj}|f_j(0)|+\hat{c}_{kj}|g_j(0)|)\right\}$ *and* $\hat{m}=\min\limits_{k\in N_n}\left\{\eta_k-\sum\limits_{j=1}^n(\hat{b}_{kj}|f_j(0)|+\hat{c}_{kj}|g_j(0)|)\eta_j\right\}$.

We mention here that our conditions are less restrictive, i.e., $\alpha_i(t)$ can be zero at some time and the delay functions can be other types of delay functions. Besides our results also improve the results in [5]. Especially when condtions $(\mathbf{C.4}')$ and $(\mathbf{C.5}')$ hold, obvious,

$$\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k(\theta)|}{\eta_k}\right\}\le\frac{\|\psi\|_\infty}{\min\{\eta_1,\ldots,\eta_n\}},$$

and for each $i\in N_n$, we get

$$\begin{aligned}\sup_{\{t|t\ge t_0\}}\left\{\frac{\sum\limits_{j=1}^n(|\beta_{ij}(t)f_j(0)|+|\gamma_{ij}(t)g_j(0)|)+|h_i(t)|}{\eta_i\alpha_i(t)-\sum\limits_{j=1}^n(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)\eta_j}\right\}&=\sup_{\{t|t\ge t_0\}}\left\{\frac{\sum\limits_{j=1}^n\left(\frac{|\beta_{ij}(t)f_j(0)|}{\alpha_i(t)}+\frac{|\gamma_{ij}(t)g_j(0)|}{\alpha_i(t)}\right)+\frac{|h_i(t)|}{\alpha_i(t)}}{\eta_i-\sum\limits_{j=1}^n\left(\frac{|\beta_{ij}(t)|F_j}{\alpha_i(t)}+\frac{|\gamma_{ij}(t)|G_j}{\alpha_i(t)}\right)\eta_j}\right\}\\&\le\frac{\hat{h}_i+\sum\limits_{j=1}^n(\hat{\beta}_{ij}|f_j(0)|+\hat{\gamma}_{ij}|g_j(0)|)}{\eta_i-\sum\limits_{j=1}^n(\hat{b}_{ij}|f_j(0)|+\hat{\gamma}_{ij}|g_j(0)|)\eta_j}\\&\le\frac{\hat{\Upsilon}}{\hat{m}}.\end{aligned}$$

So we have $\max\limits_{k\in N_n}\{\mu_k\}\le\frac{\hat{\Upsilon}}{\hat{m}}$, this means that our estimate is sharper than [5]. The above discussion shows that this paper improves and extends the results in [5].

Theorem 3. *Let $q^{(1)}(t)=(q_1^{(1)}(t),\ldots,q_n^{(1)}(t))^T$ and $q^{(2)}(t)=(q_1^{(2)}(t),\ldots,q_n^{(2)}(t))^T$ denote two solutions of (1) with distinct initial values $\psi^{(1)},\psi^{(2)}\in BC((-\infty,t_0],\mathbb{R}^n)$. Assume that conditions ($\mathbf{C.1}$), ($\mathbf{C.2}$), and the following conditions are satisfied:*

($\mathbf{C.6}$) *For $i\in N_n$, there exist positive constants $\eta_1,\eta_2,\ldots,\eta_n$, $(\max\{\eta_1,\eta_2,\ldots,\eta_n\}=1)$ such that*

$$\sum_{j=1}^n(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)\eta_j\le\eta_i\alpha_i(t),\quad t\in[t_0,+\infty),$$

and there exists $T\ge t_0$ such that

$$\sup_{\{t|t\ge T\}-\{t|\eta_i\alpha_i(t)=\sum\limits_{j=1}^n(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)\eta_j=0\}}\left\{\frac{\sum\limits_{j=1}^n(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)\eta_j}{\eta_i\alpha_i(t)}\right\}:=\rho<1.$$

($\mathbf{C.7}$) *For $i,j\in N_n$, there exist constants $\alpha_i>0$ and $\alpha(t)$ such that*

$$0\le\alpha_i\alpha(t)\le\alpha_i(t)\ \text{for}\ t\ge T,\quad\lim_{t\to+\infty}\int_T^t\alpha(s)ds\to+\infty,\quad\sup_{t-\tau_{ij}(t)\ge T}\left\{\int_{t-\tau_{ij}(t)}^t\alpha(s)ds\right\}:=\tau_{ij}<+\infty.$$

Then,

$$|q_i^{(1)}(t)-q_i^{(2)}(t)|\le\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}\eta_i,\quad t\in[t_0,T],$$

and

$$|q_i^{(1)}(t)-q_i^{(2)}(t)|\le\max_{k\in N_n}\left\{\frac{\sup\limits_{t\le t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}\eta_i e^{-\lambda^*\int_T^t\alpha(s)ds},\quad t\ge T.$$

Proof. For each $i \in N_n$ and $t \in \mathbb{R}$, define $l_i(t) := \left|\frac{q_i^{(1)}(t)-q_i^{(2)}(t)}{\eta_i}\right|$. Then for each $i \in N_n$ and $t \geq t_0$, we get

$$\begin{aligned} D^+ l_i(t) &\leq -\alpha_i(t) l_i(t) + \eta_i^{-1}\left(\sum_{j=1}^{n} |\beta_{ij}(t)| F_j \eta_j l_j(t) + \sum_{j=1}^{n} |\gamma_{ij}(t)| G_j \eta_j \sup_{t-\tau_{ij}(t)\leq s\leq t} l_j(s)\right) \\ &\leq -\alpha_i(t) l_i(t) + \eta_i^{-1} \sum_{j=1}^{n} (|\beta_{ij}(t)| F_j + |\gamma_{ij}(t)| G_j) \eta_j \sup_{t-\tau_{ij}(t)\leq s\leq t} l_j(s). \end{aligned} \tag{15}$$

Firstly, we prove

$$l_i(t) \leq \max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0} |\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}, \quad t \geq t_0, \quad i \in N_n.$$

Obviously, $l_i(t) \leq \max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}$ for $t \leq t_0$ and $i \in N_n$. Suppose there exist $i_3 \in N_n$, $\epsilon_3 > 0$ and $t_3 > t_0$ such that $l_{i_3}(t_3) = \max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\} + \epsilon_3$, and $l_j(t) \leq \max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\} + \epsilon_3$ for $t \in (-\infty, t_3]$ and $j \in N_n$, then $D^+ l_{i_3}(t)\Big|_{t=t_3} > 0$. In contrast

$$\begin{aligned} D^+ l_{i_3}(t)\Big|_{t=t_3} &\leq -\alpha_{i_3}(t_3) l_{i_3}(t_3) + \eta_{i_3}^{-1}\sum_{j=1}^{n}|\beta_{i_3 j}(t_3)| f_j \eta_j l_j(t_3) + \eta_{i_3}^{-1}\sum_{j=1}^{n}|\gamma_{i_3 j}(t_3)| G_j \eta_j \sup_{t_3-\tau_{i_3 j}(t_3)\leq s\leq t_3} l_j(s) \\ &= -\alpha_{i_3}(t_3)\left(\max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}+\epsilon_3\right) \\ &\quad + \eta_{i_3}^{-1}\sum_{j=1}^{n}|\beta_{i_3 j}(t_3)| F_j \eta_j \left(\max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}+\epsilon_3\right) \\ &\quad + \eta_{i_3}^{-1}\sum_{j=1}^{n}|\gamma_{i_3 j}(t_3)| G_j \eta_j \left(\max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}+\epsilon_3\right) \\ &= -\left[\alpha_{i_3}(t_3) - \eta_{i_3}^{-1}\sum_{j=1}^{n}(|\beta_{i_3 j}(t_3)| F_j + |\gamma_{i_3 j}(t_3)| G_j)\eta_j\right]\left(\max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}+\epsilon_3\right) \leq 0, \end{aligned}$$

This is a contradiction. Then we get $l_i(t) \leq \max_{k\in N_n}\left\{\frac{\sup_{t\leq t_0}|\psi_k^{(1)}(\theta)-\psi_k^{(2)}(\theta)|}{\eta_k}\right\}$ for $t \geq t_0$ and $i \in N_n$. Construct the following inequalities:

$$\begin{cases} D^+ l_i(t) \leq -\alpha_i(t) l_i(t) + \eta_i^{-1}\sum_{j=1}^{n}|\beta_{ij}(t)| F_j \eta_j l_j(t) + \eta_i^{-1}\sum_{j=1}^{n}|\gamma_{ij}(t)| G_j \eta_j \sup_{t-\tau_{ij}(t)\leq s\leq t} l_j(s), \quad t \geq T, \\ l_i(t) = l_i(t), \quad t \in (-\infty, T], \end{cases}$$

and define

$$\Gamma_i(\lambda) := \frac{\lambda}{\alpha_i} + \sum_{j=1}^{n} \rho e^{\lambda \tau_{ij}} - 1.$$

Similar to the proof of Theorem 2, one can find a $\lambda > 0$ such that

$$l_i(t) \leq \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_T^t \alpha(u)du}, \quad t \in [T, +\infty).$$

then we have following estimates

$$|q_i^{(1)}(t) - q_i^{(2)}(t)| \leq \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} \eta_i, \quad t \in [t_0, T],$$

and

$$|q_i^{(1)}(t) - q_i^{(2)}(t)| \leq \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} \eta_i e^{-\lambda^* \int_T^t \alpha(s)ds}, \quad t \geq T.$$

This completes the proof. $\square$

Remark 6. *Theorem 3 implies system (1) is globally generalized exponential stable. In fact that for $t \in [t_0, T)$, from the nonnegativity of $\alpha(u)$, we get*

$$\begin{aligned}
\|q^{(1)}(t) - q^{(2)}(t)\| &\leq \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_{t_0}^t \alpha(u)du} e^{\lambda \int_{t_0}^t \alpha(u)du} \\
&= C_T \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_{t_0}^t \alpha(u)du},
\end{aligned}$$

where $C_T = e^{\lambda^* \int_{t_0}^T \alpha(s)ds}$. For $t \geq T$, we get

$$\begin{aligned}
\|q^{(1)}(t) - q^{(2)}(t)\| &\leq \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_{t_0}^t \alpha(u)du} e^{\lambda \int_{t_0}^T \alpha(u)du} \\
&= C_T \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_{t_0}^t \alpha(u)du}.
\end{aligned}$$

So from the above, we get

$$\|q^{(1)}(t) - q^{(2)}(t)\| \leq C_T \max_{k \in N_n} \left\{ \frac{\sup\limits_{t \leq t_0} |\psi_k^{(1)}(\theta) - \psi_k^{(2)}(\theta)|}{\eta_k} \right\} e^{-\lambda \int_{t_0}^t \alpha(u)du}, \quad t \geq t_0.$$

Then system (1) is globally generalized exponential stable.

Remark 7. *Lu et al. [38] considered the globally generalized exponential stability of (1). Under condition (**C.2**) and the following conditions*

(**C.1′**) *For each $i, j \in N_n$, $\alpha_i(s) > 0$, $\beta_{ij}(s)$, $\gamma_{ij}(s)$ and $I_i(s)$ are all continuous functions defined on $[t_0, +\infty)$.*

(**C.6′**) *For each $i \in N_n$,*

$$\sum_{j=1}^{n} (|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j) \leq \alpha_i(t), \quad t \in [t_0, +\infty),$$

and

$$\limsup_{t\to+\infty}\left\{\frac{\sum_{j=1}^{n}(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)}{\alpha_i(t)}\right\}<1.$$

$(\mathbf{C.7}')$ *For $i,j\in N_n$, there exists a $l\in N_n$ such that*

$$\lim_{t\to+\infty}\int_0^t \alpha_l(s)ds\to+\infty,\quad \sup_{t-\tau_{ij}(t)\geq 0}\left\{\int_{t-\tau_{ij}(t)}^{t}\alpha_l(s)ds\right\}<+\infty,\quad \text{and}\quad \sup_{t\geq t_0}\left\{\frac{\alpha_l(s)}{\alpha_i(s)}\right\}<\infty.$$

Then, system (1) is globally generalized exponential stable. We mention here that if we choose $\eta_1=\eta_2=\ldots=\eta_n=1$, then our condtions are similar to the conditions in [38], but less conservative, the results in [38] do not work if $\alpha_i(t)=0$ at some time, or $\sup_{t-\tau_{ij}(t)\geq t_0}\left\{\int_{t-\tau_{ij}(t)}^{t}\alpha_i(s)ds\right\}=+\infty$ for all $i\in N_n$. Besides, $\sup_{t-\tau_{ij}(t)\geq t_0}\left\{\int_{t-\tau_{ij}(t)}^{t}\alpha_i(s)ds\right\}=+\infty$ is quite restrictive. For instance, when $\alpha_i(t)=c>0$, and the delay functions are infinite, then the condition $\sup_{t-\tau_{ij}(t)\geq t_0}\left\{\int_{t-\tau_{ij}(t)}^{t}\alpha_i(s)ds\right\}=+\infty$ is not satisfied. However, in such cases, we have the flexibility to select a suitable $\alpha(t)$ that aligns with our condtions. so this paper enhances and broadens the results in [38].

4. Examples

This section gives four illustrative examples to demonstrate the practical applicability of the theoretical results. To enhance the clarity of the obtained results, we employ a linear representation instead of a nonlinear one.

Example 1. *Consider the following NNs with proportional delays:*

$$\frac{dq_i(t)}{dt}=-\alpha_i(t)q_i(t)+\sum_{j=1}^{2}\left[\beta_{ij}(t)f_j(q_j(t))+\gamma_{ij}(t)g_j(q_j(0.5t))\right]+h_i(t),\quad i=1,2,\quad t\in[0,+\infty),\tag{16}$$

where $\alpha_1(t)=6(t^2+3t+1)$, $\alpha_2(t)=4(t^2+4t+1)$, $\beta_{11}(t)=t^2+4t+1$, $\beta_{12}(t)=2(t^2+2t+1)$, $\beta_{21}(t)=t^2+6t+1$, $\beta_{22}(t)=t^2+5t+1$, $\gamma_{11}(t)=2(t^2+t+1)$, $\gamma_{12}(t)=t^2+5t+1$, $\gamma_{21}(t)=t^2+3t+1$, $\gamma_{22}(t)=t^2+1$, $h_1(t)=60t$, $h_2(t)=60t$, $\tau_{11}(t)=\tau_{21}(t)=\tau_{12}(t)=\tau_{22}(t)=0.5t$, $f_1(q_1)=f_2(q_1)=g_1(q_1)=g_2(q_1)=|q_1|$, $\psi(0)=(15,15)$. It can be verified that, $F_1=F_2=G_1=G_2=1$. Obviously, $\eta_1=\eta_2=1$, $\mu_1=20$ and $\mu_2=20$. we can find conditions $(\mathbf{C.1})$–$(\mathbf{C.3})$ are satisfied, from Theorem 1, we get

$$|q_1(t)|\leq 20,\ |q_2(t)|\leq 20,\ t\geq 0.$$

Then system (16) is dissipative, while the ball $B(0,20)$ serves as both a globally attracting and an absorbing set, as depicted in Figure 1.

Remark 8. *All the coefficient and delay functions of Example (1) are unbounded.*

Example 2. *Consider the following NNs with proportional delays:*

$$\frac{dq_i(t)}{dt}=-\alpha_i(t)q_i(t)+\sum_{j=1}^{2}\left[\beta_{ij}(t)f_j(q_j(t))+\gamma_{ij}(t)g_j(q_j(0.5t))\right]+h_i(t),\quad i=1,2,\quad t\in[0,+\infty),\tag{17}$$

where $\alpha_1(t)=8(t+1)$, $\alpha_2(t)=6(t+2)$, $\beta_{11}(t)=t+1$, $\beta_{12}(t)=4(t+1)$, $\beta_{21}(t)=0.5(t+2)$, $\beta_{22}(t)=t+2$, $\gamma_{11}(t)=t+1$, $\gamma_{12}(t)=2(t+1)$, $\gamma_{21}(t)=0.25(t+2)$, $\gamma_{22}(t)=t+2$, $h_1(t)=60(t+1)$, $h_2(t)=25(t+2)$, $\tau_{11}(t)=\tau_{21}(t)=\tau_{12}(t)=\tau_{22}(t)=0.5t$, $f_1(q_1)=f_2(q_1)=g_1(q_1)=g_2(q_1)=|q_1|$ and $\psi^{(1)}(0)=(40,20)$ and $\psi^{(2)}(0)=(10,10)$.

It can be verified that, $F_1 = F_2 = G_1 = G_2 = 1$, $\rho_{11}^{(1)} = \frac{1}{8}$, $\rho_{12}^{(1)} = \frac{1}{2}$, $\rho_{21}^{(1)} = \frac{1}{12}$, $\rho_{22}^{(1)} = \frac{1}{6}$, $\rho_{11}^{(2)} = \frac{1}{8}$, $\rho_{12}^{(2)} = \frac{1}{4}$, $\rho_{21}^{(2)} = \frac{1}{24}$, $\rho_{22}^{(2)} = \frac{1}{6}$.

Choose $\eta_1 = 1$, $\eta_2 = 0.5$ *and* $\alpha(t) = \frac{1}{t+1}$, *then* $\sup_{t\geq 0}\left\{\int_{0.5t}^{t}\frac{1}{s+1}ds\right\} = \ln 2$, $\alpha_1 = 8$ *and* $\alpha_2 = 6$. *One can find* $\lambda_1 = \lambda_2 = 1$, *and* $\mu_1 = \mu_2 = 20$. *Then conditions of* (**C.1**)–(**C.5**) *are satisfied, for different initial values* $\psi^{(1)}$ *and* $\psi^{(2)}$, *from Theorem 2, we get*

$$\begin{aligned} &|q_1^{(1)}(t)| \leq \frac{20}{t+1} + 20, \quad |q_2^{(1)}(t)| \leq \frac{10}{t+1} + 10, \\ &|q_1^{(2)}(t)| \leq 20, \quad |q_2^{(2)}(t)| \leq 10, \\ &|q_1^{(1)}(t) - q_1^{(2)}(t)| = \frac{30}{t+1}, \quad |q_2^{(1)}(t) - q_2^{(2)}(t)| = \frac{15}{t+1}, \end{aligned}$$

which are shown in Figures 2–4, respectively.

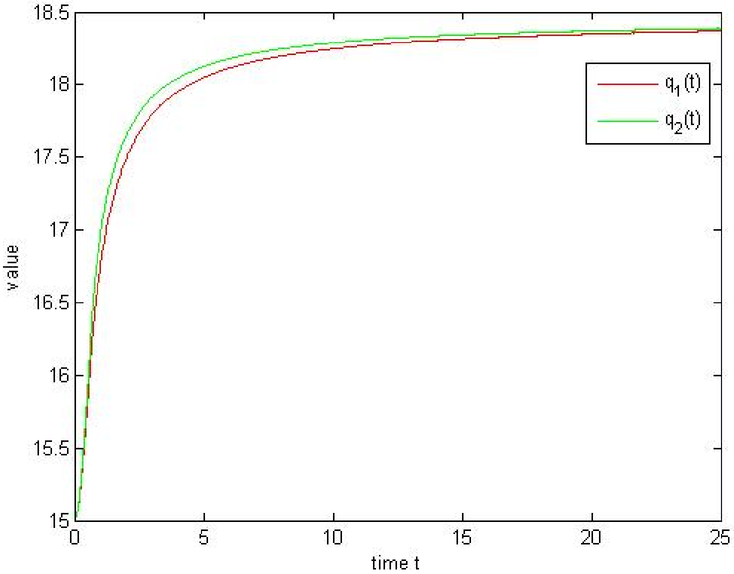

Figure 1. $q_1(t)$ and $q_2(t)$ of Example 1.

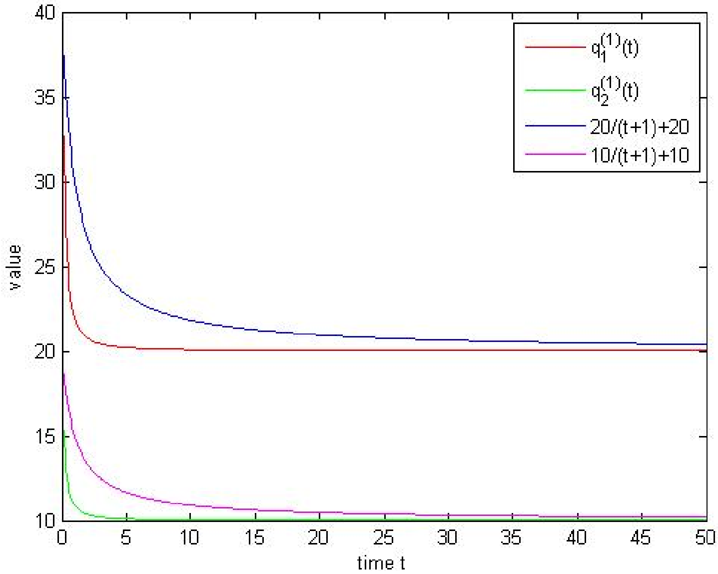

Figure 2. $q_1^{(1)}(t)$ and $q_2^{(1)}(t)$ of Example 2 and their estimates.

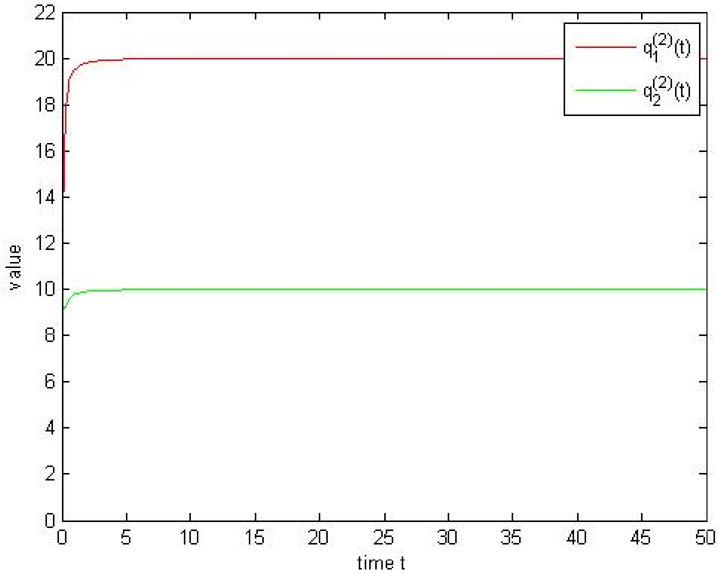

Figure 3. $q_1^{(2)}(t)$ and $q_2^{(2)}(t)$ of Example 2.

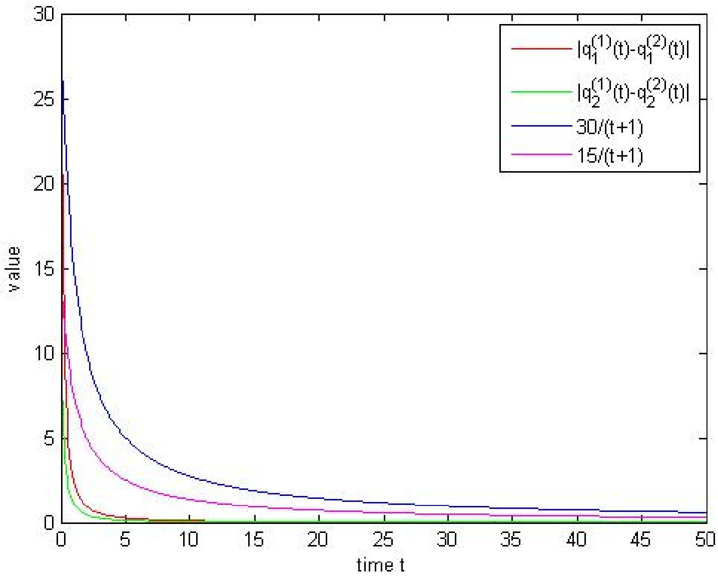

Figure 4. $|q_1^{(1)}(t) - q_1^{(2)}(t)|$ and $|q_2^{(1)}(t) - q_2^{(2)}(t)|$ of Example 2 and their estimates.

Remark 9. *All the coefficient, activation and delay functions in Example 2 are unbounded, and* $\sup\limits_{t\geq 0} \int_{0.5t}^{t} \alpha_i(s)ds = +\infty$*, for* $i = 1, 2$*, which means that the results in [22,26,27,32,33,35–38] can not solve this case.*

Example 3. *Consider the following 2-dimensional NNs with time-varying delays:*

$$\frac{dq_i(t)}{dt} = -\alpha_i(t)q_i(t) + \sum_{j=1}^{2} \left[\beta_{ij}(t)f_j(q_j(t)) + \gamma_{ij}(t)g_j(q_j(t - \tau_{ij}(t)))\right] + h_i(t), \quad i = 1, 2, \quad t \in [0, +\infty), \tag{18}$$

where, $\alpha_1(t) = 5(1 - \sin t)$, $\alpha_2(t) = 7(1 - \sin t)$, $\beta_{11}(t) = 1 - \sin t$, $\beta_{12}(t) = 5(1 - \sin t)$, $\beta_{21}(t) = 0.4(1 - \sin t)$, $\beta_{22}(t) = 2(1 - \sin t)$, $\gamma_{11}(t) = (1 - \sin t)e^{-\pi-2}$, $\gamma_{12}(t) = 5(1 - \sin t)e^{-\pi-2}$, $\gamma_{21}(t) = 0.4(1 - \sin t)e^{-\pi-2}$, $\gamma_{22}(t) = (1 - \sin t)e^{-\pi-2}$, $h_1(t) = 20(1 -$

$\sin t)(3 - 2e^{-\pi-2})$, $h_2(t) = 12(1 - \sin t)(1 - e^{-\pi-2})$, $\tau_{11}(t) = \tau_{21}(t) = \tau_{12}(t) = \tau_{22}(t) = \pi|\cos t|$, $f_1(q_1) = f_2(q_1) = g_1(q_1) = g_2(q_1) = |q_1|$, $\psi^{(1)}(0) = (40, 8)$ *and* $\psi^{(2)}(0) = (1, 1)$.

It can be verified that, $F_1 = F_2 = G_1 = G_2 = 1$, $\rho_{11}^{(1)} = \frac{1}{5}$, $\rho_{12}^{(1)} = 1$, $\rho_{21}^{(1)} = \frac{2}{35}$, $\rho_{22}^{(1)} = \frac{2}{7}$, $\rho_{11}^{(2)} = \frac{1}{5e^{\pi+2}}$, $\rho_{12}^{(2)} = \frac{1}{e^{\pi+2}}$, $\rho_{21}^{(2)} = \frac{2}{35e^{\pi+2}}$, $\rho_{22}^{(2)} = \frac{1}{7e^{\pi+2}}$.

Choose $\eta_1 = 1$, $\eta_2 = 0.2$ *and* $\alpha(t) = 1 - \sin t$, *then* $\sup\limits_{t\geq 0} \int_{t-\tau_{ij}(t)}^{t} \Big(1 - \sin s\Big)^{*} ds = \pi + 2$, $\alpha_1 = 5$ *and* $\alpha_2 = 7$. *We can find* $\lambda_1 = \lambda_2 = 1$, *and* $\mu_1 = \mu_2 = 20$. *Then conditions* (**C.1**)–(**C.5**) *are satisfied, for different initial values* $\psi^{(1)}$ *and* $\psi^{(2)}$, *from Theorem 2, we get*

$$
\begin{aligned}
&|q_1^{(1)}(t)| \leq 20e^{-t+1-cost} + 20, \quad |q_2^{(1)}(t)| \leq 4e^{-t+1-cost} + 4,\\
&|q_1^{(2)}(t)| \leq 20, \quad |q_2^{(2)}(t)| \leq 4,\\
&|q_1^{(1)}(t) - q_1^{(2)}(t)| = 39e^{-t+1-cost}, \quad |q_2^{(1)}(t) - q_2^{(2)}(t)| = 7.8e^{-t+1-cost},
\end{aligned}
$$

which are shown by Figures 5–7, respectively.

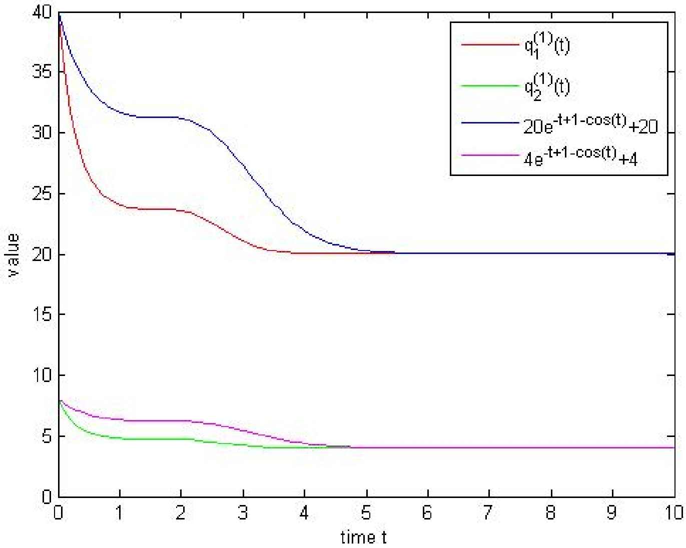

Figure 5. $q_1^{(1)}(t)$ and $q_2^{(1)}(t)$ of Example 3 and their estimates.

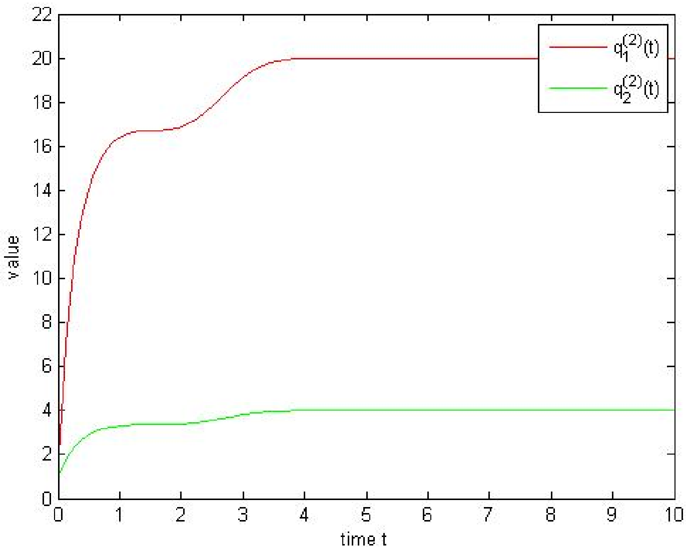

Figure 6. $q_1^{(2)}(t)$ and $q_2^{(2)}(t)$ of Example 3.

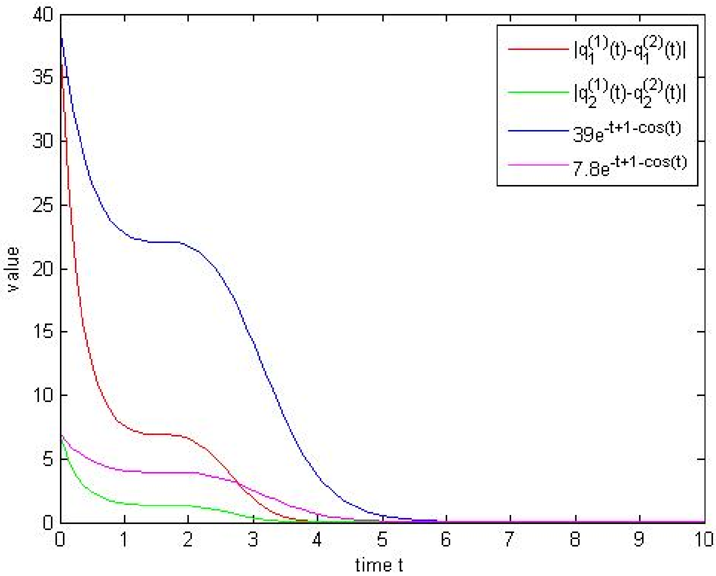

Figure 7. $|q_1^{(1)}(t) - q_1^{(2)}(t)|$ and $|q_2^{(1)}(t) - q_2^{(2)}(t)|$ of Example 3 and their estimates.

Remark 10. *It is worth noting that $\alpha_i(t) = 0$, for $t = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{N}$ and $i = 1, 2$ as well as the delay functions $\pi|\cos t|$ lack differentiability at points where $t = k\pi + \frac{\pi}{2}$ for $k \in \mathbb{N}$, which make the results in [22,26,28,32,33,36,37] be invalid.*

Example 4. *Consider the following 2-dimensional NNs with proportional delays:*

$$\frac{dq_i(t)}{dt} = -\alpha_i(t)q_i(t) + \sum_{j=1}^{2}\Big[\beta_{ij}(t)f_j(q_j(t)) + \gamma_{ij}(t)g_j(q_j(t-\tau_{ij}(t)))\Big] + h_i(t), \quad i = 1,2, \quad t \in [0,+\infty), \tag{19}$$

where $\alpha_1(t) = 8$, $\alpha_2(t) = 6$, $\beta_{11}(t) = \beta_{12}(t) = \beta_{21}(t) = \gamma_{11}(t) = \gamma_{12}(t) = \gamma_{21}(t) = 2$ for $t \in [0,5)$, $\beta_{11}(t) = \beta_{12}(t) = \beta_{21}(t) = \gamma_{11}(t) = \gamma_{12}(t) = \gamma_{21}(t) = 1$ for $t \geq 5$, $\beta_{12}(t) = \gamma_{22}(t) = 1$ for $t \in [0,5)$, $\beta_{12}(t) = \gamma_{22}(t) = 0.5$ for $t \geq 10$, $h_1(t) = 5$, $h_2(t) = 6$, $\tau_{11}(t) = \tau_{21}(t) = \tau_{12}(t) = \tau_{22}(t) = \sqrt{t+1}$, $f_1(q_1) = f_2(q_1) = g_1(q_1) = g_2(q_1) = |q_1|$, $\psi^{(1)}(t) = (40,8)$ and $\psi^{(2)}(t) = (1,1)$ for $t \in [-1,0]$.

It can be verified that, $F_1 = F_2 = G_1 = G_2 = 1$. Obviously, $\eta_1 = \eta_2 = 1$, then

$$\sum_{j=1}^{n}(|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j) \leq \alpha_i(t), \quad t \in [0,+\infty),$$

and

$$\sup_{\{t|t\geq 5\}-\{t|\eta_i\alpha_i(t)=\sum\limits_{j=1}^{n}(|\beta_{ij}(t)|F_j+|\gamma_{ij}(t)|G_j)\eta_j=0\}} \left\{\frac{\sum\limits_{j=1}^{n}(|\beta_{ij}(t)|F_j + |\gamma_{ij}(t)|G_j)\eta_j}{\eta_i\alpha_i(t)}\right\} = 0.5 < 1.$$

*Choose $\alpha(t) = \sqrt{\frac{5}{t}}$, $t \geq 5$, then $\alpha_1 = 8$, $\alpha_2 = 6$ and $\sup\limits_{t-\sqrt{t+1}\geq 5}\left\{\int_{t-\sqrt{t+1}}^{t}\sqrt{\frac{5}{s}}\right\} = 4\sqrt{10} - 10$. We can find $\lambda^* \leq 0.245$. Then conditions of (**C.1**), (**C.2**), (**C.6**) and (**C.7**) are satisfied, from Theorem 3, we get the following estimate*

$$\|q^{(1)}(t) - q^{(2)}(t)\| \leq e_1(t) := \begin{cases} 20, & t \in [0,5], \\ 20e^{-0.245(2\sqrt{5t}-10)}, & t \in (5,+\infty), \end{cases}$$

which are illustrated by Figure 8.

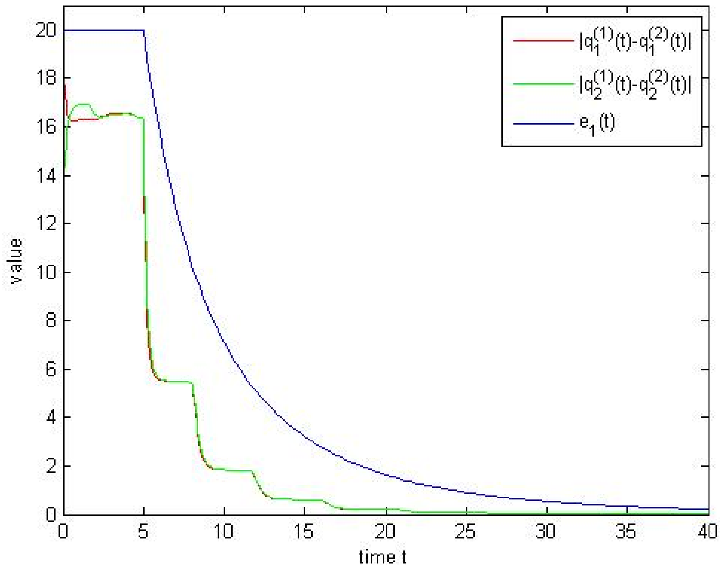

Figure 8. $q_1(t)$ and $q_2(t)$ of Example 4 and their estimate.

Remark 11. *We note that* $\sup_{t\geq 0} \int_{t-\sqrt{t+1}}^{t} \alpha_i(s)ds = +\infty$, *for* $i = 1,2$, *which makes the results in [38] be invalid.*

5. Conclusions

In this paper, we obtained some criteria on dissipativity and globally generalized exponential stability of a class of NNs with delays by constructing some generalized Halanay inequalities. We mention here that our coefficient functions and delay functions can be all unbounded, and our results improve and generalize some existing works [5,35,38]. At last, four numerical examples have shown the effectiveness of our main results.

Our method has its limitations, when the $\alpha_i(t)$ is oscillation, such as $\alpha_i(t) = 0.5 + sint$, our method is invalid in this case. The author will investigate this case in the future.

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