mathematics

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Optimal Coloring Strategies for the Max k-Cut Game

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Abstract: We explore strong Nash equilibria in the max k-cut game on an undirected and unweighted graph with a set of k colors. Here, the vertices represent players, and the edges denote their relationships. Each player, v, selects a color as its strategy, and its payoff (or utility) is determined by the number of neighbors of v who have chosen a different color. Limited findings exist on the existence of strong equilibria in max k-cut games. In this paper, we make advancements in understanding the characteristics of strong equilibria. Specifically, our primary result demonstrates that optimal solutions are seven-robust equilibria. This implies that for a coalition of vertices to deviate and shift the system to a different configuration, i.e., a different coloring, a number of coalition vertices greater than seven is necessary. Then, we establish some properties of the minimal subsets concerning a robust deviation, revealing that each vertex within these subsets will deviate toward the color of one of its neighbors.

Keywords: max k-cut problem; game theory; optimal colorings; coalitions; Nash equilibrium

MSC: 05C57; 05C15; 90C27

1. Introduction

The maximum k-cut problem entails the assignment of colors or the implementation of a cut to the vertices of a graph, ensuring that adjacent vertices have different colors, thereby maximizing color heterogeneity within the graph. This challenge is not only theoretically captivating but also holds practical significance, particularly in real-world applications involving self-interested agents.

For instance, in wireless communication networks, optimizing frequency allocation among transmitters corresponds to solving the maximum k-cut problem. Similarly, in social network analysis, where vertices represent individuals and edges signify relationships, maximizing the heterogeneity of the opinions or preferences among connected individuals aligns with addressing the maximum k-cut problem.

A strategic variation of the maximum k-cut problem is the max k-cut game played on an undirected and unweighted graph with a set of k colors. In this scenario, the vertices represent players, and the edges denote their interpersonal connections. Each player chooses one of the available colors as their strategy, and their payoff is determined by the number of neighbors who have chosen a different color [1,2].

The primary concern in the max k-cut game is the potential for players to independently reach a social optimum without external regulation. In these graph-based games, players may intentionally misalign their choices with their neighbors by selecting dissimilar colors, seeking to enhance their utility by aligning decisions within groups, referred to as coalitions.

The max k-cut game, extensively studied for its real-world applications involving selfish agents, has been proven NP-complete for finding an optimal solution [3–11].

A key challenge in the max k-cut game is establishing the existence of strong Nash Equilibria (abbreviated as SE). Strong equilibrium implies colorings where no coalition

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Citation: Garuglieri, A.; Madeo, D.; Mocenni, C.; Palma, G.; Rinaldi, S. Optimal Coloring Strategies for the Max k-Cut Game. *Mathematics* **2024**, *12*, 604. https://doi.org/10.3390/math12040604

Academic Editor: Bo Zhou

Received: 12 January 2024
Revised: 12 February 2024
Accepted: 13 February 2024
Published: 18 February 2024

can co-operatively deviate to the benefit of all its members. A less stringent version is the q-strong equilibrium (abbreviated as q-SE), where only coalitions of more than q players are permitted to co-operatively change their strategies.

Trivially, the 1-SE aligns with the Nash equilibrium (abbreviated as NE), and the n-SE corresponds to the general SE. Existing insights into the existence of strong equilibria in max k-cut games are limited. In [1], optimal strategy profiles are shown to be a SE for the max two-cut game and a 3-SE for the max k-cut game with $k \geq 2$. Additionally, it is demonstrated that an optimal strategy profile may not necessarily be a 4-SE for any $k \geq 3$. In [12], it is shown that if the number of colors is at least the same as the number of players minus two, then an optimal strategy profile is a SE. An optimal coloring in graph theory refers to a coloring of the vertices of a graph where no two adjacent vertices share the same color, and this coloring utilizes the minimum possible number of colors. The objective is to minimize the number of colors while satisfying the condition that the adjacent vertices have distinct colors. The pivotal outcome in [13] reveals that optimal colorings are 5-SE in undirected unweighted graphs. By building upon the findings of [13,14], our paper employs an innovative combinatorial approach, illustrating that optimal colorings extend to 7-SE.

This discovery holds significant conceptual implications and applications, emphasizing the resilience of optimal colorings against larger coalitions attempting to induce selfish divergence from the optimal equilibrium. In contrast to relying solely on game theory concepts and tools, our approach utilizes combinatorial properties that characterize graphs concerning optimality and the potential for achieving SE. Finally, to investigate the conjecture asserting that every optimal coloring is a SE, we propose an innovative approach incorporating combinatorial properties and algorithms on permutations.

The organization of this work is as follows. Initially, the subsequent section provides essential definitions for our investigation, highlights the pertinent findings, and introduces the central problem. Following that, in Section 2, we establish the 7-SE property of optimal colorings. Our discussion concludes by highlighting unresolved questions and presenting our preliminary findings that might prove valuable in the examination of these issues in Section 3.

A preliminary version of this paper was initially uploaded to arXiv. Subsequently, a subset of the authors presented an early version of the results, providing a strategic approach to problem resolution at the "GASCom on random generation of combinatorial structures, 12th edition" conference (GASCom2022). In [15], we presented a conceptual outline of our innovative problem-solving approach; however, it lacked the essential lemmas, propositions, and theorems crucial for a formal proof. Given the substantial interest generated by this work, we decided to revisit and expand upon it, presenting our approach with clarity and precision. Thus, in the subsequent development of our work, we have filled these gaps, providing a comprehensive and formal demonstration of the proposed strategy. The additional lemmas and theorems are not merely instrumental in proving that optimal solutions are 7-robust equilibria. They hold intrinsic value, serving as building blocks for further insights into the properties of the max k-cut game. This paper delves deeper into the subject, offering a comprehensive and formal presentation of our innovative game theory-based approach, which incorporates combinatorial techniques.

It is essential to emphasize the distinctiveness and comprehensiveness of this paper compared to its predecessors by underscoring its significance in the field since this new version provides formal and complete proofs, yielding novel results and implications that contribute to the study of strong Nash equilibria in the max k-cut game.

Additionally, some authors pursued alternative approaches within this research, yielding outcomes pertinent to a distinct class, detailed in [16]. In the context of max k-cut games on undirected and unweighted graphs, the study formally establishes that optimal colorings prevent the formation of nearly any strongly divergent coalition. This is accomplished through a novel game theory-based method, categorizing graph nodes into three subsets: the coalition itself, the coalition boundary, and unrelated nodes.

2. Materials & Methods

Consider an undirected, unweighted graph denoted as $G = \langle V, E \rangle$, where V represents a finite set comprising n vertices, and E denotes the set of m edges, forming a collection of pairs within V. Our assumption is that the graph has no self-loops, and for any two vertices, there is one edge (at most) connecting them. For $v \in V$, the degree of the vertex v is $\delta(v) = \big|\{j \in V : \{v, j\} \in E\}\big|$. Given G and a set of colors $K = \{1, \ldots, k\}$, the degree of v in the coloring σ with respect to the color a is $\delta(v, \sigma, a) := \big|\{j \in V : \{v, j\} \in E, \sigma_j = a\}\big|$. The *max k-cut problem* consists of partitioning the vertices of G into k subsets, denoted by $V_1, \ldots, V_k$, such that the number of vertices having neighbors in different sets is maximized. The max k-cut game constitutes a strategic version of the max k-cut problem, as it is defined below. There are $|V|$ players, each of which have the same strategy set, that is the set of colors K. A strategy profile, also called coloring, denoted by $\sigma \in K^n$, with n equal to the number of vertices of G, represents a labeling of vertices of G, in which a color σ_v is assigned to each player v. A strategy profile can be seen as a vector $\sigma \in K^n$ containing the strategies chosen by each vertex. Given a graph G and a coloring $\sigma \in K^n$, we define the cut of a graph G and the set $E(\sigma) := \{\{i, j\} \in E : \ \sigma_i \neq \sigma_j\}$. The cardinality of $E(\sigma)$ is called the cut value, or size of the cut, and it is denoted by $CV(\sigma)$. The cut difference between two colorings γ and σ is $\Delta CV(\sigma, \gamma) := CV(\gamma) - CV(\sigma)$. Referring to σ, for each $v \in V$, the utility, or payoff, of player v is defined as $\mu_v(\sigma) = \big|\{u \in V : \ \{u, v\} \in E, \ \sigma_u \neq \sigma_v\}\big|$. We recall that given a graph $G = (V, E)$, $|V| = n$, $|E| = m$, its adjacency matrix is the binary matrix $A = \{a_{v,w}\} \in \{0, 1\}^{n \times n}$ such that $a_{v,w} = 1$ if and only if (v, w) is an edge in E. Now, using the formalization introduced by the adjacency matrix A, we can also write the payoff as the following:

$$\mu_v(\sigma) = \sum_{\substack{u \in V \\ \sigma_u \neq \sigma_v}} a_{v,u}. \tag{1}$$

Thanks to the symmetry of A, and given a coloring σ and two disjoint subsets of the vertices $V_1, V_2 \subseteq V$, we have the following:

$$\sum_{v \in V_1} \sum_{\substack{u \in V_2 \\ \sigma_u \neq \sigma_v}} a_{v,u} = \sum_{u \in V_2} \sum_{\substack{v \in V_1 \\ \sigma_v \neq \sigma_u}} a_{u,v}. \tag{2}$$

Both Equations (1) and (2) will be used in the proof of the main results of this work.

An optimal strategy profile, also known as an optimal coloring, refers to a strategy profile that maximizes the social welfare, denoted as $SW(\sigma) = \sum_{v \in V} \mu_v(\sigma) = 2CV(\sigma)$. It is important to note that if a coloring is optimal, then all permutations of the set of colors K within it are also optimal.

In order to illustrate this concept, let us consider an example of the game and clarify the notation. Imagine a graph G, depicted in Figure 1a, with a color set $K = red, blue, green$. The six players are denoted as $v_1, v_2, v_3, v_4, v_5, v_6$. Figure 1a,b showcase two distinct colorings: $\sigma = (red, blue, red, blue, green, red)$ and $\gamma = (green, blue, green, blue, red, red)$.

Let's focus on σ. The cut of G is defined as

$$E(\sigma) = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}\},$$

and the corresponding cut value is $CV(\sigma) = 8$. The payoff of v_1 is $\mu_1(\sigma) = 2$, as only two of its three neighbors have a color different from its own. By extending this reasoning, we can determine the payoffs for the other five vertices: $\mu_2(\sigma) = 2$, $\mu_3(\sigma) = 3$, $\mu_4(\sigma) = 4$, $\mu_5(\sigma) = 3$, and $\mu_6(\sigma) = 2$. Therefore, the social welfare of σ is $SW(\sigma) = 16$.

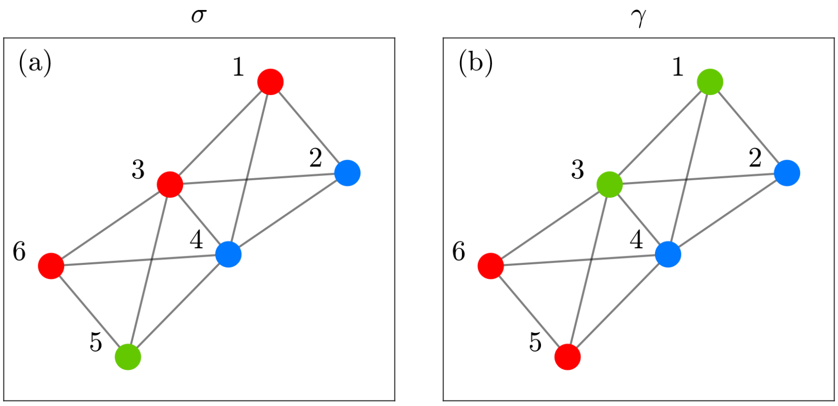

Figure 1. Main notions on colorings and coalitions. In this case, the deviating coalition from σ (**a**) to γ (**b**) is $C = \{v_1, v_3, v_5\}$.

A coalition C is a subset of nodes able to increase their own payoffs simultaneously.

For a coalition, $C \subseteq V$, and a given coloring, σ, we initially define the set of colors used by the coalition C in σ as $K_C(\sigma) = \{a \in K : \exists v \in C : \sigma_v = a\}$. Subsequently, for each color a, we establish the set of players in C that possess color a in σ as

$$C_a(\sigma) = \{v \in C : \sigma_v = a\}. \tag{3}$$

Furthermore, if for any player $v \in C$, $\gamma_v \neq \sigma_v$, and if for any player $w \notin C$, $\gamma_w = \sigma_w$, then we characterize the coalition C as deviating from the coloring σ towards the coloring γ. In order to clarify this, this deviation implies that within the coalition C, the players adopt colors different from their original ones in σ, whereas the players outside of C maintain the same colors as in σ.

A strong deviation occurs when a coalition, C, deviates in a manner that ensures all of its members strictly improve their utility. Such a strong deviation is deemed minimal if no proper subset of the deviating coalition can achieve an improvement. The coalition itself is then referred to as minimal.

Furthermore, given a coloring, σ, if a coalition, C, induces a new coloring, γ, after deviating, then we say that the set of edges $E(\sigma)\backslash E(\gamma)$ enters the cut and that the set of edges $E(\gamma)\backslash E(\sigma)$ leaves the cut.

For instance, let us consider again the graph G and coloring σ reported in Figure 1a. Let C be the coalition composed of the players $\{v_1, v_3, v_5\}$.

The set of colors utilized by the coalition C in σ is $K_C(\sigma) = \{red, green\}$. Furthermore, the set of players in C with the color *red* in σ is $C_{red}(\sigma) = \{v_1, v_3\}$, and those with the color *green* are $C_{green}(\sigma) = \{v_5\}$.

The coloring γ illustrated in Figure 1b represents a deviation from the coloring σ, which is not considered strong since not all members of the coalition experience an improvement in their utility. Specifically, v_1 maintains the same payoff in both the σ and γ colorings, while in the new coloring γ, v_5 actually worsens its payoff.

We denote $\mu_C(\sigma) = \sum_{v \in C} \mu_v(\sigma)$ as the total payoff obtained by coalition C. Additionally, σ_C represents the coloring σ restricted solely to players in C.

A Nash equilibrium (abbreviated as NE) is characterized by σ if no player can unilaterally improve their payoff by deviating from σ. For each $1 \leq q \leq n$, σ is a q-strong Equilibrium (abbreviated as q-SE) if there exists no coalition C with $|C| \leq q$ that can cooperatively deviate from σ_C to γ_C in a way that each player in C strictly improves their utility. A n-strong Equilibrium is simply referred to as a strong Equilibrium (abbreviated as SE).

Given two colorings, γ and σ, and $C \subseteq V$, we represent $P_C(\sigma, \gamma) := \big|\{\{v, j\} \in E : v, j \in C,\ \sigma_v = \sigma_j,\ \gamma_v \neq \gamma_j\}\big|$. It is worth noting that $P_C(\sigma, \gamma)$ can be expressed as follows:

$$P_C(\sigma,\gamma) = \frac{1}{2}\sum_{v\in C}\sum_{\substack{j\in C\\ \gamma_j\neq\gamma_v\\ \sigma_j=\sigma_v}} a_{v,j}. \tag{4}$$

It is straightforward to demonstrate that the set of optimal colorings of a graph encompasses all permutations of colors. This is because, through permutations, the inherent structure of the graph is maintained, particularly concerning the relationships between the vertices and their neighbors with different colors. If a coloring yields a certain payoff, then its permutation of colors also yields the same payoff. Therefore, the set of optimal colorings can be categorized into equivalence classes based on the permutation of colors. However, it is important to emphasize that not all optimal colorings are permutations of each other. In other words, among the optimal colorings, there are some that are not permutations of one another, but for each of these, all the permutations of the colors are included. We emphasize that if a coloring is not a strong deviation, then neither are all its permutations of colors.

In the forthcoming discussion, we elucidate the rationale behind Formula (4) for permutations in the context of the colorings γ and σ, focusing on the representation $P_C(\sigma,\gamma)$ and its explicit expression in Equation (2).

Consider two colorings, γ and σ, defined on a graph with vertices V and edges E. Let $C \subseteq V$ be a subset of vertices. The expression $P_C(\sigma,\gamma)$ represents the count of edges $\{v, j\} \in E$ where v and j both belong to C, $\sigma_v = \sigma_j$, and $\gamma_v \neq \gamma_j$.

The formula essentially calculates half the sum of the adjacency matrix entries $a_{v,j}$ for pairs of vertices, v and j in C, where $\gamma_v \neq \gamma_j$ but $\sigma_v = \sigma_j$.

The reasoning behind this formula is grounded in the observation that the optimal colorings of a graph encompass all possible permutations of colors. If a particular coloring yields a certain payoff, then any permutation of colors for that same coloring also results in the same payoff. Consequently, the set of optimal colorings can be grouped into equivalence classes based on the permutation of colors.

However, it is crucial to note that not all optimal colorings are permutations of each other. In other words, within the set of optimal colorings, there exist instances where some are not permutations of one another. Despite this, for each nonpermutated coloring, all possible permutations of colors are included within the set.

It is also emphasized that if a coloring is not a strong deviation, neither are any of its permutations of colors. This observation reinforces the idea that certain properties related to deviations are preserved across all permutations within the set of optimal colorings.

Lastly, we define $G(C)$ as the restriction of the graph G to the members of the coalition C, i.e., the set of vertices of $G(C)$ is C. Moreover, we say that H is an isolated component of the graph G if $G(H)$ is a connected subgraph of G and, for each vertex $i \in V(H)$, and for each vertex $j \notin V(H)$, $\{i, j\} \notin E$.

Now, we report an important result of [13] that we will use alongside the pigeon-hole principle.

Proposition 1. *Let σ be a coloring for a graph, G, and let $C \subseteq V$, with $|C| > 1$, be a minimal subset that can perform a strong deviation from one coloring, σ, to another coloring, γ; then, $K_C(\sigma) = K_C(\gamma)$.*

An Applicative Example

We conclude this subsection by defining an applicative example that we will cover in the next section. Let us consider the following situation.

Example 1. *An R&D department of a company must be divided into three teams, each of which will work on a product; however, each of the 12 employees of that department wishes not to work together with some of his colleagues. Figure 2a reports a graphical representation of the example,* *where the vertices correspond to employees, and the edges denote the desire of connected vertices not to belong to the same team. Figure 2b shows a generic assignment, α, of employees to the three projects, represented by different colors. Profit means having the greatest number of people you do not want to work with on projects other than your own. Some employees assigned to some team colors are assumed to be unhappy (for example, vertices v_5 and v_{11}) since they must work with undesired colleagues. Is it possible to find an optimal allocation, for example, the one reported in Figure 2c, which maximizes the profit by minimizing the number of unhappy people?*

This game can also be seen as an adaptation to game theory regarding the max k-cut problem. Indeed, by optimal coloring, in this case, we intend to minimize the number of pairs of employees who do not want to work together and who are assigned to the same project. To deviate means to change projects. A strong equilibrium designates the configuration such that if any number of players tried to change the project all at the same time, they would find themselves working on a project with a number of people with whom they would not want to work with (greater or equal than the previous one). In this context, then, if δ_v is the maximum number of people to which an individual v is connected, we indicate with $\delta_v(V_a(\sigma)) \le \delta_v$ the number of those she does not want to work with who are undertaking the project, a.

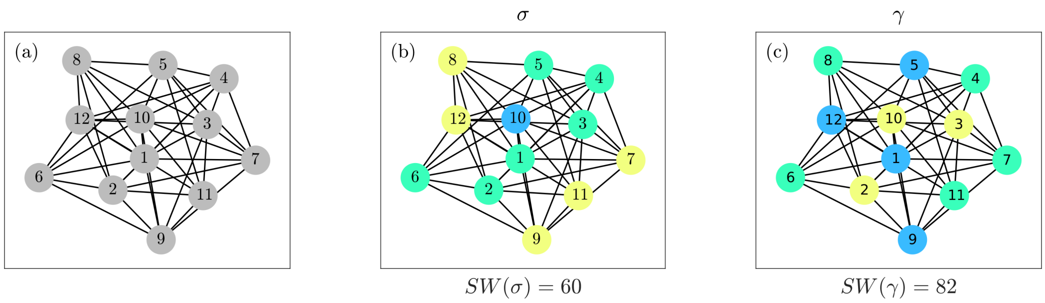

Figure 2. (**a**): A network with $n = 12$ vertices representing the R&D department described in Example 1. (**b**): A generic coloring σ with $k = 3$ colors. (**c**): An optimal coloring γ with $k = 3$ colors.

3. Results

In [13], the existence of a 5-SE in unweighted, undirected graphs is shown. In this subsection, we extend this result by proving the existence of a 7-SE.

We start by showing two propositions that form the core of our main result.

Lemma 1. *Given a graph, G, and $\sigma, \gamma \in K^n$, let σ be an NE for G, and let $C \subseteq V$ strongly deviate from σ to γ. Then, $\Delta CV(\gamma, \sigma) \ge |C| - P_C(\sigma, \gamma)$.*

Proof. We start by noticing that for both γ and σ, the cut value can be calculated by considering the vertices that belong to C and $V \setminus C$ separately:

$$CV(\sigma) = \frac{1}{2}\left[\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w} + \sum_{\substack{w\in V\setminus C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right) + \sum_{v\in V\setminus C}\left(\sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w} + \sum_{\substack{w\in V\setminus C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right)\right],$$

and

$$CV(\gamma) = \frac{1}{2}\left[\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} + \sum_{\substack{w\in V\setminus C\\ \gamma_w\neq\gamma_v}} a_{v,w}\right) + \sum_{v\in V\setminus C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} + \sum_{\substack{w\in V\setminus C\\ \gamma_w\neq\gamma_v}} a_{v,w}\right)\right],$$

Since the colors of the vertices in $V \setminus C$ do not change, we see that

$$\sum_{v\in V\setminus C}\sum_{\substack{w\in V\setminus C\\ \sigma_w\neq\sigma_v}} a_{v,w} = \sum_{v\in V\setminus C}\sum_{\substack{w\in V\setminus C\\ \gamma_w\neq\gamma_v}} a_{v,w}.$$

Additionally, we see that

$$\begin{aligned}
\Delta CV(\gamma,\sigma) &= CV(\gamma) - CV(\sigma) = \\
&= \frac{1}{2}\sum_{v\in C}\left[\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} + 2\sum_{\substack{w\in V\setminus C\\ \gamma_w\neq\gamma_v}} a_{v,w}\right) - \left(\sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w} + 2\sum_{\substack{w\in V\setminus C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right)\right] \\
&= \frac{1}{2}\sum_{v\in C}\left[\left(2\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} + 2\sum_{\substack{w\in V\setminus C\\ \gamma_w\neq\gamma_v}} a_{v,w}\right) - \left(2\sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w} + 2\sum_{\substack{w\in V\setminus C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right)\right] \\
&\quad -\frac{1}{2}\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} - \sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right).
\end{aligned}$$

Moreover, using Equation (1), we see that

$$\begin{aligned}
\Delta CV(\gamma,\sigma) &= \sum_{v\in C}(\mu_v(\gamma)-\mu_v(\sigma)) - \frac{1}{2}\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v}} a_{v,w} - \sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v}} a_{v,w}\right) \\
&= \sum_{v\in C}(\mu_v(\gamma)-\mu_v(\sigma)) - \frac{1}{2}\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v\\ \sigma_w=\sigma_v}} a_{v,w} + \sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v\\ \sigma_w\neq\sigma_v}} a_{v,w} - \sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v\\ \gamma_w=\gamma_v}} a_{v,w} - \sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v\\ \gamma_w\neq\gamma_v}} a_{v,w}\right) \\
&= \sum_{v\in C}(\mu_v(\gamma)-\mu_v(\sigma)) - \frac{1}{2}\sum_{v\in C}\left(\sum_{\substack{w\in C\\ \gamma_w\neq\gamma_v\\ \sigma_w=\sigma_v}} a_{v,w} - \sum_{\substack{w\in C\\ \sigma_w\neq\sigma_v\\ \gamma_w=\gamma_v}} a_{v,w}\right).
\end{aligned}$$

By using Equation (2), we finally obtain

$$\Delta CV(\gamma,\sigma) = \sum_{v\in C}(\mu_v(\gamma)-\mu_v(\sigma)) - P_C(\sigma,\gamma) + P_C(\gamma,\sigma).$$

On the other hand, since C strongly deviates, then

$$\mu_v(\gamma) \geq \mu_v(\sigma) + 1 \ \forall v \in C.$$

This yields

$$\sum_{v\in C}(\mu_v(\gamma)-\mu_v(\sigma)) \geq \sum_{v\in C} 1 = |C|.$$

Hence,

$$\begin{aligned}
\Delta CV(\gamma,\sigma) &\geq |C| - P_C(\sigma,\gamma) + P_C(\gamma,\sigma) \\
&\geq |C| - P_C(\sigma,\gamma).
\end{aligned}$$

The last inequality holds since $P_C(\gamma,\sigma) \geq 0$. □

Theorem 1. *Given a graph, G, let σ be an NE for it, and let $C \subseteq V$ be a minimal coalition that strongly deviates from one coloring, σ, to a different coloring γ. If $|K_C(\sigma)| \in \{|C|-3, |C|-2\}$, then $\Delta CV(\gamma,\sigma) > 0$.*

Proof. The aim is to use Lemma 1 to lower bound the ΔCV quantity and conclude the proof. First, we remark that in what follows, a, b, c, d are colors in $K_C(\sigma)$, with $|C| \geq 6$. For each feasible dimension of C, we consider the maximum $P_C(\sigma, \gamma)$ in each configuration. For the definition of the set $C_x(\sigma)$ with $x \in \{a, b, c, d\}$, we remand to Equation (3).

- Case 1: $|K_C(\sigma)| = |C| - 2$
  In this situation, for the pigeon-hole principle, we have the following subcases:
  (1.1) $|C_a(\sigma)| = 3$, $|C_b(\sigma)| = 1 \ \forall b \in K_C(\sigma) \setminus \{a\}$. By assuming that there are at least two colors in C, we see that $|C| \geq 4$.
  (1.2) $|C_a(\sigma)| = |C_b(\sigma)| = 2$, $|C_c(\sigma)| = 1 \ \forall c \in K_C(\sigma) \setminus \{a, b\}$. The presence of a single color, c, is not required. Moreover, in this case, $|C| \geq 4$ guarantees the presence of at least two colors.
- Case 2: $|K_C(\sigma)| = |C| - 3$
  In this situation, for the pigeon-hole principle, we have the following subcases:
  (2.1) $|C_a(\sigma)| = 4$, $|C_b(\sigma)| = 1 \ \forall b \in K_C(\sigma) \setminus \{a\}$. For this case, $|C| \geq 5$ in order to have at least two colors.
  (2.2) $|C_a(\sigma)| = 3$, $|C_b(\sigma)| = 2$, $|C_c(\sigma)| = 1 \ \forall c \in K_C(\sigma) \setminus \{a, b\}$. Here, the presence of single colors is not required, and $|C| \geq 5$ in order to have at least two colors.
  (2.3) $|C_a(\sigma)| = |C_b(\sigma)| = |C_c(\sigma)| = 2$, $|C_d(\sigma)| = 1 \ \forall d \in K_C(\sigma) \setminus \{a, b, c\}$. The single colors are not required. In this case, $|C| \geq 6$, and the number of colors is at least 3.

For each case and for each feasible value of $|C|$, we evaluate the maximum value of $P_C(\sigma, \gamma)$. This evaluation is graphically depicted in Figure 3, where each γ coloring maximizes the number of edges connecting the vertices with different colors, which were sharing the same colors in the corresponding σ coloring. Notice that since C is minimal, all colors in γ are the same as in σ. This is consistent with Proposition 1. The maximum value of $P_C(\sigma, \gamma)$, as well as the minimum value of $\Delta CV(\gamma, \sigma)$ are calculated in Table 1 according to Lemma 1, i.e., $\Delta CV(\gamma, \sigma) \geq |C| - P_C(\sigma, \gamma)$.

As shown in Table 1, $\Delta CV(\gamma, \sigma) > 0$ in all cases. $\square$

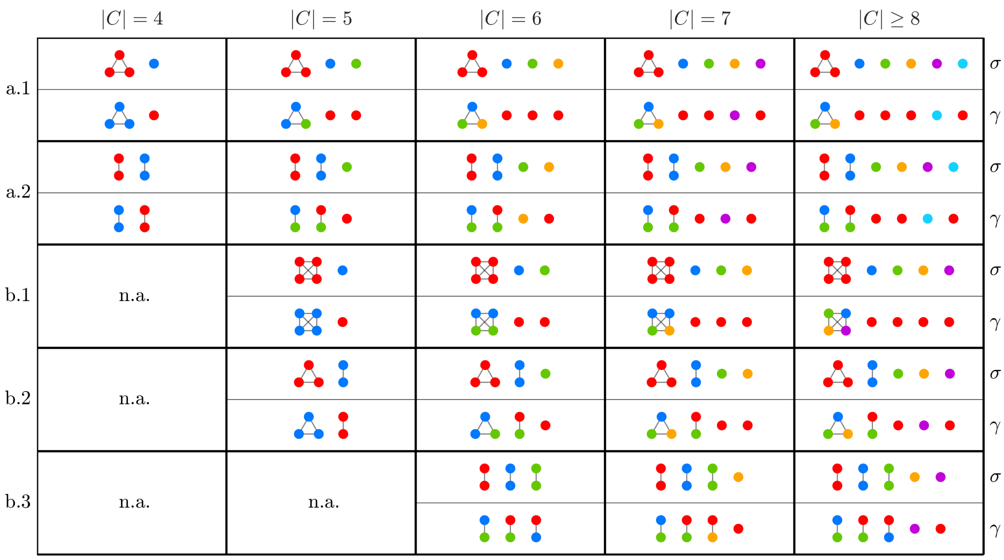

Figure 3. Graphical representation of the configurations analyzed in Theorem 1. n.a. entries indicate nonfeasible cases.

The configurations analyzed in Theorem 1, depicted below, symbolize different scenarios where a minimal coalition, C, strongly deviates from a coloring, σ, towards an alternative coloring, γ. The non-applicable (n.a.) entries in the graphical representation indicate cases that are not feasible.

In order to elaborate on what these configurations symbolize, let us delve into the details of the cases considered:

- **Case 1:** $|K_C(\sigma)| = |C| - 2$

  - **Subcase 1.1:** One color (a) has a coalition of size 3, while the other colors have single coalitions.
  - **Subcase 1.2:** Two colors (a and b) have coalitions of size 2, and the remaining color has a single coalition.
- **Case 2:** $|K_C(\sigma)| = |C| - 3$
  - **Subcase 2.1:** One color (a) has a coalition of size 4, while the other colors have single coalitions.
  - **Subcase 2.2:** One color (a) has a coalition of size 3, another color (b) has a coalition of size 2, and the remaining colors have single coalitions.
  - **Subcase 2.3:** Three colors (a, b, and c) have coalitions of size 2, and the remaining color has a single coalition.

These configurations are crucial in establishing the conditions for the existence of a Nash equilibrium (NE). The theorem highlights situations where deviations from a given coloring result in a positive change in the value of ΔCV (change in the number of edges connecting the vertices with different colors).

Table 1. Evaluation of $\Delta CV(\gamma, \sigma)$ for the different configurations analyzed in Theorem 1, depending on $|C|$ and $P_C(\sigma, \gamma)$. n.a. entries indicate nonfeasible cases.

		$\|C\| = 4$	$\|C\| = 5$	$\|C\| = 6$	$\|C\| = 7$	$\|C\| \geq 8$
a.1	$\max P_C(\sigma, \gamma)$	0	2	3	3	3
	$\Delta CV(\gamma, \sigma)$	4	3	3	4	≥5
a.2	$\max P_C(\sigma, \gamma)$	0	2	2	2	2
	$\Delta CV(\gamma, \sigma)$	4	3	4	5	≥6
b.1	$\max P_C(\sigma, \gamma)$	n.a.	0	4	5	6
	$\Delta CV(\gamma, \sigma)$	n.a.	5	2	2	≥2
b.2	$\max P_C(\sigma, \gamma)$	n.a.	0	3	4	4
	$\Delta CV(\gamma, \sigma)$	n.a.	5	3	3	≥4
b.3	$\max P_C(\sigma, \gamma)$	n.a.	n.a.	3	3	3
	$\Delta CV(\gamma, \sigma)$	n.a.	n.a.	3	4	≥5

The proposition below demonstrates that within the coalition, a rearrangement of colors occurs as a result of a significant deviation. This deviation is such that each vertex within the coalition takes on the color of another vertex within the same coalition.

Proposition 2. *Given a graph, $G = (V, E)$, let σ be an NE for G, and let $C \subseteq V$ be a minimal subset with respect to strongly deviating from one coloring, σ, to another, γ, with $\sigma \neq \gamma$. Then, for each vertex, $v \in C$, there exists a vertex $w \in C$ such that $v \neq w$, $\sigma_v \neq \sigma_w$, $\{v, w\} \in E$ and $\gamma_v = \sigma_w$.*

Proof. The proof will be split into two parts:

1. We show that each vertex in C deviates towards the color that one of its neighbors has in σ.
2. We prove that such a neighbor must be in C, meaning that each vertex deviates to a color used in C.

Concerning part 1, we show that for each vertex, $u \in C$, it holds that $\delta_u(V_{\gamma_u}(\sigma)) > 0$ (we recall that according to Equation (3) $\delta_u(V_{\gamma_u}(\sigma))$ indicates the number of neighbors of u having color σ_u in the coloring σ). Indeed, let us assume by contradiction that there exists a vertex, $t \in C$, such that $\delta_t(V_{\gamma_t}(\sigma)) = 0$. We distinguish the two subcases:

1. If $\delta_t(V_{\sigma_t}(\sigma)) = 0$, then t does not need to improve its payoff, as it is already earning its maximum payoff; however, this contradicts the hypothesis that C is a strong deviation.

2. If $\delta_t(V_{\sigma_t}(\sigma)) > 0$, since $\delta_t(V_{\gamma_t}(\sigma)) = 0$, we see that $\{t\}$ is a minimal strong deviation from σ. However, this contradicts the hypothesis that σ is an NE and that C is minimal with respect to strongly deviating from σ.

Therefore, for each vertex $u \in C$, $\delta_u(V_{\gamma_u}(\sigma)) > 0$. Notice that this fact also holds if u deviates towards the color of one of its neighbors that is not in C.

Concerning part 2, let $u \in C$ be such that $\gamma_u = \sigma_t$ for some $t \in V \backslash C$, and $\{s, t\} \in E$ with $\sigma_s \neq \sigma_t$ for each vertex $s \in C$.

Since σ is an NE, it holds that $\delta_u(V_{\sigma_t}(\sigma)) \geq \delta_u(V_{\sigma_u}(\sigma))$.

Moreover, assuming that C strongly deviates, we see that $\mu_u(\gamma) > \mu_u(\sigma)$, which means that $\delta_u(V_{\sigma_t}(\gamma)) < \delta_u(V_{\gamma_u}(\gamma))$. Hence, $\delta_u(V_{\sigma_t}(\sigma)) > \delta_u(V_{\sigma_t}(\gamma))$.

Anyway, it might happen that some of the neighbors of u in γ deviate towards σ_t, meaning that $\delta_u(V_{\sigma_t}(\gamma)) \geq \delta_u(V_{\sigma_t}(\sigma))$, thus obtaining a contradiction.

Therefore, for each vertex $v \in C$, there must be a vertex $w \in C$ such that $w \neq v$, $\{v, w\} \in E$ and $\gamma_v = \sigma_w$. This means that a vertex, $u \in C$, must deviate to the color of one of its neighbors in C. $\square$

When referring to Example 1, Proposition 2 asserts that one member of the coalition is so disliked by another that he leaves the project to him.

Lemma 2. *Let $G = (V, E)$ be a graph, σ be an NE for $G = (V, E)$, and $C \subseteq V$ be a strongly deviating coalition from one coloring, σ, to another, γ, with $\sigma \neq \gamma$. Then, $|C| \geq 2$.*

Proof. Suppose that $C = \{v\}$ for a vertex $v \in V$. If C is a strong deviation, then v is able to change its color in order to improve its payoff unilaterally, namely $\mu_v(\gamma) > \mu_v(\sigma)$. This contradicts the fact that σ is NE. $\square$

The following proposition allows for ignoring nonconnected subgraphs in the proof of Theorem 1.

Proposition 3. *Let G be a graph, K be the set of all possible colors, σ be an NE for G, and $C \subseteq V$ be a minimal subset with respect to a strong deviation from one coloring, σ, to another, γ, with $\sigma \neq \gamma$. Then, $G(C)$ is an isolated component of G.*

Proof. First, we show that no isolated component of $G(C)$ can be monochromatic, i.e., every isolated component $H = (V_H, E_H)$ of $G(C)$ is such that there is no color, a, in K such that for every vertex, v, in V_H, $\sigma_v = a$. In other words, we will show that each isolated component of $G(C)$ must have representatives of at least two colors.

From Lemma 2, $|V_H| \geq 2$. Assuming (by absurdity) that all members of V_H have the same color, a, we see that in the profitable deviation, each vertex of V_H helps the other vertices of V_H to increase their profits. This means that each vertex of V_H would make a greater profit by deviating itself towards γ, contradicting the minimality of C.

Now, we are able to show that under the considered assumptions, $G(C)$ is an isolated component. We reason via a contrapositive. Suppose that the number of isolated components of C is greater than 1. We have just shown that none of these isolated components can be monochromatic. Let us consider two of these isolated components, C_1 and C_2, and by using $d(u, v)$, denote the length of the shortest path between the vertices $u \in C_1$ and $v \in C_2$ in terms of edges. If $\min_{u \in C_1, v \in C_2} d(u, v) \geq 1$, then the shortest path between any vertex of C_1 and any vertex of C_2 passes through at least one vertex of $V \backslash C$. According to Theorem 1, each vertex of C can only deviate towards the color of one of its neighbors; therefore, a vertex of C_1 deviates towards the color in σ of another vertex of C_1. The same situation happens for C_2. Hence, C_1 does not need C_2 to deviate and vice versa. This contradicts the minimality of C in strongly deviating. Therefore, $G(C)$ must be an isolated component of G. $\square$

Now, we prove that for all the cases where a minimal strongly deviating coalition, C, has, at most, 7 vertices, the cut value increases. This result was presented at the conference mentioned in the introduction [15].

Theorem 2. *Let σ be an NE for G, and let $C \subseteq V$ be a minimal subset that strongly deviates from one coloring, σ, to another γ. If $|C| \leq 7$, then $\Delta CV(\sigma,\gamma) > 0$.*

Proof. We need to consider all the cases for which $|C| \leq 7$. Notice that the cases $|C| \leq 5$ are already covered by [13]. Moreover, in Proposition 4 of [13], it was proven that C is a minimal strongly deviating coalition for $|K_C(\sigma)| \in \{2,5,6\}$ with $|C| = 6$ and $|K_C(\sigma)| \in \{2,6,7\}$ with $|C| = 7$.

From Theorem 1, we know that in the cases $|C| = 6$ and $|C| = 7$, if $|K_C(\sigma)| \geq 6-3 = 3$ or $|K_C(\sigma)| \geq 7-3 = 4$, then $\Delta CV(\gamma,\sigma) > 0$. Then, only the case $|K_C(\sigma)| = 3$ must be accounted for.

Let us assume that $K_C(\sigma) = \{a,b,c\}$. The only possible configurations up to the permutations of the colors are the following four:

Configuration 1 . $|C_a(\sigma)| = 3$, $|C_b(\sigma)| = |C_c(\sigma)| = 2$.
Configuration 2. $|C_a(\sigma)| = |C_b(\sigma)| = 3$, $|C_c(\sigma)| = 1$.
Configuration 3. $|C_a(\sigma)| = 4$, $|C_b(\sigma)| = 2$, and $|C_c(\sigma)| = 1$.
Configuration 4. $|C_a(\sigma)| = 5$, $|C_b(\sigma)| = |C_c(\sigma)| = 1$.

These configurations are briefly reported in Figure 4.

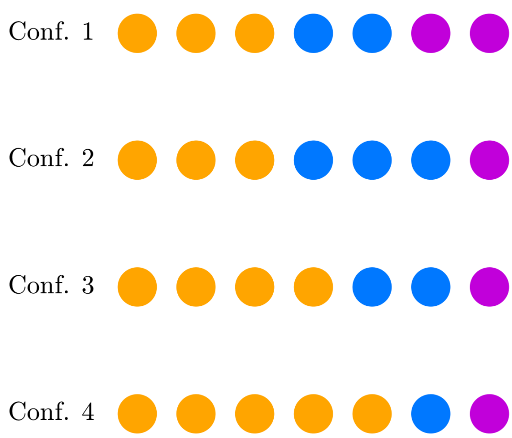

Figure 4. Pictorial representation of the coalition colorings covered by Theorem 2.

Concerning Configurations 1 and 2, the upper limit for $P_C(\sigma,\gamma)$ is less than 7. Specifically, $P_C(\sigma,\gamma) = 5$ for Configuration 1 and $P_C(\sigma,\gamma) = 6$ in Configuration 2. As a result, $\Delta CV(\gamma,\sigma) \geq 1$.

For Configuration 3, $P_C(\sigma,\gamma) = 7$ only if each of the four vertices of color a in σ deviates toward a different color from the other three. However, it is evident that at least two other colors and corresponding vertices are required to ensure this deviation. Even if a vertex of color b and one of color a exchange colors, and the remaining vertex of color b deviates, with another vertex of color a, toward the color c, $P_C(\sigma,\gamma) < 7$. Consequently, $\Delta CV(\gamma,\sigma) \geq 1$.

Finally, in Configuration 4, the feasible values for $P_C(\sigma,\gamma)$ are $P_C(\sigma,\gamma) \in \{10,9,8,7\}$. For $P_C(\sigma,\gamma) \in \{10,9,8\}$, where similar reasoning as in Configuration 3 applies. Conversely, if $P_C(\sigma,\gamma) = 7$, it is impossible for C_a to represent a subgraph without loops of length 3. The presence of all possible graphs with 5 vertices and 7 edges is illustrated in Figure 5. The existence of a loop of length 3, coupled with only two available colors, implies that $P_C(\sigma,\gamma) \neq 7$. Hence, $P_C(\sigma,\gamma) \leq 6$, and consequently, in any case, $\Delta CV(\gamma,\sigma) > 0$, indicating that $CV(\gamma) > CV(\sigma)$. □

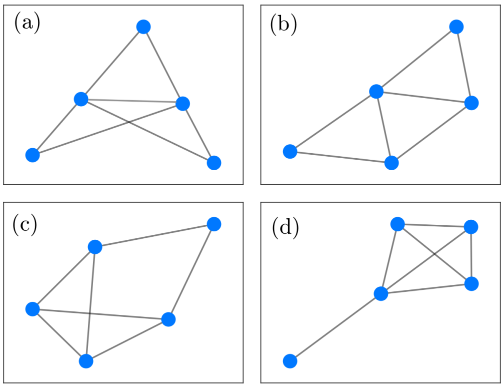

Figure 5. All possible ways in which 5 vertices can be connected, according to Configuration 4 of Theorem 2 for $P_C(\sigma, \gamma) = 7$.

We are now ready to prove the main result of the section. Note that this result was presented at the conference mentioned in the introduction [15].

Theorem 3. *Let σ be an optimal coloring for G. Then, σ is also a 7-SE.*

Proof. Assume, for the sake of contradiction, that the optimal coloring σ is not a 7-SE. According to Theorem 2, this assumption would lead to a contradiction.

Recall that Theorem 2 establishes that for any minimal subset, for a C that strongly deviates from the coloring σ to an alternative coloring γ with $|C| \leq 7$, the change in the number of edges connecting the vertices to different colors, denoted by $\Delta CV(\sigma, \gamma)$, is greater than 0.

Now, suppose σ is not a 7-SE. This implies the existence of a coalition, C, with $|C| \leq 7$ such that there is a co-operative deviation from σ to an alternative coloring γ where $\Delta CV(\sigma, \gamma) \leq 0$. However, Theorem 2 contradicts this scenario, stating that for such a C and γ, $\Delta CV(\sigma, \gamma) > 0$.

Therefore, our assumption that σ is not a 7-SE leads to a contradiction. Hence, we conclude that the optimal coloring σ must be a 7-SE. $\square$

When referring to Example 1, Theorem 3 can be read as follows: if management allocates employees to projects in order to minimize the number of pairs of people who do not want to work together and who are assigned to the same project, then groups of at most seven employees are unable to apply for a transfer to other projects to minimize the number of undesired collaborations. Therefore, it can be seen that the seven employees have no way to move to other projects without having to work alongside people they do not want to work with. Figure 6 depicts all the optimal colorings available for the network introduced in Example 1. In particular, we assign them the following names: σ is the benchmark case, reported in Figure 2c and in Figure 6a, γ^1, γ^2, γ^3, γ^4, and γ^5 are all the other optimal configurations, as shown in Figure 6b–f. The last five optimal colorings were compared to σ, and the corresponding deviating coalition, C, between each couple of optimal colorings were computed in terms of deviating colors. Specifically, the members of the coalition σ that deviate to another optimal coloring γ^i, with $i \in \{1, \ldots, 5\}$ are highlighted by bold black borders. We notice that in all cases, the payoffs, $\mu_C(\gamma^i)$, collected by the members of these coalitions are equal to $\mu_C(\sigma)$ for the one obtained in the benchmark σ. This result shows that no strong deviating coalitions of 7 (or more) members exist in the considered game. Additionally, we also verified that $\mu_v(\sigma) = \mu_v(\gamma^i)$ $\forall v \in C$ and for $i \in \{1, \ldots, 5\}$. Furthermore, the figure shows that the colorings that are color permutations of each other have the same payoff.

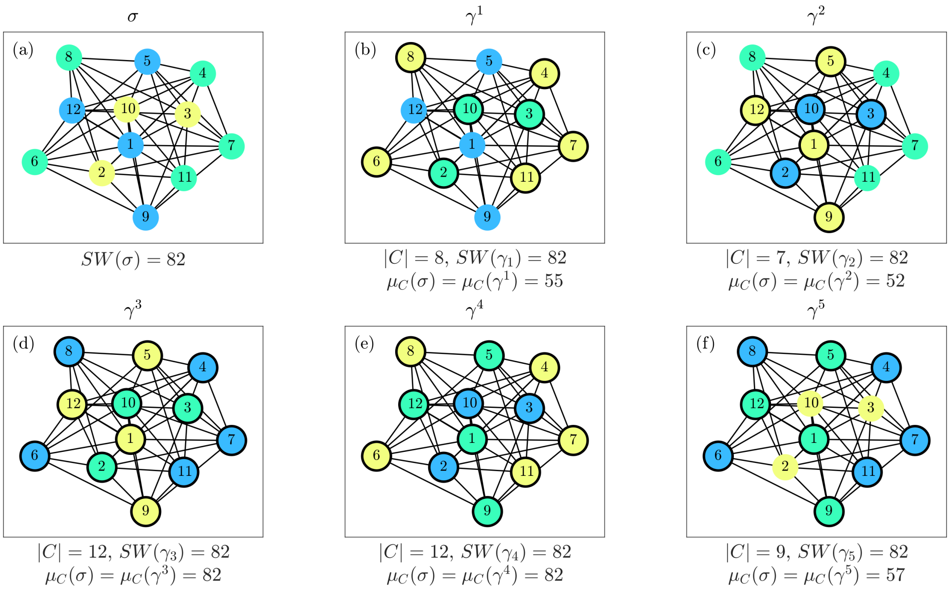

Figure 6. Comparison of the six optimal colorings (σ, γ^1, γ^2, γ^3, γ^4, and γ^5) in a network with 12 vertices and 3 colors. (**a**): Optimal coloring σ. (**b**–**f**): Comparison of the optimal colorings σ, γ^1, γ^2, γ^3, γ^4, and γ^5. The vertices with black edges represent the deviating coalitions from σ.

4. Conclusions

In this work, some results on the robustness to 7-deviating coalitions of optimal colorings are proposed. The method is innovative, leveraging game theory as its foundation. This allows us to apply the Nash equilibrium concept to analyze the emergence of coalitions deviating from optimal configurations. However, we also employ combinatorial properties specific to graphs in relation to optimality and the potential for achieving a SE.

The central outcome of this study is encapsulated in Theorem 3, establishing that the optimal colorings of graphs are 7-strong equilibria. This implies that, for the system to transition from one configuration to another, a coalition exceeding seven divergent elements must be identified. Nonetheless, smaller coalitions within optimal colorings may still deviate toward other existing optimal solutions.

Extending our proof strategy to the case of m-strong equilibria with $m > 7$ presents challenges due to the substantial increase in the number of color configurations.

Despite these challenges, we posit that it might be feasible to extend our findings to larger coalitions, reinforcing the strength of optimal colorings to prevent divergent solutions. We reiterate a conjecture previously formulated (see also [13]):

Conjecture 1. *Every optimal coloring is a strong equilibrium.*

It is noteworthy that a random graph, G, is a graph with n vertices where each potential edge has a probability p of existence. The number of edges in such a graph is a random variable with an expected value of $\binom{n}{2}p$. Specifically, an Erdös-Renyì graph on the vertex set V is a random graph that connects each pair of vertices $\{i, j\}$ with a probability, p, independently. This model is defined by the number of vertices $N = |V|$ and p.

Figure 7 serves as motivation for the conjecture. A numerical experiment, considering two sets of Erdös-Renyì graphs with 10 instances each, involving $n = 15$ vertices with average degrees of 5 and 10, respectively, was conducted. The distribution of optimal colorings across 10 runs for each case is presented. It is evident that optimal colorings of various sizes are present, and notably, none of them qualify as strong, as the payoff of the deviating coalitions consistently fails to increase.

In order to substantiate Conjecture 1, further exploration of the properties of optimal colorings is imperative [17–20]. For instance, a deeper examination of how the neighborhood of a deviating subset is colored in an optimal coloring or the clustering of colors in an optimal coloring could provide valuable insights.

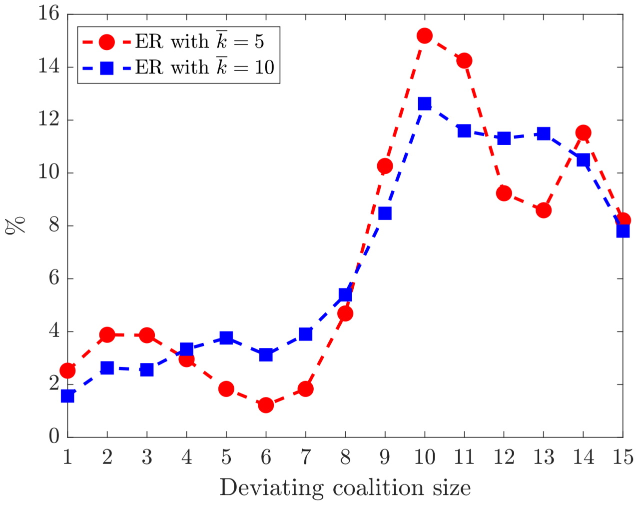

Figure 7. Distribution of the size of the deviating coalitions for different average degrees of $n = 15$ Erdös-Renyì random graphs. Notably, this analysis considers all possible permutations of colorings within each coalition, providing a comprehensive view of the distribution. The results reveal that optimal colorings of diverse sizes are present, and interestingly, none of them qualify as strong deviations, as the payoff of deviating coalitions consistently fails to increase.

Author Contributions: All authors have contributed equally to the research and the writing of the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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