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Branching Random Walks in a Random Killing Environment with a Single Reproduction Source

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Abstract: We consider a continuous-time branching random walk on \mathbb{Z} in a random non-homogeneous environment. The process starts with a single particle at initial time $t = 0$. This particle can walk on the lattice points or disappear with a random intensity until it reaches the certain point, which we call the reproduction source. At the source, the particle can split into two offspring or jump out of the source. The offspring of the initial particle evolves according to the same law, independently of each other and the entire prehistory. The aim of this paper is to study the conditions for the presence of exponential growth of the average number of particles at every lattice point. For this purpose, we investigate the spectrum of the random evolution operator of the average particle numbers. We derive the condition under which there is exponential growth with probability one. We also study the process under the violation of this condition and present the lower and upper estimates for the probability of exponential growth.

Keywords: branching processes; random walks; branching random walks; random environments

MSC: 60J27; 60J80; 05C81; 60J85



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1. Introduction

In physical models with a random environment, phenomena can occur that differ substantially from what is usually encountered in statistical physics. In particular, the mean energy of the quantity under consideration can grow slower than the root of the mean square of this quantity, and both of these growth rates, in turn, are larger than the growth rate of a typical realization of the quantity under study. Such a phenomenon has been called intermittency (see, e.g., [1,2]). An example of such behavior is considered, in particular, in [3], where a model of particle population is considered. The intensity of particles splitting was assumed to be stationary in time and random in the spatial variables, with its mean value equal to zero. In addition, particle diffusion was included in the model.

The processes considered in papers [2,3] can be regarded as a special case of a branching random walk (BRW) in a random environment, which apparently was first presented in [1]. The authors introduced basic concepts for BRWs in a random environment and developed approaches to BRW analysis. The model studied in this work coincides with the previously introduced the parabolic Anderson model [4], where paper [1] itself is recognized as fundamental and has sparked active research on applications of the Anderson model in various fields [5].

The phenomenon of intermittency in the case of BRWs in random environments required the study of the asymptotic behavior of particle number moments averaged over the environment. In particular, it was required to study the regularity of the growth of such moments. The required asymptotics were obtained in [6] under the assumption of an asymptotically Weibull distribution of the right tail of the random potential, i.e., the difference between the splitting rate and the death rate. For the same potential but for a non-homogeneous model, similar results were obtained in [7]. For the case of a random subexponential potential results were obtained in [8,9]. The case of the Pareto potential was studied in [10]. Thus, the general question of the existence of intermittency in the BRW model has been practically fully investigated.

Further investigations focused on the non-averaged characteristics of BRWs, e.g., non-averaged moments in [11] or survival probability in [12]. These characteristics are more difficult to study, but they provide an opportunity to describe not only the qualitative but also the quantitative behavior of individual realizations of the process. One of the main tools for studying such problems is the study of the spectrum of the corresponding random operator.

The present paper extends the study of the spectrum of a random operator to a model of the BRW in a non-homogeneous random environment. We investigate the simplest characteristic of a random spectrum, the spectral bifurcation, consisting of the existence and non-existence of a positive eigenvalue. We also investigate the conditions for the occurrence of this bifurcation and estimate its probabilistic characteristics. Note that some of the results for this model are announced in [13].

The BRW model is in demand in various natural sciences and humanities, at least in demography, where branching processes are often considered to demonstrate a realistic model for the distribution of people, despite its obvious simplicity, and in biology in similar problems (see, e.g., [14–18]). The introduction of a random environment into the BRW model expands the range of biological problems that can be modeled [19].

We consider a BRW as a model of a population process rather than a physical model. A simple but extremely important characteristic of such a process is the criticality of the growth rate of the particle population. It is known that in the case of a random environment the particle population can be exponentially decreasing or exponentially increasing [20]. Exponential growth of the particle population entails the presence of a positive eigenvalue in the spectrum of the evolution operator for the average number of the particles. Therefore, we consider the spectral bifurcation study as a tool for qualitative assessment of changes in process behavior rather than as the purpose of this paper.

We note separately that many authors have considered the branching process in a random environment; see, e.g., [21–23]. In such processes there is no spatial structure, so the developed research methods are not suitable for the study of our problem. We also note the early work [24], where a BRW in a random medium in discrete time was considered. However, this model has no connection with the parabolic Anderson model, and, thus, is far from the BRW model under consideration.

2. Model Description

Let us consider a branching random walk (BRW) on a one-dimensional lattice \mathbb{Z} with continuous time. On the lattice we define a field of independent identically distributed random variables $\mathcal{M} = \{\mu(x, \cdot), x \in \mathbb{Z} \setminus \{0\}\}$, which are defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that each random variable $\mu(x, \cdot)$ takes values from the closed interval $[0, c]$ with $c \geq 0$ and is a mixture of discrete and absolutely continuous random variables (r.v.). Also, we assume that the continuous component of $\mu(x, \cdot)$ has a positive density on $[0, c]$. The field \mathcal{M} forms on \mathbb{Z} a “random killing environment” that determines the intensity of particle death in the BRW. In addition, we introduce the parameter $\Lambda \geq 0$, which is responsible for the intensity of particle multiplication at zero, and parameter $\varkappa > 0$, which controls the intensity of particle walking on the lattice.

Suppose that at time $t = 0$ there is a realization of the \mathcal{M} field denoted by $\mathcal{M}(\omega) = \{\mu(x, \omega), x \in \mathbb{Z} \setminus \{0\}, \omega \in \Omega\}$. Also, assume that the process at time $t = 0$ starts with a single particle at some point $x \in \mathbb{Z}$. The further evolution proceeds as follows. Suppose the particle is at zero, then in time $h \rightarrow 0$ it can split into two particles with probability $\Lambda h + o(h)$, move equally likely to one of the neighboring points with probability $\varkappa h + o(h)$, and remain in place with probability $1 - \Lambda h - \varkappa h + o(h)$. Suppose the particle is at the point $x \neq 0$, then in time $h \rightarrow 0$ it can disappear with probability $\mu(x, \omega)h + o(h)$, move equally likely to one of the neighboring points with probability $\varkappa h + o(h)$, and remain in place with probability $1 - \mu(x, \omega)h - \varkappa h + o(h)$. The new particles evolve according to the same law independently of each other and of all prehistory.

The introduced process is Markovian, and can be described in terms of a set of exponential and polynomial variables. This description may be more convenient for the perception of the model. Let us introduce the average waiting time $\tau(x)$ at an arbitrary point $x \in \mathbb{Z}$:

$$\tau(x) = \begin{cases} (\varkappa + \Lambda)^{-1}, & \text{if } x = 0; \\ (\varkappa + \mu(x, \omega))^{-1}, & \text{if } x \neq 0. \end{cases}$$

The evolution of a particle located at point x is as follows. If the particle is at zero, it waits for an exponentially distributed time with parameter $\tau(0)^{-1}$, and then instantly splits in two or moves equiprobably to one of the neighboring lattice points. The choice between these two events is made with corresponding probabilities $\Lambda\tau(0)$ and $\varkappa\tau(0)$. If the particle is at a point x outside zero, it also waits an exponentially distributed time with parameter $\tau(x)^{-1}$ and then vanishes instantaneously or moves equiprobably to one of the neighboring lattice points. The choice between these two events is made with corresponding probabilities $\mu(x, \omega)\tau(x)$ and $\varkappa\tau(x)$. The evolution of particles occurs independently of each other and of all prehistory.

We will show later on that the branching process of particles at the point $x \in \mathbb{Z}$ can be conveniently described by the potential $V(x, \omega)$, which reflects the criticality of the branching process at each point:

$$V(x, \omega) = \begin{cases} \Lambda, & x = 0; \\ -\mu(x, \omega), & x \neq 0 \end{cases}$$

or

$$V(x, \omega) = \Lambda\delta_0(x) - \mu(x, \omega)(1 - \delta_0(x)).$$

The BRW at time t due to the Markov property can be completely described by the set of particle numbers at time t at the points $y \in \mathbb{Z}$ denoted by $N_t(y, \omega)$. However, $N_t(y, \omega)$ is a random variable and hence is difficult to investigate. Therefore, it is common to consider the average particle number [1,6]:

$$m_1(t, x, y, \omega) = \mathbb{E}_x N_t(y, \omega),$$

where \mathbb{E}_x is the mathematical expectation under the condition that at time $t = 0$ there is one particle at point x .

By F_μ we further denote the distribution function of $\mu(x)$. In this paper, we are interested in the probability $P(\Lambda, \varkappa, F_\mu)$ of the realization of an environment in which there is an exponential growth of $m_1(t, x, y, \omega)$ for given parameters Λ , \varkappa , and F_μ . We will refer to such exponential growth as “supercriticality”. The formal definition is as follows:

$$P(\Lambda, \varkappa, F_\mu) = \mathbb{P} \left\{ \omega \in \Omega : \exists \lambda, C(x, y) > 0 : \lim_{t \rightarrow \infty} \frac{m_1(t, x, y, \omega)}{C(x, y)e^{\lambda t}} = 1, \quad \forall x, y \in \mathbb{Z} \right\},$$

where $C(x, y) = C(x, y, \omega, \Lambda, \varkappa, F_\mu)$, $\lambda = \lambda(\omega, \Lambda, \varkappa, F_\mu)$. Note that we require exponential growth of the average particle population simultaneously in all points of the lattice. However, further we will show that this condition is equivalent to exponential growth at least

in one point. Intuitively speaking, the exponential growth at one point “is spread” over the whole lattice with the help of random walk.

The purpose of this paper is to estimate $P(\Lambda, \varkappa, F_\mu)$ as a function of Λ , \varkappa , and F_μ . To achieve this, we first use the standard approach described, e.g., in [6,7,25], and write the Cauchy problem for $m_1(t, x, y, \omega)$:

$$\begin{aligned} \frac{\partial m_1(t, x, y, \omega)}{\partial t} &= (\varkappa \Delta m_1)(t, x, y, \omega) + V(x, \omega) m_1(t, x, y, \omega), \\ m_1(0, x, y) &= \delta_y(x), \end{aligned} \tag{1}$$

where $\varkappa \Delta f(x) = \frac{\varkappa}{2} \sum_{|x'-x|=1} (f(x') - f(x))$ is the discrete Laplace operator on \mathbb{Z} , and the sign $|\cdot|$ denotes the lattice distance on the l_1 norm. Here and below we assume that all operators are defined on $l_2(\mathbb{Z})$.

Let us introduce a random self-adjoint operator $H(\omega) = \varkappa \Delta + V(x, \omega)$ to rewrite the problem (1) in a simpler form:

$$\begin{aligned} \frac{\partial m_1(t, x, y, \omega)}{\partial t} &= H(\omega) m_1(t, x, y, \omega), \\ m_1(0, x, y) &= \delta_y(x). \end{aligned} \tag{2}$$

In problems of this kind, the behavior of $m_1(t, x, y, \omega)$ depends on the spectrum structure of the random operator $H(\omega)$. Therefore, the present work is mainly devoted to the study of the spectrum of the $H(\omega)$. In Sections 3 and 4, it is shown that the spectrum of $\sigma(H(\omega))$ consists of a non-positive non-random part and can contain a positive random eigenvalue; in Section 5 we derive the condition under which $P(\Lambda, \varkappa, F_\mu) = 1$; we address violation of this condition in Sections 6 and 7, where we present the lower and upper estimates for $P(\Lambda, \varkappa, F_\mu)$. The main proofs are given in the text of the paper after the corresponding statements, while the auxiliary proofs are placed in Section 9.

3. The Non-Random Part of the Spectrum of the Evolutionary Operator

We obtained the results of this and the next section using the technique described in [20]. In these sections, we prove the results for the Cauchy problem in arbitrary dimension $d \in \mathbb{N}$, although the case $d = 1$ is sufficient to study our model. Consider the following Cauchy problem for $m_1(t, x, y, \omega)$:

$$\begin{aligned} \frac{\partial m_1(t, x, y, \omega)}{\partial t} &= (\varkappa \Delta m_1)(t, x, y, \omega) + V(x, \omega) m_1(t, x, y, \omega), \\ m_1(0, x, y) &= \delta_y(x), \end{aligned} \tag{3}$$

where $\varkappa \Delta f(x) = \frac{\varkappa}{2d} \sum_{|x'-x|=1} (f(x') - f(x))$ is the discrete Laplace operator on \mathbb{Z}^d , and the sign $|\cdot|$ denotes the lattice distance on the l_1 norm.

For convenience of reasoning, let us introduce the averaging operator:

$$(\varkappa \bar{\Delta} f)(x) = \frac{\varkappa}{2d} \sum_{|x'-x|=1} f(x'),$$

where $\varkappa \bar{\Delta} f(x) = \frac{\varkappa}{2d} \sum_{|x'-x|=1} f(x')$. The Laplace operator $\varkappa \Delta$ can be represented as the difference of the averaging operator and the multiplication operator:

$$(\varkappa \Delta f)(x) = \varkappa \bar{\Delta} f(x) - \varkappa f(x).$$

Consider an auxiliary operator $H_\mu(\omega)$ for which the splitting intensity at zero is absent and the death intensity at zero $\mu(0, \omega)$ is defined in the same way as $\mu(x, \omega)$ for $x \in \mathbb{Z} \setminus \{0\}$.

$$H_\mu(\omega) = \varkappa \Delta - \mu(x, \omega) = \varkappa \bar{\Delta} - \varkappa - \mu(x, \omega).$$

The operator $H(\omega)$ can be viewed as a random one-point perturbation of the operator $H_\mu(\omega)$ at zero. Therefore, the essential spectra of these operators coincide [26]:

$$\sigma_{ess}(H(\omega)) = \sigma_{ess}(H_\mu(\omega)).$$

Furthermore, a single-point perturbation can only produce at most one positive eigenvalue. Thus, the first problem is to investigate the essential spectrum of the operator $H_\mu(\omega)$.

For convenience, we give the formulations of the lemmas from the works of [26,27], which will be needed to study the spectrum of the operator $H_\mu(\omega)$.

Lemma 1 (see, e.g., [26]). *The number λ belongs to the essential spectrum of the operator H_μ if we can construct a sequence of “almost eigenfunctions”, i.e.,*

$$\exists \{f_n \in l_2(\mathbb{Z}^d) : \|f_n\| = 1, (f_n, f_m) = \delta(n, m), \|H_\mu f_n - \lambda f_n\| \rightarrow 0, n \rightarrow \infty\}. \quad (4)$$

Lemma 2 (see, e.g., [27]). *The spectrum of the operator $\varkappa\Delta$ is equal to $[-2\varkappa; 0]$. For an eigenvalue $\lambda \in [-2\varkappa; 0]$, there exists a representation*

$$\lambda = \frac{\varkappa}{d} \sum_{i=1}^d \cos(\phi_i) - \varkappa,$$

for some $\vec{\phi} = (\phi_1, \dots, \phi_d)$, $\phi_i \in [-\pi, \pi]$. The corresponding function $\psi_\lambda(x) = \exp\{i(\vec{\phi}, x)\}$ is an eigenfunction for λ . As a consequence, the spectrum of the operator $\varkappa\bar{\Delta}$ is equal to $[-\varkappa; \varkappa]$.

Using these lemmas and the proof scheme from [20], we obtain the following result.

Lemma 3. *The spectrum of the operator $H_\mu(\omega)$ almost surely consists of only the essential part, which is equal to the interval $[-2\varkappa - c; 0]$.*

Proof. The operator $H_\mu(\omega)$ is the sum of the averaging operator $\varkappa\bar{\Delta}$ and the multiplication by the function $-\mu(x, \omega) - \varkappa$. Due to Lemma 2, the operator $\varkappa\bar{\Delta}$ has a spectrum equal to $[-\varkappa; \varkappa]$ and a norm equal to \varkappa . In turn, the spectrum of the operator of the multiplication by the $-\mu(x, \omega)$ is equal to the closure of the set of values of this function. For almost sure (a.s.) any ω , this closure is equal to the interval $[-c; 0]$ by virtue of the definition of $\mu(x, \omega)$. Therefore, the spectrum of the combined operator $-\mu(x, \omega) - \varkappa$ is equal to $[-\varkappa - c; -\varkappa]$.

The operator $H_\mu(\omega)$ can be considered as a perturbation of the self-adjoint operator $-\mu(x, \omega) - \varkappa$ by the self-adjoint operator $\varkappa\bar{\Delta}$. In such a case, according to perturbation theory [28], the spectrum of the operator H_μ will differ from the interval $[-\varkappa - c, -\varkappa]$ by at most \varkappa , leading to the following inclusion:

$$\sigma(H_\mu) \subseteq [-2\varkappa - c; 0]. \quad (5)$$

To show the inverse inclusion, we use Lemma 1, and for each $\lambda \in [-2\varkappa - c; 0]$ we will construct a sequence of “almost eigenfunctions” $\{f_n(x)\}$, $f_i(x) \in l_2(\mathbb{Z}^d)$. Note that we construct a sequence function for each fixed ω , i.e., $\{f_n(x)\} = \{f_n(x, \omega)\}$.

Let us represent λ as $\lambda = a + b$, $a \in [-2\varkappa; 0]$ $b \in [-c; 0]$. We need to construct f_n such that the approximations $\varkappa\Delta f_n \approx a f_n$ and $-\mu(x, \omega) f_n \approx b f_n$ are true in some sense because then,

$$(\varkappa\Delta - \mu(x, \omega)) f_n \approx (a + b) f_n = \lambda f_n.$$

The condition $\varkappa\Delta f_n \approx a f_n$ requires that the function be “almost everywhere”, similar to $\exp\{i(\vec{\phi}, x)\}$ of Lemma 2 with a suitable $\vec{\phi}$. The condition $-\mu(x, \omega) f_n \approx b f_n$ requires that the function be non-zero only on the region where $-\mu(x, \omega) \approx b$.

The functions satisfying both conditions are indicators of the balls on which $-\mu(x, \omega) \in [b - \varepsilon; b + \varepsilon]$ for sufficiently small $\varepsilon > 0$. Therefore, $-\mu(x, \omega) \approx b$ and the multiplication operator will act “almost eigen-like”. The diffusion operator $\varkappa\Delta$ will act “almost eigen-like” inside and outside such balls, but not at the boundary. Therefore, the radius of the balls must increase to infinity so that the “non-eigen” action of $\varkappa\Delta$ tends to zero. The exact construction of the system of functions $\{f_n\}$ and the proof they are “almost eigenfunctions” is given in Section 9.1.

In summary, for any $\lambda \in [-2\varkappa - c; 0]$ we can construct a sequence of “almost eigenfunctions” $\{f_n\}$, and hence

$$\sigma(H_\mu) \supseteq [-2\varkappa - c; 0]. \tag{6}$$

Inclusions (5) and (6) complete the proof of the lemma. \square

Let us summarize the result of this chapter. The auxiliary operator $H_\mu(\omega)$ a.s. has a non-random essential spectrum $[-2\varkappa - c; 0]$.

4. The Random Part of the Spectrum of the Evolutionary Operator

Let us return to the operator $H(\omega) = \varkappa\Delta + V(x, \omega)$. As we have already mentioned, it can be viewed as a random one-point perturbation of the previously described operator H_μ with an essential spectrum $\sigma(H_\mu) = [-2\varkappa - c; 0]$. By the Weyl criterion [26], under compact perturbation the essential spectrum of the operator does not change, while one positive eigenvalue may appear, which we will denote by $\lambda(\omega)$:

$$\sigma(H(\omega)) = [-2\varkappa - c; 0] \cup \lambda(\omega).$$

As we mentioned, under Equation (2) the structure of the $\sigma(H(\omega))$ defines the behavior of BRW. In particular, if $\lambda(\omega) > 0$, an exponential growth of the average particle number is observed; see, e.g., [27]. Thus, the study of the probability of exponential growth is reduced to the study of the probability of the appearance of a positive eigenvalue:

$$P(\Lambda, \varkappa, F_\mu) = \mathbb{P}\{\omega : \exists \lambda(\omega) \in \sigma(H(\omega)) : \lambda(\omega) > 0\}.$$

Let us formulate the problem of finding the eigenvalue of $\lambda(\omega)$ with the corresponding eigenfunction $u(x)$. Note that from $u(0) = 0$ it follows that $u(x) \equiv 0$. Therefore, without restricting generality, let $u(0) = 1$:

$$\begin{aligned} (\varkappa\Delta + V(x, \omega))u(x) &= \lambda u(x), \\ u(0) &= 1. \end{aligned} \tag{7}$$

For convenience, Equation (7) can be decomposed into two equations. When $x = 0$ it takes the following form:

$$\begin{aligned} (\varkappa\Delta + \Lambda - \lambda)u(0) &= 0, \\ u(0) &= 1. \end{aligned} \tag{8}$$

When $x \neq 0$, it takes the following form:

$$\begin{aligned} (\varkappa\Delta - \mu(x, \omega) - \lambda)u(x) &= 0, \\ u(0) &= 1. \end{aligned} \tag{9}$$

For simplicity of the formulas, we introduce the following notations:

$$E = \varkappa + \lambda.$$

Due to this notation, Equation (9) takes the following form:

$$\begin{aligned} (\varkappa\bar{\Delta} - (\mu(x, \omega) + E))u(x) &= 0, \quad x \neq 0, \\ u(0) &= 1. \end{aligned} \tag{10}$$

Let us move on to finding the solution to this system of equations.

Lemma 4. *The solution to Equation (10) is given by the following formula:*

$$u(x) = \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2d}{\mu(z, \omega) + E} \right), \tag{11}$$

where by $\gamma : a \rightsquigarrow b = \{a = x_1, \dots, x_n \neq b\}$ we denote the path from point a to point b through the neighboring points of the lattice and $a)$ the path does not intersect 0 and $b)$ point b is considered to be excluded from the path γ . The solution given by Formula (11) makes sense for any $\lambda > 0$ in any dimension $d \in \mathbb{N}$.

The first part of the lemma is verified by directly substituting Formula (11) into the problem (10). The correct definiteness of the expression (11) for any $\lambda > 0$ in dimension is checked by combinatorial reasoning and asymptotic methods. The full proof of Lemma 4 is given in Section 9.2.

Remark 1. *Lemma 4 is a special case of the popular (especially in the physics literature) path expansion of the resolvent, but it is usually applied to the λ from an essential Laplacian spectrum, i.e., $\lambda < 0$ in our case. For such λ formula, (11) is incorrect due to the small denominators. Therefore, one has to study complex λ and later pass to the limit $\text{Im } \lambda \rightarrow 0$; see details, e.g., in Lecture 6 of [20] or in [29].*

Our goal is to understand under what conditions there exists an isolated positive eigenvalue λ_0 of the operator H_μ , perturbed by the reproduction potential $(\Lambda - \mu)\delta_0(x)$. In such a case (11) is well defined and Lemma 4 is probably new. Let us stress that $u(x)$ is the resolvent kernel of H_μ with some normalization. In fact, $u(x) = \frac{R_\lambda(0,x)}{R_\lambda(0,0)}$ for $\lambda > 0$.

Remark 2. *The essential spectrum is the non-random support of the random spectral measure of $H_\mu(\omega)$. Under the condition that r.v. $\mu(x)$ has absolutely continuous distribution, it follows from the general theory of one-dimensional random Schrödinger operator on $l_2(\mathbb{Z})$ that the spectral measure is pure point and eigenfunctions are almost surely exponentially decreasing (exponential localization).*

This result is very old (see details in [20,29–31]), and we will not discuss this topic. Our case is the analysis of the spectral bifurcation: existence and non-existence of the positive eigenvalue.

5. Condition for Almost Certainly Supercritical BRW Behavior

Let us calculate the environment-independent interval in which the eigenvalue of the problem (7) lies. For this purpose, let us consider the “best” and the “worst” realizations of the environments. Namely, by setting $\mu = 0$ at all points, then setting $\mu = c$ at all points, we can obtain the following result.

Theorem 1. *The value $P(\Lambda, \varkappa, F_\mu)$ is equal to one if and only if the following condition is satisfied:*

$$\Lambda \geq \sqrt{(\varkappa + c)^2 - \varkappa^2} - c. \tag{12}$$

If the condition (12) is satisfied, then for any realization of the environments ω , the eigenvalue of $\lambda(\omega)$ lies in the interval

$$\lambda(\omega) \in \left[\sqrt{(\Lambda + c)^2 + \varkappa^2} - (\varkappa + c); \sqrt{\Lambda^2 + \varkappa^2} - \varkappa \right].$$

Proof. Consider Equation (8):

$$\begin{aligned} (\varkappa\Delta + \Lambda - \lambda)u(0) &= 0, \\ u(0) &= 1. \end{aligned} \tag{13}$$

Given the notation $E = \varkappa + \lambda$, it can be rewritten as follows:

$$(\varkappa \bar{\Delta} u)(0) + \Lambda - E = 0. \tag{14}$$

In dimension $d = 1$, the expression (14) takes a simpler form:

$$\frac{\varkappa}{2}(u(1) + u(-1)) + \Lambda = E.$$

Moreover, if μ is equal to some constant at all points, then $u(1) = u(-1)$ and the expression is further simplified:

$$\varkappa u(1) + \Lambda = E. \tag{15}$$

Note that for arbitrary environment ω , the solution $u(1)$ is bounded above and below by the solutions for the environments in which $\mu \equiv 0$ and $\mu \equiv c$. Let us find these estimates.

Let $\mu(x, \omega)$ be equal to some constant c_1 at all points. In this case, $u(1)$ is defined using Equation (11) as follows:

$$u(1) = \sum_{\gamma:1 \rightarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2}{\mu(z, \omega) + E} \right) = \sum_{\gamma:1 \rightarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2}{E + c_1} \right) = \sum_{\gamma:1 \rightarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2}{E_1} \right), \tag{16}$$

where $E_1 = E + c_1$.

Reasoning using the reflection principle as in the proof of Lemma 4 (see Section 9.2) allows us to write out the series in the expression (16) exactly. First, let us compute $L(1, 0, n)$, that is, the number of paths that start at 1, end at 0, contain n points, and do not intersect 0. Note that $L(1, 0, 1) = 1$, and for the remaining odd n according to reasoning (37) in Section 9.2 the following is true:

$$L(1, 0, n) = C_{n-1}^{\frac{n-1}{2}} - C_{n-1}^{\frac{n+1}{2}} = \frac{2}{n+1} C_{n-1}^{\frac{n-1}{2}}, \quad n = 3, 5, \dots$$

Thus, let us write out $u(1)$ from the expression (16):

$$\begin{aligned} u(1) &= \sum_{\gamma:1 \rightarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2}{E_1} \right) = \sum_{n=1,3,\dots} L(1, 0, n) \cdot \left(\frac{\varkappa/2}{E_1} \right)^n \\ &= \sum_{n=1,3,\dots} \frac{2}{n+1} C_{n-1}^{\frac{n-1}{2}} \cdot \left(\frac{\varkappa/2}{E_1} \right)^n = \sum_{m=0,2,\dots} \frac{2}{m+2} C_m^{m/2} \cdot \left(\frac{\varkappa/2}{E_1} \right)^{m+1} \\ &= \frac{\varkappa/2}{E_1} \sum_{k=0}^{\infty} \left(\frac{C_{2k}^k}{1+k} \right) \cdot \left(\frac{\varkappa/2}{E_1} \right)^{2k}. \end{aligned} \tag{17}$$

For convenience, we denote $\frac{\varkappa/2}{E_1}$ by a . The coefficient $\frac{C_{2k}^k}{1+k}$ is the Catalan number, so the series (17) can be calculated exactly; see, e.g., [32]:

$$\begin{aligned} u(1) &= a \sum_{k=0}^{\infty} \left(\frac{C_{2k}^k}{1+k} \right) \cdot a^{2k} = a \frac{1 - \sqrt{1 - 4a^2}}{2a^2} \\ &= \frac{1}{2a} - \sqrt{\frac{1}{4a^2} - 1} = \frac{E_1}{\varkappa} - \sqrt{\left(\frac{E_1}{\varkappa} \right)^2 - 1}. \end{aligned} \tag{18}$$

After substituting (18), the expression (15) takes the following form:

$$E = \Lambda + E_1 - \sqrt{E_1^2 - \varkappa^2}.$$

Since $E_1 = E + c_1$, the expression takes the form

$$E = \Lambda + (E + c_1) - \sqrt{(E + c_1)^2 - \varkappa^2}.$$

From here, we can calculate that

$$\Lambda + c_1 = \sqrt{(E + c_1)^2 - \varkappa^2}$$

or, finally,

$$\lambda = \sqrt{(\Lambda + c_1)^2 + \varkappa^2} - (c_1 + \varkappa).$$

Substituting $c_1 = 0$ and $c_1 = c$ completes the proof of the lemma. \square

6. Upper Estimate for $P(\Lambda, \varkappa, F_\mu)$

In the previous section, we found that under the condition $\Lambda \geq \sqrt{(\varkappa + c)^2 - \varkappa^2} - c$ the BRW is a.s. supercritical, i.e., $P(\Lambda, \varkappa, F_\mu) = 1$. The goal of this and the next section is to give estimates for $P(\Lambda, \varkappa, F_\mu)$ when this condition is violated.

To obtain an estimate from above, let us fix a non-random “poor” environment and find out when it does not generate a positive eigenvalue. The poorer environments also do not generate an eigenvalue. If the probability of generating a family of poor environments is P_1 , then $P(\Lambda, \varkappa, F_\mu) < 1 - P_1$. In this paper, we consider the simplest case: an environment that takes some negative values at points neighboring zero.

Lemma 5. Consider an environment ω_1 in which points neighboring from zero have killing intensities equal to μ_1 and μ_{-1} . A positive eigenvalue in this environment exists if and only if the following condition is met:

$$\Lambda > \frac{\mu_1 + \mu_{-1} + 2\sigma\mu_1\mu_{-1}}{(1 + \sigma\mu_1)(1 + \sigma\mu_{-1})}, \tag{19}$$

where $\sigma = \frac{1}{\varkappa/2}$ for $z \in \mathbb{R}$.

Let us give the general idea of the proof. The eigenvalue problem for the considered environment has the following form:

$$\begin{aligned} (\varkappa\Delta + V(x, \omega))u(x) &= \lambda u(x), \\ u(0) &= 1, \end{aligned} \tag{20}$$

where

$$V(x, \omega) = \begin{cases} \Lambda, & x = 0; \\ -\mu_1, & x = 1; \\ -\mu_{-1}, & x = -1; \\ 0, & |x| \geq 2. \end{cases}$$

In the appendix, we show that $u(x)$ must have the following form:

$$u(x) = \begin{cases} 1, & x = 0; \\ C_1 e^{-kx}, & x \geq 1; \\ C_{-1} e^{kx}, & x \leq -1, \end{cases} \tag{21}$$

where $C_{\pm 1}$ and k are some positive constants.

Then, we substitute (21) into (20) and derive the condition that is equivalent to the existence of positive eigenvalue λ with corresponding eigenfunction ψ_λ . It turns out that this is the condition (20), which completes the proof of the lemma. The proof is quite technical and is given in Section 9.3.

Now consider the general set of environments

$$\Omega_1 = \{\omega \in \Omega : \mu(1, \omega) = \mu_1, \mu(-1, \omega) = \mu_{-1}\}.$$

Note that the average number of particles in the non-random environment ω_1 is a.s. greater than the average number of particles in a population in any environment from Ω_1 . Suppose that the condition (19) is satisfied for ω_1 . In such a case, nothing can be said about the eigenvalues of the environments from Ω_1 . Suppose that the condition (19) is not satisfied for ω_1 . Then, according to the previous Lemma 5, there is no positive eigenvalue for ω_1 and hence there is no positive eigenvalue for all environments from Ω_1 .

Let us denote the event “condition (19) is met” by A and write the previous reasoning more formally:

$$\begin{aligned} P(\Lambda, \varkappa, F_\mu) &= \mathbb{P}\{\exists \lambda(\omega) > 0\} = \mathbb{P}\{\exists \lambda(\omega) > 0|A\}\mathbb{P}(A) + \mathbb{P}\{\exists \lambda(\omega) > 0|\bar{A}\}\mathbb{P}(\bar{A}) \\ &= \mathbb{P}\{\exists \lambda(\omega) > 0|A\}\mathbb{P}(A) + 0 \cdot \mathbb{P}(\bar{A}) \leq \mathbb{P}(A). \end{aligned} \tag{22}$$

The event “condition (19) is met” for a random environment is written as follows:

$$\mathbb{P}(A) = \mathbb{P}\left\{ \Lambda > \frac{\xi_1 + \xi_2 + 2\sigma\xi_1\xi_2}{(1 + \sigma\xi_1)(1 + \sigma\xi_2)} \right\},$$

where ξ_i are independent copies $\mu(x, \omega)$. Thus, we obtain the following theorem.

Theorem 2. *The following upper bound estimate is true:*

$$P(\Lambda, \varkappa, F_\mu) \leq \mathbb{P}\left\{ \Lambda > \frac{\xi_1 + \xi_2 + 2\sigma\xi_1\xi_2}{(1 + \sigma\xi_1)(1 + \sigma\xi_2)} \right\},$$

where ξ_i are independent copies $\mu(x, \omega)$.

7. Lower Estimate for $P(\Lambda, \varkappa, F_\mu)$

The first method for obtaining a lower estimate for $P(\Lambda, \varkappa, F_\mu)$ is to consider some convenient function $\psi(x)$ and examine the quadratic form $(H(\omega)\psi, \psi)$. If for some $a > 0$ the quadratic form $(H(\omega)\psi, \psi)$ is positive with probability p_a , then the operator $H(\omega)$ has a positive eigenvalue with probability p_a at least. We have chosen the simple function $\psi(x) = 2^{-a|x|}, x \in \mathbb{Z}$ and this reasoning leads to the following theorem.

Theorem 3. *The following estimate from below is true:*

$$P(\Lambda, \varkappa, F_\mu) \geq \max_{a \in (0; \infty)} \mathbb{P}\left(\omega : \Lambda > \varkappa \frac{(2^a - 1)}{(2^a + 1)} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{4^{a|x|}} \right).$$

In particular, for $a = 1$:

$$P(\Lambda, \varkappa, F_\mu) \geq \mathbb{P}\left(\omega : \Lambda > \frac{\varkappa}{3} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{4^{|x|}} \right).$$

The proof of the theorem requires direct investigation of the quadratic form $(H(\omega)\psi, \psi)$ for the function $\psi(x) = 2^{-a|x|}, x \in \mathbb{Z}$, which is a technical task, and so the proof is placed in Section 9.4.

The second way to obtain an upper estimate of $P(\Lambda, \varkappa, F_\mu)$ uses the idea of Lemma 5. We consider a non-random killing environment of simplified form that can form “islands” around zero without killing. For this environment, we study the eigenvalue problem

and then generalize the conclusion to all environments that are “better” than the one under consideration.

First, let us denote $P(\mu(x, \omega)) = 0$ by p . Random variables $\mu(x, \omega)$ can form an “island” around zero with probability p^{2l} . Let us denote such a case by Ω_l :

$$\Omega_l = \{\omega \in \Omega : \mu(i, \omega) = 0, \forall i \in -l, \dots, l\}.$$

Let us use an idea from Lemma 5 and consider a non-random environment ω_l of the following form:

$$\mu(x, \omega_l) = \begin{cases} 0 & \text{for } x \in -l, \dots, l; \\ c & \text{for } x \notin -l, \dots, l. \end{cases}$$

The environment ω_l admits a direct calculation of the condition on the positivity of the eigenvalue of the corresponding operator, which is presented by the following lemma. The proof of the lemma is technical and is therefore placed in Section 9.5.

Lemma 6. *If a positive eigenvalue exists for all $\omega \in \Omega_l$, then it is bounded from below by a solution with respect to λ of the following equation:*

$$\frac{2\alpha\kappa}{1 + \sqrt{1 - 4\alpha^2}} + \kappa\alpha^{2l} \cdot R(\alpha, \beta) + \Lambda - \kappa - \lambda = 0, \tag{23}$$

where $\alpha = \frac{\kappa/2}{\kappa+\lambda}$, $\beta = \frac{\kappa/2}{c+\kappa+\lambda}$, and the expression R is defined as

$$R(\alpha, \beta) = \sum_{k=0}^{\infty} (\beta^{2k+1} - \alpha^{2k+1}) C_{k+l}, \tag{24}$$

where C_n denotes the n -th Catalan number. If the series in Equation (24) does not converge then there exists a $\omega \in \Omega_l$ for which there is no positive eigenvalue.

Now, using Lemma 6 we find the smallest positive number \hat{l} such that Equation (23) admits a positive solution. By the lemma, all environments of $\Omega_{\hat{l}}$ will have positive eigenvalues. Therefore the probability $P(\Lambda, \kappa, F_{\mu})$ is at least equal to the probability of generating an environment from $\Omega_{\hat{l}}$ or, equivalently, the probability of generating a \hat{l} -island. Finally, the probability of generating a \hat{l} -island is $p^{2\hat{l}} = (\mathbb{P}\{\mu(x, \omega) = 0\})^{2\hat{l}}$, which leads to the following theorem.

Theorem 4. *There is the following estimation from below:*

$$P(\Lambda, \kappa, F_{\mu}) \geq (\mathbb{P}\{\mu(x, \omega) = 0\})^{2\hat{l}},$$

where $\hat{l} \in \mathbb{N}$ is the smallest number for which the expression described below admits a positive solution. If there is no such \hat{l} , then $P(\Lambda, \kappa, F_{\mu}) = 0$.

$$\frac{2\alpha\kappa}{1 + \sqrt{1 - 4\alpha^2}} + \kappa\alpha^{2l} \cdot R(\alpha, \beta) + \Lambda - \kappa - \lambda = 0, \tag{25}$$

where $\alpha = \frac{\kappa/2}{\kappa+\lambda}$, $\beta = \frac{\kappa/2}{c+\kappa+\lambda}$, and the expression R is defined as follows:

$$R(\alpha, \beta) = \sum_{k=0}^{\infty} (\beta^{2k+1} - \alpha^{2k+1}) C_{k+l}, \tag{26}$$

where C_n denotes the Catalan number.

At first sight, Theorem 4 is useless due to its excessive complexity. However, unlike Theorem 3, it offers a concrete numerical algorithm for estimating $P(\Lambda, \varkappa, F_\mu)$ based on a non-Monte Carlo method. Moreover, this algorithm will run fast because of the exponentially fast convergence of the series used in the theorem.

8. Conclusions

We have studied a previously unconsidered model of branching random walk with a single branching source and a killing random environment. The introduction of a random environment into the BRW model expands the range of biological problems that can be modeled using BRWs. In the present work, we investigated the probability of the presence of supercritical BRW growth. The developed approaches made it possible to estimate the spectrum of the corresponding random evolution operator.

The generalization of the obtained results can be carried out in several directions. For example, in Theorem 2 one can consider not two but several points in the neighborhood of zero, and in Theorem 3 one can consider a function of a more general form. A rather interesting problem is the generalization of Theorem 4, in which instead of an “island” with zero killing intensity, one can consider an “island” consisting of points with small but non-zero positive killing intensity. Also, one of the directions of further research is the numerical evaluation of the accuracy of the estimates obtained in this paper.

9. Proofs

9.1. Continuation of the Proof of Lemma 3

Let us recall that we need to prove the inclusion $\sigma(H_\mu) \supseteq [-2\varkappa - c; 0]$.

Proof. Let us use Lemma 1, and for each $\lambda \in [-2\varkappa - c; 0]$ construct a sequence of “almost eigenfunctions” $\{f_n(x)\}$, $f_i(x) \in L_2(\mathbb{Z}^d)$. Note that we construct a sequence function for each fixed ω , i.e., $\{f_n(x)\} = \{f_n(x, \omega)\}$. Let us represent λ as $\lambda = a + b$, $a \in [-2\varkappa; 0]$ $b \in [-c; 0]$. Let us construct f_n such that in some sense $\varkappa\Delta f_n \approx af_n$ and simultaneously $-\mu(x, \omega)f_n \approx bf_n$. Then,

$$(\varkappa\Delta - \mu(x, \omega))f_n \approx (a + b)f_n = \lambda f_n.$$

The condition $\varkappa\Delta f_n \approx af_n$ requires that the function be “almost everywhere”, similar to $\exp\{i(\vec{\phi}, x)\}$ of Lemma 2 with a suitable $\vec{\phi}$. The condition $-\mu(x, \omega)f_n \approx bf_n$ requires that the function be non-zero only on the region where $-\mu(x, \omega) \approx b$.

A candidate function satisfying both conditions looks like this:

$$f_n(x) = f_n(x, \omega) = \frac{1}{\sqrt{|B_n(\omega)|}} \exp\{i(\vec{\phi}, x)\} I\{B_n(\omega)\},$$

where the random set $B_n(\omega) = B_n$ contains the points $x \in \mathbb{Z}^d$, such that $-\mu(x, \omega) \in [b - \frac{1}{n}, b + \frac{1}{n}]$, and the multiplier $1/\sqrt{|B_n|}$ is needed to normalize the function.

Lemma 1 additionally requires orthogonality of almost eigenfunctions. Hence, it should be required that $B_m \cap B_n = 0$ for $n \neq m$. Furthermore, it will turn out in the proof that it should be required in advance that $|B_n| \rightarrow \infty$.

Let us prove the existence of the required sets $\{B_n\}$. For this, we fix an arbitrary number n and recall that the density of $-\mu(x, \omega)$ is positive on the interval $[-c; 0]$. According to the Borel–Cantelli lemma, for an arbitrary realization of ω there exists a system of non-intersecting balls $\{C_i(n)\}_{i=1}^\infty$ consisting of lattice points $x \in \mathbb{Z}^d$ such that

$$x \in C_i(n) \Rightarrow -\mu(x, \omega) \in \left[b - \frac{1}{n}, b + \frac{1}{n} \right], \tag{27}$$

$$|C_i(n)| \rightarrow \infty \text{ for } i \rightarrow \infty.$$

Now, the system of sets $\{B_n\}$ can be constructed by induction. Let the system $\{B_n\}$ be constructed up to the number k . Let us construct the system $\{C_i(k+1)\}$ described above. As the set B_{k+1} , we take any set of $\subset \{C_i(k+1)\}$ that is farther from zero than all points from B_1, \dots, B_n . Thus, the induction is complete and the system $\{B_n\}$ is constructed.

Let us verify that the functions $\{f_n\}$ are almost eigenfunctions. First, consider the action of the operator $\varkappa\Delta$ on the function f_n :

$$\varkappa\Delta f_n = \frac{1}{\sqrt{|B_n|}} \varkappa\Delta \exp\{i(\vec{\phi}, x)\} I\{B_n\}.$$

If the points $x - 1, x, x + 1$ lie inside B_n , then the diffusion operator acts on its eigenfunction:

$$\varkappa\Delta f_n(x) = a \frac{1}{\sqrt{|B_n|}} \exp\{i(\vec{\phi}, x)\} = a f_n(x).$$

If all points $x - 1, x, x + 1$ lie outside B_n , then the diffusion operator acts on the null function and also $\varkappa\Delta f_n(x) = 0 = a f_n(x)$.

Let at least one of their points $x - 1, x, x + 1$ lie on the boundary of B_n . In this case, there remains a non-zero function f_n^{res} after applying the operator:

$$\varkappa\Delta f_n(x) = a f_n(x) + f_n^{res}(x). \tag{28}$$

The function f_n^{res} reflects the “error” of the operator on the boundary of B_n with respect to the operator multiplying by a . This function is non-zero only at a finite number of points C_d , depending on the dimensionality but not on n . The norm f_n^{res} is bounded from above by $C_d/\sqrt{|B_n|}$, which tends to zero when $n \rightarrow \infty$.

Now consider the action of the operator $-\mu(x, \omega)$ on the function f_n . On the region $\{B_n\}$, the function $\mu(x, \omega)$ takes values in the interval $[b - \frac{1}{n}, b + \frac{1}{n}]$, so

$$\begin{aligned} -\mu(x, \omega) f_n(x) &= -\mu(x, \omega) \frac{1}{\sqrt{|B_n|}} \exp\{i(\vec{\phi}, x)\} I\{B_n\} \\ &= b \frac{1}{\sqrt{|B_n|}} \exp\{i(\vec{\phi}, x)\} I\{B_n\} + g_n^{res}(x) = b f_n(x) + g_n^{res}(x). \end{aligned} \tag{29}$$

The function g_n^{res} reflects the “error” of the operator on the area B_n with respect to the multiplication operator on b . The norm g_n^{res} is bounded from above by the value $1/n$, which tends to zero when $n \rightarrow \infty$.

Putting together the expressions (28) and (29), we obtain the following:

$$\|H_\mu f_n - \lambda f_n\| = \|(a + b) f_n - (a + b) f_n + f_n^{res} + g_n^{res}\| \rightarrow 0, \quad n \rightarrow \infty.$$

Thus, $\{f_n\}$ is the desired sequence of almost eigenfunctions, and $\lambda \subset \sigma(H)$. The number λ was taken arbitrarily from the interval $[-2\varkappa - c, 0]$, hence

$$\sigma(H_\mu) \supseteq [-2\varkappa - c; 0], \tag{30}$$

which completes the proof of the lemma. \square

9.2. Proof of Lemma 4

Let us recall the formulation of Lemma 4: The solution to Equation (10) when $x \neq 0$ is given by the following formula:

$$u(x) = \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2d}{\mu(z, \omega) + E} \right), \tag{31}$$

where $\gamma : a \rightsquigarrow b = \{a = x_1, \dots, x_n \neq b\}$ denotes the path from point a to point b through the neighboring points of the lattice, and a) the path does not intersect 0 and b) point b is considered to be excluded from the path γ . The solution given by formula (31) makes sense for any $\lambda > 0$ in any dimension $d \in \mathbb{N}$.

Proof. Note that, for a path $\gamma : x \rightsquigarrow 0$, the symbol $|\gamma|$ denotes the length of the path in the sense of “number of points in γ excluding zero” or “number of steps from x to 0” which are the same. For simplicity of notation, let us prove the lemma for the case of a one-dimensional lattice $d = 1$. Let us study the action of the operator $\varkappa\bar{\Delta}$ on the function $u(x)$ when $x \neq 0$:

$$\varkappa\bar{\Delta}u(x) = \frac{\varkappa}{2}(u(x + 1) + u(x - 1)), \tag{32}$$

Note that the set of paths $\gamma : x \rightsquigarrow 0$ included in $u(x)$ decomposes into two subsets: paths $\gamma_+ : x \rightsquigarrow x + 1 \rightsquigarrow 0$ and paths $\gamma_- : x \rightsquigarrow x - 1 \rightsquigarrow 0$. Thus,

$$\begin{aligned} u(x) &= \sum_{\gamma}(\cdot) = \frac{\varkappa/2}{\mu(x, \omega) + E} \sum_{\gamma_+}(\cdot) + \frac{\varkappa/2}{\mu(x, \omega) + E} \sum_{\gamma_-}(\cdot) \\ &= \frac{\varkappa/2}{\mu(x, \omega) + E}(u(x + 1) + u(x - 1)). \end{aligned} \tag{33}$$

Or the following, which is the same thing:

$$u(x + 1) + u(x - 1) = \frac{\mu(x, \omega) + E}{\varkappa/2}u(x). \tag{34}$$

Combining (32) and (34) we obtain:

$$\varkappa\bar{\Delta}u(x) = \frac{\varkappa}{2}(u(x + 1) + u(x - 1)) = (\mu(x, \omega) + E)u(x). \tag{35}$$

By virtue of (35), the proof of the lemma in the one-dimensional case is complete:

$$(\varkappa\bar{\Delta} - (\mu(x, \omega) + E))u(x) = (\mu(x, \omega) + E)u(x) - (\mu(x, \omega) + E)u(x) = 0.$$

In the multidimensional case, the reasoning remains exactly the same, except that the expression (33) will contain paths on all lattice points neighboring x .

Now let us show the correctness of (31) for any $\lambda > 0$. For simplicity, we first consider the one-dimensional case $d = 1$. We investigate the convergence of the series (31). Note that the following upper bound estimate is true and achievable:

$$u(x) = \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2}{\mu(z, \omega) + E} \right) \leq \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa}{2E} \right) = \sum_{\substack{\gamma: x \rightsquigarrow 0, \\ |\gamma|=n}} \left(\frac{\varkappa}{2E} \right)^n L(x, 0, n), \tag{36}$$

where $L(x, 0, n)$ is the number of paths of the form $x \rightsquigarrow 0$ that contain n points. Note that if the parity of x and n does not coincide, then $L(x, 0, n)$ converges to zero.

Without restricting generality, let us assume $x > 0$. Finding $L(a, b, k)$ is a standard problem for applying the reflection principle to discrete random walks; see, e.g., [33]. The answer is as follows:

$$L(a, b, k) = C_k^{\frac{k+b-a}{2}} - C_k^{\frac{k+b+a}{2}}, \quad a, b, n > 0.$$

Therefore,

$$L(x, 0, n) = L(x, 1, n - 1) = C_{n-1}^{\frac{n-x}{2}} - C_{n-1}^{\frac{n+x}{2}} = c_1 x^{c_2} 2^n, \quad n \rightarrow \infty, \tag{37}$$

where c_1 and c_2 are positive constants.

Thus, the series in the (36) inequalities are geometric series:

$$\sum_{\substack{\gamma: x \rightsquigarrow 0, \\ |\gamma|=n}} \left(\frac{\varkappa}{2E}\right)^n L(x, 0, n) < c_3 + c_4 \sum_{i=1}^n \left(\frac{\varkappa}{2E}\right)^i \cdot 2^i, \tag{38}$$

where c_3 and c_4 are positive constants.

The series (38) converges when $\varkappa/E < 1$. Which, given the notation $E = \varkappa + \lambda$, can be rewritten as follows:

$$\lambda > 0.$$

In the case $d > 1$, the estimation of (36) takes the following form:

$$u(x) = \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa/2d}{\mu(z, \omega) + E}\right) \leq \sum_{\gamma: x \rightsquigarrow 0} \prod_{z \in \gamma} \left(\frac{\varkappa}{2dE}\right) = \sum_{\substack{\gamma: x \rightsquigarrow 0, \\ |\gamma|=n}} \left(\frac{\varkappa}{2dE}\right)^n L(x, 0, n), \tag{39}$$

where again C_n is the number of γ paths of length n , where the length is counted with respect to zero and the conditions from Lemma 4 are imposed on the path.

Let us consider $n \gg x$, since the convergence of the series (39) depends on them alone. Note that when $n \gg x$, the number of trajectories $L(x, 0, n) \sim L(0, 0, n)$, $n \rightarrow \infty$. We denote by $L_0(0, 0, n)$ the number of trajectories starting and ending at zero without the condition of non-intersection of zero. The event of a trajectory crossing the zero point in dimension $d > 1$ is rare, so $L(0, 0, n) \sim L_0(0, 0, n)$, $n \rightarrow \infty$.

Let us fix d movements “up” along each of the coordinates. Each path in $L_0(0, 0, n)$ is defined by only $n/2$ steps, each of which can have one of the coordinate movements, i.e.,

$$L_0(0, 0, n) = d^{n/2} C_n^{n/2} \sim (2\sqrt{d})^n, \quad n \rightarrow \infty.$$

Proceeding as in the one-dimensional case, we obtain that the series in the estimation of (39) is a geometric series:

$$\sum_{\substack{\gamma: x \rightsquigarrow 0, \\ |\gamma|=n}} \left(\frac{\varkappa}{2dE}\right)^n L(x, 0, n) < c_5 + \sum_{i=1}^n \left(\frac{\varkappa}{2dE}\right)^i \cdot (2\sqrt{d})^i, \tag{40}$$

where c_5 is a positive constant. The series (40) converges when $\varkappa/\sqrt{d}E < 1$. Which, given the notation $E = \varkappa + \lambda$, can be rewritten as follows:

$$\lambda > \varkappa \left(\frac{1}{\sqrt{d}} - 1\right).$$

Therefore, for $\lambda > 0$ the series converges, which completes the proof of the lemma. \square

9.3. Proof of Lemma 5

Let us recall the formulation of Lemma 5:

Consider an environment ω_1 in which points neighboring from zero have killing intensities equal to μ_1 and μ_{-1} . A positive eigenvalue in this environment exists if and only if

$$\Lambda > \frac{\mu_1 + \mu_{-1} + 2\sigma\mu_1\mu_{-1}}{(1 + \sigma\mu_1)(1 + \sigma\mu_{-1})},$$

where $\sigma = \frac{1}{\varkappa/2}$ for $z \in \mathbb{R}$.

Proof. The eigenvalue problem for the considered environment is as follows:

$$\begin{aligned} (\varkappa\Delta + V(x, \omega))u(x) &= \lambda u(x), \\ u(0) &= 1, \end{aligned} \tag{41}$$

where

$$V(x, \omega) = \begin{cases} \Lambda, & x = 0; \\ -\mu_1, & x = 1; \\ -\mu_{-1}, & x = -1; \\ 0, & |x| \geq 2. \end{cases}$$

First, let us show how the eigenfunction for this problem looks in general form. We will use the forward and inverse discrete Fourier transforms; see, e.g., [27]. The Fourier transform of the function f is defined as follows:

$$\tilde{f}(\theta) = \sum_{x \in \mathbb{Z}} e^{i\theta x} f(x).$$

The inverse Fourier transform is defined as follows:

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(\theta) e^{-i\theta x} d\theta.$$

Let us write for the operator H the eigenvalue problem λ with the corresponding eigenfunction u :

$$\varkappa\Delta u(x) + \Lambda\delta_0(x)u(x) - \mu_{-1}\delta_{-1}(x)u(x) - \mu_1\delta_1(x)u(x) = \lambda u(x). \tag{42}$$

After applying the Fourier transform, the expression (42) takes the form

$$\varkappa(\cos(\theta) - 1)\tilde{u}(x) + \Lambda u(0) - \mu_{-1}u(-1)e^{-i\theta} - \mu_1u(1)e^{i\theta} - \lambda\tilde{u}(x).$$

The Fourier transform of the eigenfunction $\tilde{u}(x)$ is as follows:

$$\tilde{u}(x) = \frac{\Lambda u(0) - \mu_{-1}u(-1)e^{-i\theta} - \mu_1u(1)e^{i\theta}}{\lambda + \varkappa - \varkappa \cos \theta},$$

and, finally, the solution $u(x)$ can be represented as

$$u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Lambda u(0) - \mu_{-1}u(-1)e^{-i\theta} - \mu_1u(1)e^{i\theta}}{\lambda + \varkappa - \varkappa \cos \theta} e^{-i\theta x} d\theta.$$

Calculating here the integral for $x \geq 1$, we obtain

$$u(x) = -\mu_{-1}u(-1)\frac{w^{x-1}}{r} + \Lambda u(0)\frac{w^x}{r} - \mu_1u(1)\frac{w^{x+1}}{r}, \tag{43}$$

where $r = \sqrt{\lambda(\lambda + 2\varkappa)}$ and $w = \frac{\lambda + \varkappa - r}{\varkappa}$.

The expression (43) can be rewritten in a more convenient form:

$$u(x) = w^x \left(-\mu_{-1}u(-1)\frac{1}{wr} + \Lambda u(0)\frac{1}{r} - \mu_1u(1)\frac{w}{r} \right) = B_1 \cdot e^{-kx}, \tag{44}$$

where $B_1 = \left(-\mu_{-1}u(-1)\frac{1}{wr} + \Lambda u(0)\frac{1}{r} - \mu_1u(1)\frac{w}{r} \right)$ and $e^{-k} = w$. Function w^x decreases as x tends to infinity, since $u \in L_2(\mathbb{Z})$, therefore $k > 0$.

Let us do exactly the same for $x \leq 1$ and put $f(0) = 1$ by normalization. We determine that the eigenfunction must have the following form:

$$\psi(x) = \begin{cases} 1, & x = 0; \\ C_1 e^{-kx}, & x \geq 1; \\ C_{-1} e^{kx}, & x \leq -1, \end{cases}$$

where $C_{\pm 1}$ and k are positive constants. Let us show that there is a positive eigenvalue for this function if and only if the lemma condition is satisfied.

Let us write the problem (41) for the point $x \in [2; \infty)$:

$$\frac{\varkappa}{2} \psi(x + 1) + \frac{\varkappa}{2} \psi(x - 1) - \varkappa \psi(x) = \lambda \psi(x);$$

from here we can calculate that

$$\lambda = \frac{\varkappa}{2} (e^{-k} + e^k - 2) = \varkappa (\cosh k - 1) = 2\varkappa \sinh^2(k/2) = k^2 + O(k^4), \quad k \rightarrow 0.$$

In particular, when $\lambda \rightarrow 0+$, it follows from the condition $k > 0$ that $k \rightarrow 0+$, that is

$$\lambda \rightarrow 0+ \quad \rightarrow \quad e^k \rightarrow 1+ . \tag{45}$$

Now let us write the problem (41) for the point $x = 1$:

$$\frac{\varkappa}{2} \psi(2) + \frac{\varkappa}{2} \psi(0) - \varkappa \psi(1) - \mu \psi(1) = \lambda \psi(1).$$

From here, we can calculate that

$$C_1 = \frac{1}{1 + e^{-k} \frac{\mu_1}{\varkappa/2}}.$$

For simplicity we denote $\frac{1}{\varkappa/2}$ by σ . Let us perform similar reasoning for $x = -1$, obtaining

$$C_{\pm 1} = \frac{1}{1 + \sigma \mu_{\pm 1} e^{-|k|}}.$$

Finally, let us write the problem (41) for $x = 0$:

$$\frac{\varkappa}{2} \psi(1) + \frac{\varkappa}{2} \psi(-1) - \varkappa \psi(0) + \lambda \psi(0) = \lambda \psi(0).$$

From here, we can calculate that

$$(C_1 + C_{-1} - 1)e^{-k} + \sigma \Lambda - e^k = 0$$

or, finally,

$$e^{2k} - \sigma \Lambda e^k - \left(\frac{1}{1 + \sigma \mu_1 e^{-k}} + \frac{1}{1 + \sigma \mu_{-1} e^{-k}} - 1 \right) = 0.$$

First, for simplicity, let us make $\mu = \mu_{-1} = \mu_1$, also, denote e^k by z and obtain the following expression:

$$z^2 - \sigma \Lambda z - \frac{2}{1 + \sigma \mu \frac{1}{z}} + 1 = 0$$

or

$$z^3 - z^2 \sigma (\Lambda - \mu) - z(\sigma^2 \Lambda \mu + 1) + \sigma \mu = 0. \tag{46}$$

The current problem is to write out conditions on Λ, μ , and σ that guarantee the positivity of λ . Let us use the expression (45) and note that (46) is a smooth function with respect to z , so we can make $z = 1$ to find the limit solution at $z \rightarrow 1+$:

$$1 - \sigma(\Lambda - \mu) - (\sigma^2\Lambda\mu + 1) + \sigma\mu = 0 \Leftrightarrow \Lambda = \frac{2\mu}{1 + \sigma\mu}.$$

The eigenvalue $\lambda > 0$ exists when this condition is violated to the “supercritical side”, i.e.,

$$\Lambda > \frac{2\mu}{1 + \sigma\mu}.$$

We obtain the conditions of the lemma under the assumption of $\mu_1 = \mu_{-1}$.

In the case of unequal μ_1 and μ_{-1} , the same solution method gives the condition of the lemma:

$$\Lambda > \frac{\mu_1 + \mu_{-1} + 2\sigma\mu_1\mu_{-1}}{(1 + \sigma\mu_1)(1 + \sigma\mu_{-1})}.$$

□

9.4. Proof of Theorem 3

Let us recall the formulation of Theorem 3:

The following estimate from below is true:

$$P(\Lambda, \varkappa, F_\mu) \geq \max_{a \in (0; \infty)} \mathbb{P} \left(\omega : \Lambda > \varkappa \frac{(2^a - 1)}{(2^a + 1)} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{4^{|x|}} \right).$$

In particular, for $a = 1$:

$$P(\Lambda, \varkappa, F_\mu) \geq \mathbb{P} \left(\omega : \Lambda > \frac{\varkappa}{3} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{4^{|x|}} \right).$$

Proof. Consider the function $\psi(x) = 2^{-a|x|}$. Let us denote $\varphi(x) = (H(\omega)\psi)(x)$ and directly calculate the quadratic form $(\varphi, \psi) = (\varphi(x, \omega), \psi(x))$. First, let us calculate the function $\varphi(x)$:

$$\begin{aligned} \varphi(x, \omega) &= \varkappa\Delta\psi(x) + \Lambda\delta_0(x)\psi(x) - (1 - \delta_0(x))\mu(x, \omega)\psi(x) \\ &= \frac{\varkappa}{2}(\psi(x + 1) + \psi(x - 1) + 2\psi(x)) + \Lambda\delta_0(x)\psi(0) - (1 - \delta_0(x))\mu(x, \omega)\psi(x). \end{aligned} \tag{47}$$

Let us substitute into (47) the expression for $\psi(x)$ and consider separately the points 0 and $x > 0$:

$$\varphi(0, \omega) = \frac{\varkappa}{2}(2^{-a} + 2^{-a} - 2) + \Lambda = \varkappa(2^{-a} - 1) + \Lambda; \tag{48}$$

$$\begin{aligned} \varphi(x, \omega) &= \frac{\varkappa}{2}(2^{-a} \cdot 2^{-ax} + 2^a \cdot 2^{-ax} - 2 \cdot 2^{-ax}) - \mu(x, \omega)2^{-ax} \\ &= \frac{\varkappa}{2}2^{-ax}(2^{-a} + 2^a - 2) - \mu(x, \omega)2^{-ax}. \end{aligned} \tag{49}$$

Using (48) and (49), we calculate the required quadratic form:

$$\begin{aligned}
 (\varphi(x, \omega), \psi(x)) &= \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \varphi(x)\psi(x) + \varphi(0)\psi(0) \\
 &= \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \left(\frac{\varkappa}{2} 2^{-a|x|} (2^{-a} + 2^a - 2) - \mu(x, \omega) 2^{-a|x|} \right) \cdot 2^{-a|x|} + \varkappa(2^{-a} - 1) + \Lambda \\
 &= \frac{\varkappa}{2} (2^{-a} + 2^a - 2) \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} 2^{-2a|x|} - \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{2^{2a|x|}} + \varkappa(2^{-a} - 1) + \Lambda \\
 &= -\varkappa \frac{2^{-a} - 1}{(2^{-a} + 1)2^a} + \varkappa(2^{-a} - 1) + \Lambda - \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{2^{2a|x|}} \\
 &= -\varkappa \frac{(2^a - 1)}{(2^a + 1)} + \Lambda - \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{2^{2a|x|}}. \tag{50}
 \end{aligned}$$

If $(\varphi(x, \omega), \psi(x)) > 0$, then by virtue of Section 4, the operator $H(\omega)$ has a positive eigenvalue. Given the expression (50), the condition for the positivity of the quadratic form can be rewritten in the following form:

$$\Lambda > \varkappa \frac{(2^a - 1)}{(2^a + 1)} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{2^{2a|x|}}.$$

By substituting $a = 1$ we obtain:

$$\Lambda > \frac{\varkappa}{3} + \sum_{\substack{x=-\infty; \\ x \neq 0}}^{\infty} \frac{\mu(x, \omega)}{4^{|x|}}.$$

□

9.5. Proof of Lemma 6

Recall the formulation of Lemma 6:

Consider a set of Ω_l including environments that have l -islands around zero:

$$\Omega_l = \{\omega \in \Omega : \mu(i, \omega) = 0, \forall i \in -l, \dots, l\}.$$

If a positive eigenvalue exists for all $\omega \in \Omega_l$, then it is bounded from below by a solution with respect to λ of the following equation:

$$\frac{2\alpha\varkappa}{1 + \sqrt{1 - 4\alpha^2}} + \varkappa\alpha^{2l} \cdot R(\alpha, \beta) + \Lambda - \varkappa - \lambda = 0,$$

where $\alpha = \frac{\varkappa/2}{\varkappa+\lambda}$, $\beta = \frac{\varkappa/2}{\varepsilon+\varkappa+\lambda}$, and the expression R is defined as follows:

$$R(\alpha, \beta) = \sum_{k=0}^{\infty} (\beta^{2k+1} - \alpha^{2k+1}) C_{k+l}, \tag{51}$$

where C_n denotes the Catalan number. If the series in Equation (51) does not converge then there exists a $\omega \in \Omega_l$ for which there is no positive eigenvalue.

Proof. For convenience in the proof, let us rewrite Formula (4) for $d = 1$:

$$u(x) = \sum_{\gamma: x \rightarrow 0} \prod_{z \in \gamma} \left(\frac{x/2}{\mu(z, \omega) + E} \right). \tag{52}$$

By virtue of Equation (15), we are interested in the quantity $u(1)$, for which the paths from 1 to 0 are important. By the lemma condition, for any $\omega \in \Omega_l$ there exists an l -island around zero. Therefore, every trajectory from 1 to 0 of length less than $2l$ will not leave this island, and every trajectory of length greater than $2l$ must spend at least $2l$ steps in this island. We divide all trajectories into two families: trajectories of length less than or equal to $2l$ trajectories of length greater than $2l$. The contribution of each of the smaller trajectories to the sum (52) is exactly $\left(\frac{x/2}{0+E}\right)^{|\gamma|}$. The contribution of each of the large trajectories to the sum (52) is at least $\left(\frac{x/2}{0+E}\right)^{2l} \cdot \left(\frac{x/2}{c+E}\right)^{|\gamma|-2l}$. Thus,

$$\begin{aligned} u(1) &= \sum_{\gamma: 1 \rightarrow 0} \prod_{z \in \gamma} \left(\frac{x/2}{\mu(z, \omega_l) + E} \right) \\ &\geq \sum_{\substack{\gamma: 1 \rightarrow 0, \\ |\gamma| < 2l}} \left(\frac{x/2}{0+E} \right)^{|\gamma|} L(0, 1, |\gamma|) + \sum_{\substack{\gamma: 1 \rightarrow 0, \\ |\gamma| \geq 2l}} \left(\frac{x/2}{0+E} \right)^{2l} \left(\frac{x/2}{c+E} \right)^{|\gamma|-2l} L(0, 1, |\gamma|). \end{aligned} \tag{53}$$

Let us rewrite Equation (53) by introducing the notations $\alpha = \frac{x/2}{E}$, $\beta = \frac{x/2}{c+E}$:

$$u(1) \geq \sum_{\substack{\gamma: 1 \rightarrow 0, \\ |\gamma| < 2l}} \alpha^{|\gamma|} L(0, 1, |\gamma|) + \sum_{\substack{\gamma: 1 \rightarrow 0, \\ |\gamma| \geq 2l}} \alpha^{2l} \beta^{|\gamma|-2l} L(0, 1, |\gamma|). \tag{54}$$

Let us simplify Equation (54) in the way (17) does, yielding the following:

$$\begin{aligned} u(1) &\geq \sum_{k=0}^{l-1} \alpha^{2k+1} C_k + \alpha^{2l} \sum_{k=l}^{\infty} \beta^{2k+1-2l} C_k \\ &= \sum_{k=0}^{\infty} \alpha^{2k+1} C_k + \alpha^{2l} \sum_{k=l}^{\infty} \left(\beta^{2k+1-2l} - \alpha^{2k+1-2l} \right) C_k. \end{aligned} \tag{55}$$

Using (18) and (55) we obtain:

$$u(1) \geq \frac{2\alpha}{1 + \sqrt{1 - 4\alpha^2}} + \alpha^{2l} \sum_{k=0}^{\infty} \left(\beta^{2k+1} - \alpha^{2k+1} \right) C_{k+l}. \tag{56}$$

Substituting (56) into (15) completes the proof of the lemma. \square

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Abbreviations

The following abbreviations are used in this manuscript:

BRW	Branching random walk
r.v.	Random variable
a.s.	Almost sure

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