mathematics

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On Enriched Suzuki Mappings in Hadamard Spaces

Teodor Turcanu [1,†] and Mihai Postolache [1,2,3,*,†]

[1] Department of Mathematics and Informatics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania; teodor.turcanu@upb.ro
[2] The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, Health Big Data Research Institute, North Minzu University, Yinchuan 750021, China
[3] Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711 Bucharest, Romania
* Correspondence: mihai.postolache@upb.ro
† These authors contributed equally to this work.

Abstract: We define and study enriched Suzuki mappings in Hadamard spaces. The results obtained here are extending fundamental findings previously established in related research. The extension is realized with respect to at least two different aspects: the setting and the class of involved operators. More accurately, Hilbert spaces are particular Hadamard spaces, while enriched Suzuki nonexpansive mappings are natural generalizations of enriched nonexpansive mappings. Next, enriched Suzuki nonexpansive mappings naturally contain Suzuki nonexpansive mappings in Hadamard spaces. Besides technical lemmas, the results of this paper deal with (1) the existence of fixed points for enriched Suzuki nonexpansive mappings and (2) Δ and strong (metric) convergence of Picard iterates of the α-averaged mapping, which are exactly Krasnoselskij iterates for the original mapping.

Keywords: enriched Suzuki mapping; Hadamard space; fixed point; Picard iteration; convergence

MSC: 47H10; 54H25; 37C25

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Citation: Turcanu, T.; Postolache, M. On Enriched Suzuki Mappings in Hadamard Spaces. *Mathematics* **2024**, *12*, 157. https://doi.org/10.3390/math12010157

Academic Editor: Janusz Brzdęk

Received: 28 November 2023
Revised: 22 December 2023
Accepted: 25 December 2023
Published: 3 January 2024

1. Introduction

In some recent papers [1–4], Berinde and Păcurar introduced a technique for extending the scope of several well-known classes of mappings (such as contractions, nonexpansive, Kannan, and Chatterjea) for which the authors have used the term *enrichment*. The idea is to consider mappings, say, T, which do not necessarily obey any contraction-type condition but satisfy such a condition for pairs of points obtained as affine combinations of x and Tx and, respectively, y and Ty. For instance, in a normed space X, the enriched nonexpansiveness condition, for $\alpha \in (0,1)$, writes as

$$\|((1-\alpha)x+\alpha Tx)-((1-\alpha)y+\alpha Ty)\| \leq \|x-y\|, \text{ for all } x,y \in X.$$

In other words, the mapping T is enriched nonexpansive if and only if its associated α-averaged mapping $T_\alpha = (1-\alpha)Id + \alpha T$ is nonexpansive. The usefulness of this approach, when it comes to the existence of fixed points, stems from the fact that the enriched versions have the same fixed point sets as the original mappings. Moreover, both mappings oftentimes have similar properties with respect to associated common iterates, a fact that plays a key role while obtaining convergence results.

A natural task is to extend these fruitful methods and ideas to other settings, such as geodesic spaces, or, more precisely, to complete CAT(0) spaces also known as Hadamard spaces, which can be seen as important nonlinear generalizations of Hilbert spaces, and provide a suitable setting for nonlinear analysis and optimization problems (see also [5,6]). The fact that this is a suitable setting for developing fixed point theoretic results has been indicated in the pioneering works of Kirk [7,8] (also, for a basic introduction into the subject, we refer the reader to [9]).

In a recent paper [10], enriched contractions and enriched nonexpansive mappings were studied in the setting of Hadamard spaces by Salisu et al. The approach in [10] was based on taking the squares in the respective inequalities and exploiting the relationship between the square of the norm and the inner product, which, for the setting of Hadamard spaces, was replaced by the quasi-linearization map (for details, please see [11]). However, one can argue that invoking the quasi-linearization map while extending the enrichment techniques to Hadamard spaces is not necessary at all, at least for mappings whose definition does not involve the inner product. More precisely, the natural analogue of the averaged mapping in the setting of Hadamard spaces is actually the mapping defined by $T_\alpha = (1-\alpha)I \oplus \alpha T$, which should be used directly in the definition. More details in this respect are presented in the sequel.

Returning now to the question of extending the enrichment techniques of Berinde and Păcurar to mappings in Hadamard spaces, we can address the class of Suzuki nonexpansive mappings [12] initially introduced in the setting of normed spaces, which represents an important generalization of nonexpansive mappings (i.e., mappings such that $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in X$). This class was extended to the setting of Hadamard spaces in [13] and more recently was extended to the enriched version in [14] in the setting of Hilbert spaces. A natural development would be defining and studying enriched Suzuki mappings in the setting of Hadamard spaces, which is precisely the main goal of this paper, which generalizes the main results of Salisu et al. [10], Nanjaras et al. [13], Ullah et al. [14]. More precisely, the generalization is realized with respect to three different aspects. With respect to the setting, Hilbert spaces are particular Hadamard spaces, and in this sense, we have a generalization of [14]. With respect to the classes of mappings involved, on the one hand, enriched Suzuki nonexpansive mappings are natural generalizations of enriched nonexpansive mappings studied in [10]. On the other hand, enriched Suzuki nonexpansive mappings naturally contain Suzuki nonexpansive mappings in Hadamard spaces, and in this respect, we have a generalization of the results from [13].

The main results of this paper, besides technical lemmas, are concerned with (1) the existence of fixed points for enriched Suzuki nonexpansive mappings and (2) Δ and strong (metric) convergence of the Picard iterates of the α-averaged mapping, which are exactly Krasnoselskij iterates for the original mapping. Last but not least, we provide an example in order to illustrate our findings. It is worth mentioning that the mapping proposed as an example is neither enriched nonexpansive (since it is discontinuous) nor Suzuki nonexpansive.

2. Preliminaries

A continuous mapping $c\colon [0,t] \to M$, where (M,d) is a metric space, with $c(0) = x \in M$ and $c(t) = y \in M$ is called *geodesic* if

$$d(c(\tau_1), c(\tau_2)) = |\tau_1 - \tau_2|,$$

for any $\tau_1, \tau_2 \in [0,t]$. Its image, denoted by $[x,y]$, is called *geodesic segment*. If any pair of distinct points can be joined by a geodesic, then (M,d) is called *geodesic space* and is said to be *uniquely geodesic* if the geodesic is unique. Three distinct points, x, y, and z, in a uniquely geodesic metric space (M,d) determine a unique *geodesic triangle* denoted by $\Delta(x,y,z)$. A *comparison triangle* for $\Delta(x,y,z)$ is a triangle in the Euclidean plane $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ such that

$$d(x,y) = d_E(\overline{x},\overline{y}), \quad d(y,z) = d_E(\overline{y},\overline{z}), \quad d(z,x) = d_E(\overline{z},\overline{x}),$$

where d_E is the Euclidean metric.

Definition 1 ([15,16])**.** *Let (M, d) be a geodesic space and let Δ be a geodesic triangle in M with $\overline{\Delta}$ as its corresponding comparison triangle. Then, the triangle Δ satisfies the* CAT(0) *inequality if*

$$d(x, y) \leq d_E(\overline{x}, \overline{y}),$$

for all $x, y \in \Delta$ and the corresponding $\overline{x}, \overline{y} \in \overline{\Delta}$. A geodesic space is said to be a CAT(0) *space if all its geodesic triangles satisfy the* CAT(0) *inequality.*

A complete CAT(0) space is called a Hadamard space.
Below are some fundamental properties of CAT(0) spaces.

Lemma 1 ([15])**.** *Let (M, d) be a* CAT(0) *space. Then*

(i) *(M, d) is uniquely geodesic.*
(ii) *For a given pair of distinct points x, y in M and some $t \in [0, 1]$, there exists a unique point $z \in [x, y]$, such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote this point by $z = tx \oplus (1 - t)y$.*
(iii) *$[x, y] = \{tx \oplus (1 - t)y \colon t \in [0, 1]\}$.*
(iv) *$d(x, z) + d(z, y) = d(x, y)$ if and only if $z \in [x, y]$.*
(v) *The mapping $f \colon [0, 1] \to [x, y]$, $f(t) = tx \oplus (1 - t)y$ is continuous and bijective.*

Lemma 2 ([15])**.** *Let (M, d) be a* CAT(0) *space. Then*

$$d(z, tx \oplus (1 - t)y) \leq td(z, x) + (1 - t)d(z, y)$$

and

$$d^2(z, tx \oplus (1 - t)y) \leq td^2(z, x) + (1 - t)d^2(z, y) - t(1 - t)d^2(x, y), \tag{1}$$

for all $x, y, z \in M$ and $t \in [0, 1]$.

Definition 2. *Given a bounded sequence $\{x_n\}$ in a* CAT(0) *space (M, d), one can associate the function*

$$r(\cdot, \{x_n\}) : M \to [0, \infty), \quad r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n),$$

which defines the asymptotic radius

$$r(\{x_n\}) = \inf\{x \in M : r(x, \{x_n\})\}$$

and, respectively, the asymptotic center

$$A(\{x_n\}) = \{x \in M : r(x, \{x_n\}) = r(\{x_n\})\}$$

of the sequence $\{x_n\}$.

A remarkable property of CAT(0) spaces is that the asymptotic center of a given sequence is unique (Proposition 7, [17]). This fact is the basis for a notion of convergence which has similar properties with weak convergence in Banach spaces and is weaker than metric convergence.

Definition 3 ([18])**.** *A sequence $\{x_n\}$ in a CAT(0) space (M, d) is said to be Δ-convergent to some point $x \in M$, if x is the unique asymptotic center for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

Lemma 3 ([15,18])**.** *In a CAT(0) space (M, d), the following assertions are true:*

i. *Any bounded sequence in M has a Δ-convergent subsequence.*
ii. *If $\{x_n\}$ is a bounded sequence in a closed and convex subset $C \subseteq M$, then $A(\{x_n\}) \in C$.*
iii. *If $\{x_n\}$ is a bounded sequence in M with $A(\{x_n\}) = \{x\}$ and $\{p_n\}$ is a subsequence of $\{x_n\}$ with $A(\{p_n\}) = \{p\}$ and the sequence $\{d(x_n, p)\}$ converges, then $x = p$.*

The following two properties will play a key role in the sequel.

Definition 4 ([19])**.** *Let (M, d) be a uniquely geodesic space and C a nonempty closed and convex subset. Then, a mapping $T\colon M \to M$ is called asymptotically regular if and only if, for any $x_0 \in C$, $\lim\limits_{n\to\infty} d\left(T^{n+1}x_0, T^n x_0\right) = 0$.*

Definition 5 ([20])**.** *Let C be a subset in a metric space (M, d). A mapping $T\colon C \to M$ is called demicompact if it has the property that whenever $\{x_n\}$ is a bounded sequence such that $\{d(x_n, Tx_n)\}$ converges, then there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ converging to some point in C.*

3. Enriched Suzuki Nonexpansive Mappings

Definition 6 ([12])**.** *Let C be a nonempty set in a normed space X. A mapping $T\colon C \to X$ is said to be Suzuki nonexpansive* (SN) *if*

$$\|Tx - Ty\| \le \|x - y\|,$$

for all $x, y \in C$ such that $\frac{1}{2}\|x - Tx\| \le \|x - y\|$.

Definition 7 ([14])**.** *Let C be a nonempty set in a normed space X. A mapping $T\colon X \to X$ is said to be enriched Suzuki nonexpansive* (ESN) *if there exists $b \in [0, \infty)$ such that*

$$\|b(x - y) + Tx - Ty\| \le (b + 1)\|x - y\|, \tag{2}$$

for all $x, y \in C$ satisfying $\frac{1}{2}\|x - Tx\| \le (b + 1)\|x - y\|$.

Remark 1. *Rewriting the inequality* (2) *as*

$$\left\| \left(\frac{b}{b+1}x + \frac{1}{b+1}Tx \right) - \left(\frac{b}{b+1}y + \frac{1}{b+1}Ty \right) \right\| \le \|x - y\|,$$

we see that the left term is the norm of a difference of terms of the form $T_\alpha x = (1 - \alpha)x + \alpha Tx$, for $\frac{1}{b+1} = \alpha \in (0, 1]$, i.e., an affine combination of the vectors x and Tx, respectively. Moreover, the condition $\frac{1}{2}\|x - Tx\| \le (b + 1)\|x - y\|$ rewrites as $\frac{1}{2}\|x - T_\alpha x\| \le \|x - y\|$.

The above remark suggest the natural way to define enriched Suzuki nonexpansive mappings in the setting of a uniquely geodesic space (M, d). More precisely, the natural analogue of the vector $(1 - \alpha)x + \alpha Tx$ is the unique point on the geodesic segment $[x, Tx]$, denoted by

$$T_\alpha x = (1 - \alpha)x \oplus \alpha Tx, \quad \alpha \in (0, 1). \tag{3}$$

The neat thing about the mapping T_α, called the *α-averaged mapping of T*, is that it has the same fixed points as the mapping T in the virtue of the identity

$$d(x, T_\alpha x) = \alpha d(x, Tx). \tag{4}$$

Clearly, for $\alpha = 1$ and $\alpha = 0$, we have, respectively, $T_1 = T$ and $T_0 = Id$. For obvious reasons, we exclude theses two cases from our further discussion, and henceforth, we shall adopt the notation (3).

Definition 8. *Let C be a nonempty set in a uniquely geodesic space (M, d). A mapping $T\colon C \to M$ is said to be enriched Suzuki nonexpansive* (ESN) *if*

$$d(T_\alpha x, T_\alpha y) \le d(x, y),$$

for all $x, y \in C$ such that $d(x, T_\alpha x) \le 2d(x, y)$.

In other words, the mapping T is enriched Suzuki nonexpansive if and only if the mapping T_α is Suzuki nonexpansive.

There are two very important subclasses that are given below.

Definition 9. *Let C be a nonempty set in a uniquely geodesic space (M, d). A mapping $T\colon C \to M$ is said to be (α, γ)-enriched contraction if there exist $\alpha \in [0, 1]$ and $\gamma \in [0, 1)$ such that*

$$d(T_\alpha x, T_\alpha y) \le \gamma d(x, y), \ \ \forall x, y \in C.$$

Definition 10. *Let C be a nonempty set in a uniquely geodesic space (M, d). A mapping $T\colon C \to M$ is said to be α-enriched nonexpansive mapping if there exists $\alpha \in [0, 1]$ such that*

$$d(T_\alpha x, T_\alpha y) \le d(x, y), \ \ \forall x, y \in C.$$

Notice that these definitions do not involve the quasi-linearization map and are simpler, more natural, and *more general* (it follows from the fundamental inequality (1)) as compared with those given in [10]. Moreover, the fact that these are indeed subclasses becomes obvious if we accept the proposed definitions.

On the other hand, classical Suzuki nonexpansive mappings with a fixed point are also quasi-nonexpansive mappings, i.e., mappings such that $d(Tx, p) \le d(x, p)$, for all $x \in C$ and $p \in \mathrm{Fix}(T)$. A similar relationship holds for the enriched analogs.

Definition 11. *Let C be a nonempty set in a uniquely geodesic space (M, d). A mapping $T\colon C \to M$ such that $\mathrm{Fix}(T) \neq \emptyset$ is said to be enriched quasi-nonexpansive if and only if, for any $p \in \mathrm{Fix}(T)$,*

$$d(T_\alpha x, p) \le d(x, p), \text{ for all } x \in C.$$

Proposition 1. *With the above notations and definitions, every enriched Suzuki nonexpansive mapping is an enriched quasi-nonexpansive mapping.*

Proof. As $d(p, T_\alpha p) = 0$, for $p \in \mathrm{Fix}(T)$, the ESN condition implies $d(Tx, p) \le d(x, p)$ for all $x \in C$. □

Proposition 2 (Theorem 4.1, [13])**.** *Let C be a nonempty bounded closed convex subset of a Hadamard space. If $T\colon C \to C$ satisfies condition (C), then T has a fixed point in C.*

Lemma 4 (Lemma 3.5 [13])**.** *Let C be a nonempty subset of a uniquely geodesic space. If $T\colon C \to C$ satisfies condition (C), then*

$$d(x, Ty) \le 3d(x, Tx) + d(x, y), \text{ for all } x, y \in C.$$

4. Main Results

We start this section with the following existence result, which extends Theorem 4.1 of [13].

Theorem 1. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping, then T has a fixed point in C.*

Proof. As noticed above, T being ESN is equivalent to T_α, $\alpha \in (0,1]$, being Suzuki nonexpansive. According to Proposition 2, the mapping T_α has a fixed point in $p \in C$, and from (4), it follows that p is a fixed point for T as well. □

Due to the coincidence of the fixed point sets of mappings T and T_α, we also have the following extension of Corollary 4.2 from [13].

Corollary 1. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping, then the set* $\mathrm{Fix}(T)$ *is nonempty closed, convex, and hence contractible.*

Before discussing Δ and strong convergence results, we need to establish some technical lemmas first.

Lemma 5. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$ and suppose that $T\colon C \to C$ is an ESN mapping. Define the sequence $\{x_n\}_{n\geq 0}$*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T_\alpha x_n, \quad n \geq 0. \end{cases} \tag{5}$$

Then, the limit $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in \mathrm{Fix}(T)$.

Proof. As the mapping T_α is Suzuki nonexpansive, it is also quasi-nonexpansive, implying that

$$d(x_{n+1}, p) = d(T_\alpha x_n, p) \leq d(x_n, p),$$

for any $p \in C$; i.e., the sequence is nonincreasing and bounded and, hence, convergent. □

The following result establishes the fact that ESN mappings satisfy the demiclosedness principle.

Lemma 6. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$ and suppose that $T\colon C \to C$ is an ESN mapping. If $\{x_n\}_{n\geq 0}$ is a sequence in C, such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $x_n \xrightarrow{\Delta} p \in \mathcal{H}$, then $p \in C$ and $Tp = p$.*

Proof. The fact that $p \in C$ is established by Lemma 3 (ii).

Turning to the second part, $T\colon C \to C$ being an ESN mapping is equivalent to the α-averaged T_α satisfying condition (C). On the other hand, according to Lemma 4, we have

$$d(x_n, T_\alpha p) \leq 3d(x_n, T_\alpha x_n) + d(x_n, p),$$

which, by taking lim sup, yields

$$\limsup_{n\to\infty} d(x_n, T_\alpha p) \leq \limsup_{n\to\infty} d(x_n, p).$$

By the uniqueness of the asymptotic centers, it follows that $T_\alpha p = p$, implying $Tp = p$, as $\mathrm{Fix}(T) = \mathrm{Fix}(T_\alpha)$. □

Lemma 7. *Let (M, d) be a uniquely geodesic space and C a nonempty closed and convex subset. Then, for any quasi-nonexpansive mapping $T\colon C \to C$ such that* $\mathrm{Fix}(T) \neq \emptyset$*, the corresponding α-averaged mapping T_α for arbitrary $\alpha \in (0,1)$ is asymptotically regular.*

Proof. Take an arbitrary $x_0 \in C$ and consider the sequence of Picard iterates $x_n = T_\alpha^n x_0$, $n \geq 1$. For any $p \in \mathrm{Fix}(T)$, we have, according to the fundamental inequality (1) and quasi-nonexpansiveness of T,

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha)d^2(T_\alpha^n x_0, p) + \alpha d^2(TT_\alpha^n x_0, p) - (1-\alpha)\alpha d^2(T_\alpha^n x_0, TT_\alpha^n x_0) \\ &\leq (1-\alpha)d^2(T_\alpha^n x_0, p) + \alpha d^2(T_\alpha^n x_0, p) - (1-\alpha)\alpha d^2(T_\alpha^n x_0, TT_\alpha^n x_0) \\ &= d^2(T_\alpha^n x_0, p) - (1-\alpha)\alpha d^2(T_\alpha^n x_0, TT_\alpha^n x_0) \\ &\leq d^2(T_\alpha^n x_0, p), \end{aligned}$$

which means that the sequence $\{d^2(T_\alpha^n x_0, p)\}$ is nonincreasing and that we also have

$$d^2(T_\alpha^n x_0, TT_\alpha^n x_0) = \frac{1}{(1-\alpha)\alpha}\left[d^2(T_\alpha^n x_0, p) - d^2(T_\alpha^{n+1} x_0, p)\right],$$

implying that

$$\lim_{n\to\infty} d^2(T_\alpha^n x_0, T_\alpha^{n+1} x_0) = \lim_{n\to\infty} \alpha d^2(T_\alpha^n x_0, TT_\alpha^n x_0) = 0, \text{ for } n \to \infty,$$

which in turn implies the desired result. □

The above proof suggests two important facts, which we underline in the following.

Remark 2. *The sequence of Picard iterates is an approximate fixed point sequence for both the mapping T and its associated α-averaged mapping T_α, i.e.,*

$$\lim_{n\to\infty} d(x_n T x_n) = 0, \textit{ and } \lim_{n\to\infty} d(x_n T_\alpha x_n) = 0. \tag{6}$$

Now we are in a position to state the first Δ-convergence result.

Theorem 2. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping, then the sequence $\{x_n\}$ of Picard iterates (5) is Δ-convergent to a fixed point of T.*

Proof. Denote the set of all associated asymptotic centers by $\omega_A(x_n) := \bigcup A(\{u_n\})$, with union taken over all subsequences of $\{x_n\}$. Now let $u \in \omega_A(x_n)$ and consider the sequence $\{u_n\}$ such that $A(\{u_n\}) = u$. Since $\{u_n\}$ is bounded, according to Lemma 3 (i) and (ii), it has a subsequence u_{n_k} that is Δ-convergent to some $u' \in C$. As $\lim_{n\to\infty} d(u_{n_k}, T_\alpha u_{n_k}) = 0$, according to Lemma 6 $u' \in \mathrm{Fix}(T) = \mathrm{Fix}(T_\alpha)$ and according to Lemma 5, the limit $\lim_{n\to\infty} d(x_n, u')$ exists. Let us show that $u = u'$. Suppose the opposite; then we have the following inequalities in which we use the properties of lim sup and the uniqueness of the asymptotic center

$$\begin{aligned} \limsup_{n\to\infty} d(u_{n_k}, u') &< \limsup_{n\to\infty} d(u_{n_k}, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, u') \\ &= \limsup_{n\to\infty} d(u_{n_k}, u'), \end{aligned}$$

leading to a contradiction. Thus, $u = u' \in \mathrm{Fix}(T)$. Now let $A(\{x_n\}) = p$. According to Lemma 5, the limit $\lim_{n\to\infty} d(x_n, u)$ exists, and hence, due to Lemma 3 (iii), $p = u$, implying that $\{x_n\}$ is Δ-convergent to $p \in \mathrm{Fix}(T)$. □

Theorem 3. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping and is demicompact, then the sequence $\{x_n\}$ of Picard iterates (5) converges to a fixed point of T.*

Proof. We have, according to (6), that $\lim_{n\to\infty} d(x_n, Tx_n)$ converges, and since, by assumption, T is demicompact, there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ converging to some point $p \in C$. On the other hand, p is also the Δ-limit of $\{x_{n_k}\}$, and hence, $p \in \mathrm{Fix}(T)$. Lastly, the fact that the entire sequence $\{x_n\}_{n\geq 0}$ converges to p follows from the inequality

$$d(x_{n+1}, p) \leq d(x_n, p), \quad n \geq 0,$$

established above. □

Theorem 4. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping and C is additionally a compact subset in $\mathcal{H}$, then the sequence $\{x_n\}$ of Picard iterates (5) converges to a fixed point of T.*

Proof. The compactness of C implies the existence of a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ converging to some $p \in C$. From Lemma 4, we have

$$d(x_{n_k}, Tp) \leq 3d\big(x_{n_k}, Tx_{n_k}\big) + d(x_{n_k}, p), \text{ for all } k \geq 0.$$

Letting $k \to \infty$ and keeping in mind that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ yields $\lim_{n\to\infty} d(x_{n_k}, Tp) = 0$, and by the uniqueness of the limit, it follows that $Tp = p$. The fact that the whole sequence $\{x_n\}_{n\geq 0}$ converges to p can be deduced from the existence of the limit $\lim_{n\to\infty} d(x_n, p)$, which exists according to Lemma 5. □

According to [21], a mapping $T\colon C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $f(\xi) > 0$, for all $\xi > 0$ and $d(x, Tx) \geq f(d(x, \mathrm{Fix}(T)))$, for all $x \in C$, where $d(x, \mathrm{Fix}(T)) = \inf_{z\in \mathrm{Fix}(T)} d(x, z)$.

Theorem 5. *Let C be a nonempty bounded closed convex subset of a Hadamard space $(\mathcal{H}, d)$. If $T\colon C \to C$ is an enriched Suzuki nonexpansive mapping that satisfies condition (I), then the sequence $\{x_n\}$ of Picard iterates (5) converges to a fixed point of T.*

For proof, please notice that if T is an ESN mapping with the corresponding function f, then T_α is a Suzuki nonexpansive mapping with the corresponding function $g := \alpha f$. The rest of the proof can be followed in [13], Theorem 5.5.

5. Example

The setting of our choice in which we provide our illustrative example is the Poincaré half-plane, i.e., the set $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, in which the distance is defined as

$$d(p, q) = 2\ln\left(\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}\right),$$

for $p = (x_1, y_1)$ and $q = (x_2, y_2)$. On subset $C = \{(0, y) \in \mathbb{H}^2 \mid y \in [e^{1/2}, e^2]\}$, which is closed and convex, define the mapping

$$T\colon C \to C, \ \ T(0, y) = \begin{cases} (0, e^{1/\ln y}) & y \neq e^2 \\ (0, e^{3/4}) & y = e^2 \end{cases}.$$

Due to its particular form, the distance between two points from C, say, $p = (0, x)$ and $q = (0, y)$, is computed with the simplified formula $d(p, q) = |\ln x - \ln y|$. Additionally, since we are working on the vertical axis only, for the simplicity of notations, we shall identify the points with the second coordinate in the sequel.

Let us show that T is an enriched Suzuki nonexpansive mapping for $\alpha = \dfrac{2}{5}$. For this, we need to show that

$$d(T_{\frac{2}{5}}p, T_{\frac{2}{5}}q) \leq d(p,q) \tag{7}$$

for all p and q such that

$$\frac{1}{5}d(p,Tp) = \frac{1}{2}d(p,T_{\frac{2}{5}}p) \leq d(p,q). \tag{8}$$

We have the following cases:

Case I: For $p = e^x$ and $q = e^y$, such that $(e^x, e^y) \in [e^{1/2}, e^2) \times [e^{1/2}, e^2)$, we have $Tp = e^{1/x}$, $Tp = e^{1/y}$, $T_{\frac{2}{5}}p = \frac{3}{5}e^x \oplus \frac{2}{5}e^{1/x} = e^{\frac{3}{5}x + \frac{2}{5}\frac{1}{x}}$, and $T_{\frac{2}{5}}q = \frac{3}{5}e^y \oplus \frac{3}{5}e^{1/y} = e^{\frac{3}{5}y + \frac{2}{5}\frac{1}{y}}$. Applying the distance formula, condition (7) becomes

$$\left|\frac{3}{5}(x-y) + \frac{2}{5}\left(\frac{1}{x} - \frac{1}{y}\right)\right| \leq |x-y|,$$

which holds for any pair $(x,y) \in \left[\frac{1}{2}, 2\right) \times \left[\frac{1}{2}, 2\right)$ due to the equivalences

$$\left|3 - \frac{2}{xy}\right| \leq 5 \Leftrightarrow -5 \leq 3 - \frac{2}{xy} \leq 5 \Leftrightarrow -8 \leq -\frac{2}{xy} \leq 2 \Leftrightarrow 4 \geq \frac{1}{xy} \geq -1.$$

Case II: Now let $p = e^x$ and $q = e^y$, with $x \in \left[\frac{1}{2}, 2\right)$ and $y = 2$. Condition (8), after applying the distance formula, writes as

$$\frac{1}{5}\left|x - \frac{1}{x}\right| \leq |x-2| \tag{9}$$

and we have the following subcases:

a: For $x \geq 1$, the inequality (9) leads to $6x^2 - 10x - 1 \leq 0$, that is, $x \in \left[\dfrac{5-\sqrt{31}}{6}, \dfrac{5+\sqrt{31}}{6}\right]$.

b: For $x < 1$, similarly, we obtain $4x^2 - 10x - 1 \leq 0$, i.e., $x \in \left[\dfrac{5-\sqrt{21}}{4}, \dfrac{5+\sqrt{21}}{4}\right]$.

Therefore, since

$$\frac{5-\sqrt{31}}{6} < \frac{5-\sqrt{21}}{4} < \frac{1}{2} < \frac{5+\sqrt{31}}{6} < \frac{5+\sqrt{21}}{4} < 2,$$

it remains to be shown that condition (7) holds for $x \in \left[\dfrac{1}{2}, \dfrac{5+\sqrt{31}}{6}\right]$ and $y = 2$. Indeed, as condition (7) becomes

$$\left|\frac{3}{5}(x-2) + \frac{2}{5}\left(\frac{1}{x} - \frac{3}{4}\right)\right| \leq |x-2|, \tag{10}$$

we have, as well, two subcases:

1: If $\dfrac{3}{5}(x-2) + \dfrac{2}{5}\left(\dfrac{1}{x} - \dfrac{3}{4}\right) \geq 0$, then (10) is

$$\frac{3}{5}(x-2) + \frac{2}{5}\left(\frac{1}{x} - \frac{3}{4}\right) \leq 2 - x,$$

leading to the equivalences

$$\frac{1}{x} - \frac{3}{4} \leq 4(2-x) \Leftrightarrow 16x^2 - 35x + 4 \leq 0.$$

As $16\left(\frac{1}{2}\right)^2 - 35\frac{1}{2} + 4 = -\frac{3}{2} < 0$ and $16 \cdot 2^2 - 35 \cdot 2 + 4 = -2 < 0$, we conclude that the last inequality holds for any $x \in [1/2, 2]$.

2: On the other hand, for $\frac{3}{5}(x-2) + \frac{2}{5}\left(\frac{1}{x} - \frac{3}{4}\right) < 0$, (10) becomes

$$\frac{3}{5}(2-x) + \frac{2}{5}\left(\frac{3}{4} - \frac{1}{x}\right) \leq 2 - x,$$

that is

$$\frac{3}{4} - \frac{1}{x} \leq 2 - x \Leftrightarrow 4x^2 - 5x - 4 \leq 0 \Leftrightarrow x \in \left[\frac{5-\sqrt{89}}{8}, \frac{5+\sqrt{89}}{8}\right].$$

Therefore, since $\frac{5+\sqrt{31}}{6} < \frac{5+\sqrt{89}}{8}$, condition (7) holds for all $x \in \left[\frac{1}{2}, \frac{5+\sqrt{31}}{6}\right]$.

Case III: Lastly, for $p = e^x$ and $q = e^y$, where $x = 2$ and $y \in \left[\frac{1}{2}, 2\right)$, condition (8) writes as

$$\frac{1}{5}\left|2 - \frac{3}{4}\right| \leq |y-2| \Leftrightarrow y \leq \frac{7}{4}, \tag{11}$$

and condition (7) becomes

$$\left|\frac{3}{5}(y-2) + \frac{2}{5}\left(\frac{1}{y} - \frac{3}{4}\right)\right| \leq |y-2|, \tag{12}$$

which has already been established above but for x instead of y. Noticing as before that $\frac{7}{4} < \frac{5+\sqrt{89}}{8}$, we conclude that (12) holds for all $y \in \left[\frac{1}{2}, \frac{7}{4}\right]$, which ends the proof of the fact that T is an enriched Suzuki nonexpansive mapping.

In order to see that T is not Suzuki nonexpansive in the sense of Definition 6, it is enough to take $p = e$ and $q = e^{1/2}$. Indeed, $\frac{1}{2}d(p, Tp) = \frac{1}{2}|1-1| \leq |1-2| = d(p,q)$, but $d(Tp, Tq) = |2-1| = 1 > |\frac{1}{2} - 1| = d(p,q)$.

Lastly, since T is discontinuous at $p = e^2$, it cannot be an enriched nonexpansive mapping as in Definition 10, and therefore, T does not belong to the more particular immediate subclasses.

6. Conclusions

In this paper, we introduced the class of enriched Suzuki mappings in Hadamard spaces. The obtained results extend several existing studies in this direction if we look to at least two different aspects: the setting and the class of involved operators. On the one hand, Hilbert spaces are particular Hadamard spaces. On the other hand, enriched Suzuki nonexpansive mappings are natural generalizations of enriched nonexpansive mappings, these ones naturally containing Suzuki nonexpansive mappings in Hadamard spaces. The new results are on the existence of fixed points for enriched Suzuki nonexpansive mappings and on the Δ and strong convergence of Picard iterates of the α-averaged mapping. As further development of our study, one can consider wider classes of operators defined in geodesic spaces.

Author Contributions: Conceptualization, T.T. and M.P.; validation, T.T. and M.P.; formal analysis, T.T. and M.P.; investigation, T.T. and M.P.; writing—original draft preparation, T.T.; writing—review and editing, M.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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