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Calculating Insurance Claim Reserves with an Intuitionistic Fuzzy Chain-Ladder Method

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Abstract: Estimating loss reserves is a crucial activity for non-life insurance companies. It involves adjusting the expected evolution of claims over different periods of active policies and their fluctuations. The chain-ladder (CL) technique is recognized as one of the most effective methods for calculating claim reserves in this context. It has become a benchmark within the insurance sector for predicting loss reserves and has been adapted to estimate variability margins. This variability has been addressed through both stochastic and possibilistic analyses. This study adopts the latter approach, proposing the use of the CL framework combined with intuitionistic fuzzy numbers (IFNs). While modeling with fuzzy numbers (FNs) introduces only epistemic uncertainty, employing IFNs allows for the representation of bipolar data regarding the feasible and infeasible values of loss reserves. In short, this paper presents an extension of the chain-ladder technique that estimates the parameters governing claim development through intuitionistic fuzzy regression, such as symmetric triangular IFNs. Additionally, it compares the results obtained with this method with those derived from the stochastic chain ladder by England and Verrall.

Keywords: loss reserving; chain ladder; probability–possibility transformation; intuitionistic fuzzy numbers; symmetric triangular intuitionistic fuzzy numbers; intuitionistic fuzzy regression

MSC: 91G05; 62P05; 90C05; 62A86; 90C70

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1. Introduction

The estimation of loss reserves is a fundamental process in the management of insurance companies. It consists of setting a prudent value on claims not yet made on active policies, which will ultimately impact the financial statements and the required capital to continue with the current insurance portfolio [1]. Thus, a prudent estimation of these provisions, which ultimately requires the application of so-called actuarial judgement, needs to use a value of maximum reliability but, at the same time, estimate the possible variability around that expected value [2]. The final estimated value for reserves, although it should tend to overestimate them and cover possible unfavorable deviations from their expected value, should not be excessive [3].

Within claim-reserving methods, the actuarial literature often distinguishes between deterministic and stochastic methods. While the former provides a point value of reserves that can be considered the "expected" or maximum confidence value, stochastic methods allow the variability around that reasonable value to be measured [2]. To this commonly accepted typology, we can add fuzzy methods [4].

Among the various applications that fuzzy set theory (FST) has had in insurance mathematics, we can outline the modeling of uncertain and vague parameters with possibility distributions [5,6]. In these applications, fuzzy modeling allows the quantification of epistemic uncertainty, that is, a measure of the reliability with which a certain variable A takes a specific value x [7]. In the context of determining loss reserves, this vagueness may be induced, first, by the imprecision of some of the data available to the insurance company [8].

An additional source to consider is the scarcity of the sample used for reserve calculation; since it is not advisable to use data too far from the present, it can bias estimates due to factors such as changes in judicial practices and public awareness of liability issues [9].

The literature on the variability of loss reserves starts from a scheme used in practice that allows loss reserves to be obtained as a point value. A very common scheme for calculating the value of reliable mathematical reserves is the chain-ladder (CL) method, or variants of this method, such as the London CL or the Bornhuetter–Ferguson methods [10]. The chain-ladder method has been the subject of adaptations that allow modeling of the variability of reserves stochastically [11,12] but also with possibility distributions [13–15].

As shown in Table 1, among the most commonly used schemes for quantifying loss reserves, in addition to the CL method, we can highlight the geometric separation method [16] and methods that model incremental claims in a two-way manner, such as in [17]. The methodology for adjusting the parameters governing the evolution of claims over time can be performed heuristically [4,14,15,18] or with fuzzy regression methods that apply both the principle of minimum fuzziness [19,20] and the fuzzy least-squares approach [21,22].

Table 1. A revision of contributions to claim reserve modeling with fuzzy mathematics.

Method to Fit Fuzzy Parameters	Note Extensions	Taylor's Separation Method	Two-Way Methods
Heuristically	[4,14,15,18]	---	---
FR-MFP	[3,13]	[23,24]	[19,20,25]
FR-FLS	[26]	[21]	[22]

Note: FR-MFP stands for fuzzy regression with the minimum fuzziness principle, and FR-FLS stands for fuzzy least squares.

All of the methods reviewed in Table 1 model the uncertainty of parameters with type-one fuzzy numbers, i.e., simple fuzzy numbers (FNs), that is, through possibility distributions that allow introducing epistemic uncertainty about the real value of the parameters [27]. However, FNs do not allow the introduction of negative information about these parameters that the evaluator might have, i.e., about what the parameters "are not". This paper extends fuzzy loss reserving to the use of bipolar information, i.e., imprecise estimations about the values that the parameters of interest can take and about those they cannot take. Bipolarity does not introduce additional uncertainty but provides new information [28].

Our paper uses the chain-ladder scheme to capture the dynamics of claiming processes and the concept of intuitionistic fuzzy numbers (IFNs) that model uncertain quantities [29] within Atanassov's theory of intuitionistic fuzzy sets [30,31]. Thus, this work expands the practical applications of IFNs, which are relatively scarce in finance and insurance. Among such applications, we can highlight the following:

1. Capital budgeting [32–35];
2. Option pricing [36–38];
3. Productivity measurements [39–41];
4. Actuarial field: while Uzhga-Rebrov and Grabusts [42] use intuitionistic fuzzy values to address environmental risk analysis, Andrés-Sánchez [43] does so to price the life contingencies of people with impaired life expectancies.

This paper falls within the fourth domain, specifically in the field of claim reserving. In this regard, our aim is threefold. First, we demonstrate that the estimation of stochastic loss reserves can be interpreted as estimates made through possibility distributions with type-one fuzzy numbers. Subsequently, we introduce intuitionistic fuzzy regression in claim-reserving calculations. Although fuzzy regression has been applied in several areas of actuarial analysis, such as mortality adjustment [44,45], the use of intuitionistic fuzzy regression in actuarial science is nonexistent. We do so by employing the intuitionistic fuzzy regression method [46], which extends the possibilistic regression models [47–49].

Similarly, we compare the results of the proposed method to those obtained with the stochastic chain-ladder (SCL) method [12].

2. Intuitionistic Fuzzy Numbers

2.1. Fuzzy Numbers and Intuitionistic Fuzzy Numbers

Definition 1. *A fuzzy set (FS) in a referential set X, $\ddot{A}$, is defined as follows [50]:*

$$\ddot{A} = \{\langle x, \mu_A(x)\rangle,\ x \in X\}, \tag{1}$$

where $\mu_A : X \longrightarrow [0,1]$ is the membership function of $\ddot{A}$.

Definition 2. *The fuzzy set $\ddot{A}$ can be represented through level sets or α-cuts, A_α [50]:*

$$A_\alpha = \{x|\mu_A(x) \geq \alpha,\ 0 < \alpha \leq 1\}. \tag{2}$$

Definition 3. *A fuzzy number (FN), $\ddot{A}$, is a fuzzy subset of the real line [51] such that*

i. *is normal, i.e., $\exists x|\mu_A(x) = 1$;*
ii. *is convex, i.e.,$\forall x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1,\ \mu_A(\lambda x_1, (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$.*

Remark 1. *As a consequence, the α-cuts of $\ddot{A}$ and A_α are confidence intervals:*

$$A_\alpha = \{x|\mu_A(x) \geq \alpha,\ 0 < \alpha \leq 1\} = [\underline{A}_\alpha,\ \overline{A}_\alpha], \tag{3}$$

where $\underline{A}_\alpha$ is an increasing function of α and $\overline{A}_\alpha$ is a decreasing function.

Remark 2. *The membership function of $\ddot{A}$, $\mu_A(x)$ is also called the possibility distribution function.*

Fuzzy set theory commonly relies on fuzzy numbers (FNs) to represent imprecise quantities [51]. Specifically, triangular fuzzy numbers are very common in practical applications because the grading of the membership level is linear. This approach is reasonable because it applies the principle of parsimony when dealing with vague information [52].

Definition 4. *A symmetric triangular fuzzy number (STFN) is a particular case in which a triangular fuzzy number (TFN) can be represented by the couple $\ddot{A} = (A, r_A), r_A \geq 0$. Then, the membership function is*

$$\mu_A(x) = \begin{cases} 1 - \frac{|x-A|}{r_A} & |x - A| < r_A \\ 1 & x = A \\ 0 & otherwise \end{cases}, \tag{4}$$

with the following being its α-cut representation:

$$A_\alpha = [\underline{A}_\alpha,\ \overline{A}_\alpha] = [A - r_A(1-\alpha), A + r_A(1-\alpha)],\ 0 \leq \alpha \leq 1. \tag{5}$$

Within TFNs, shapes are of special interest when the available information about the reference variable is scarce and can be summarized in a center and plausible deviations from it [53,54]. Symmetric triangular fuzzy numbers (STFNs) allow for a good balance between comprehensiveness in capturing the available information and the use of the parsimony principle [54]. In insurance modeling, the usefulness of STFNs has been shown in several papers [13,14,55].

Definition 5. *Let us take a continuous random variable* $\boldsymbol{A}$ *and a family of confidence intervals* A_α*, such that* $P(x \in A_\alpha) \geq 1 - \alpha$ *and* $P(\cdot)$ *is a probability measure. Therefore, an equivalent fuzzy quantity* $\ddot{A}$ *has the following α-cut,* A_α *[56]:*

$$A_\alpha = \left[\underline{A}_\alpha,\ \overline{A}_\alpha\right] = \left[\left\{x|F(\boldsymbol{A} \leq x) = \frac{\alpha}{2}\right\},\ \left\{x|F(\boldsymbol{A} \leq x) = 1 - \frac{\alpha}{2}\right\}\right], \tag{6}$$

where $F(\cdot)$ *is the distribution function.*

Remark 3. *Consequently, the possibility distribution function of* $\ddot{A}$ *equivalent to* $\boldsymbol{A}$ *is* $\mu_A(x) = sup\{\alpha|x\epsilon A_\alpha\}$.

It should be emphasized that the interpretation of probabilistic confidence intervals as α-level sets of possibility distributions has been widely argued in the literature [53,54,56–61]. Buckley [58] justified the transformation of a set of probabilistic confidence intervals into fuzzy numbers with the fact that in subsequent calculations, more information is used than simple point estimates or confidence intervals.

Definition 6. *The intuitionistic fuzzy set (IFS)* $\widetilde{A}$ *defined in a referential set* X *is*

$$\widetilde{A} = \{\langle x, \mu_A(x), v_A(x)\rangle,\ x \in X\}, \tag{7}$$

where $\mu_A : X \longrightarrow [0,1]$ *measures the membership of* x *in* $\widetilde{A}$ *and* $v_A : X \longrightarrow [0,1]$ *is nonmembership. These functions must accomplish* $0 \leq \mu_A(x) + v_A(x) \leq 1$.

Remark 4. *The degree of hesitancy,* $h_A(x)$*, of* $\widetilde{A}$ *is* $h_A(x) = 1 - \mu_A(x) - v_A(x)$.

Remark 5. *An IFS generalizes the concept of an FS such that if* $h_A(x) = 0\ \forall x$*,* $\widetilde{A}$ *is a conventional FS* $\ddot{A}$.

Definition 7. *An IFN can be expressed using* $\langle\alpha, \beta\rangle$*-levels or* $\langle\alpha, \beta\rangle$*-cuts, as* $A_{\langle\alpha,\beta\rangle}$*:*

$$A_{\langle\alpha,\beta\rangle} = \{x|\mu_A(x) \geq \alpha,\ v_A(x) \leq \beta, 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in [0,1]\}. \tag{8}$$

Remark 6. $A_{\langle\alpha,\beta\rangle}$ *can be decoupled into two level sets [62], such as* $A_\alpha = \{\,x|\mu_A(x) \geq \alpha\}$ *and* $A^*_\beta = \{x|v_A(x) \leq \beta\}$*, in such a way that*

$$A_{\langle\alpha,\beta\rangle} = \left\langle A_\alpha = \{\,x|\mu_A(x) \geq \alpha\}\ ,\ A^*_\beta = \{x|v_A(x) \leq \beta\}, 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in [0,1]\right\rangle. \tag{9}$$

Definition 8. *An intuitionistic fuzzy number (IFN) is an IFS defined on real numbers, such that*

i. *is normal, i.e.,* $\exists x|\mu_A(x) = 1 \Rightarrow v_A(x) = h_A(x) = 0$;
ii. $\mu_A(x)$ *is convex,* $\forall x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1, \mu_A(\lambda x_1, (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$;
iii. $v_A(x)$ *is concave,* $\forall x_1, x_2 \in \mathbb{R}, 0 \leq \lambda \leq 1, v_A(\lambda x_1, (1-\lambda)x_2) \leq \max(v_A(x_1), v_A(x_2))$.

Remark 7. *The* $\langle\alpha, \beta\rangle$*-cuts of* $\widetilde{A}$ *and* $A_{\langle\alpha,\beta\rangle}$ *can be decoupled as follows:* $A_\alpha = \{\,x|\mu_A(x) \geq \alpha\} = \left[\underline{A}_\alpha,\ \overline{A}_\alpha\right]$ *and* $A^*_\beta = \{v_A(x) \leq \beta\} = \left[\underline{A^*}_\beta, \overline{A^*}_\beta\right]$, $0 \leq \alpha + \beta \leq 1,\ \alpha, \beta \in (0,1)$.

Remark 8. *Thus, from Remark 7, an* $\langle\alpha, \beta\rangle$*-level of* $A_{\langle\alpha,\beta\rangle}$ *can be represented as*

$$A_{\langle\alpha,\beta\rangle} = \left\langle A_\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right], A^*{}_\beta = \left[\underline{A^*}_\beta, \overline{A^*}_\beta\right], 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in (0,1)\right\rangle, \tag{10}$$

where $\underline{A}_\alpha$ and $\overline{A^*}_\beta$ increase with their arguments, α and β, respectively. Similarly, $\overline{A}_\alpha$ and $\underline{A^*}_\beta$ are decreasing with respect to these arguments.

Remark 9. *In an IFN, $\mu_A(x)$ can be interpreted as the lower possibility distribution function of the quantity of interest A, and $\mu_{A^*}(x) = 1 - v_A(x)$ is the upper distribution function of that quantity.*

The functions $\mu_{A^*}(x)$ and $\mu_A(x)$ can be interpreted as bipolar possibility distribution measurements, in such a way that $\mu_{A^*}(x)$ accounts for the potential possibility and $\mu_A(x)$ quantifies the real possibility of A being x [28].

Definition 9. *A symmetric triangular intuitionistic fuzzy number (STIFN) is a particular case of a triangular intuitionistic fuzzy number (TIFN) that can be denoted as $\widetilde{A} = (A, r_A, r_A^*)$, with membership and nonmembership functions:*

$$\mu_A(x) = \begin{cases} 1 - \frac{|x-A|}{r_A} & |x - A| < r_A \\ 1 & x = A \\ 0 & otherwise \end{cases}, \tag{11}$$

and

$$v_A(x) = \begin{cases} \frac{|x-A|}{r_A^*} & |x - A| < r_A \\ 0 & x = A \\ 1 & otherwise \end{cases}, \tag{12}$$

where $r_A \leq r_A^*$. Figure 1 depicts the shape of an STIFN and the relationship between the embedded functions $\mu_A(x)$ (the actual possibility distribution function), $v_A(x)$, $\mu_{A^*}(x)$ (the potential possibility distribution function), and $h_A(x)$.

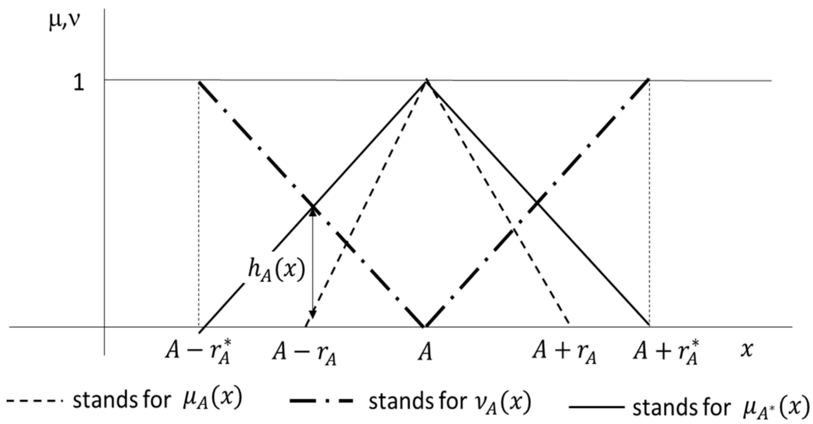

Figure 1. Triangular intuitionistic fuzzy numbers.

Remark 10. *The level sets $A_{\langle\alpha,\beta\rangle}$ of a TIFN can be decoupled into*

$$A_\alpha = \left[\underline{A}_\alpha, \overline{A}_\alpha\right] = [A - r_A(1-\alpha), A + r_A(1-\alpha)], \tag{13}$$

$$A^*{}_\beta = \left[\underline{A^*}_\beta, \overline{A^*}_\beta\right] = [A - r_A^*\beta, A + r_A^*\beta]. \tag{14}$$

Thus, STIFNs are an extension of STFNs such that if $r_A = r_A^*$, we deal with conventional TFNs [63]. Thus, the use of symmetrical TFNs based on the principle of parsimony to justify their use can be extended to the use of STIFNs.

2.2. Intuitionistic Fuzzy Number Arithmetic

The fuzzy loss-reserving methods in Table 1 calculate provisions based on the assumption that the parameters governing the claiming process are determined by fuzzy numbers. Performing arithmetic operations with FNs requires the application of Zadeh's extension principle, which can be implemented through α-cuts [64].

Zadeh's extension principle and its compatibility with α-cuts arithmetic can be extended to the evaluation of functions defined in real numbers when the parameters are IFNs instead of FNs [65]. This paper considers the case of continuous and differentiable functions $y = f(x_1, x_2, \ldots, x_n)$, such that the values of the input variables are given as the means of IFNs $\widetilde{A}_{(i)},\ i = 1, 2, \ldots, n$. This generates an IFN $\widetilde{B}$, $\widetilde{B} = f\left(\widetilde{A}_{(1)}, \widetilde{A}_{(2)}, \ldots, \widetilde{A}_{(n)}\right)$. Thus, the membership and nonmembership functions of $\widetilde{B}$ are as follows:

$$\mu_B(y) = \max_{y=f(x_1,x_2,\ldots,x_n)} \min\left\{\mu_{A_{(1)}}(x_1), \mu_{A_{(2)}}(x_2), \ldots, \mu_{A_{(n)}}(x_n)\right\}, \tag{15}$$

$$v_B(y) = \min_{y=f(x_1,x_2,\ldots,x_n)} \max\left\{v_{A_{(1)}}(x_1), v_{A_{(2)}}(x_2), \ldots, v_{A_{(n)}}(x_n)\right\}. \tag{16}$$

Therefore, if $\widetilde{A}_{(i)},\ i = 1, 2, \ldots, n$ are FNs, it is only necessary to obtain $\mu_B(y)$ using the usual max/min principle. However, we can fit $\widetilde{B}$ thoughout $B_{\langle\alpha,\beta\rangle}$ from ${A_{(i)}}_{\langle\alpha,\beta\rangle}$ by calculating $B_{\langle\alpha,\beta\rangle} = f\left({A_{(1)}}_{\langle\alpha,\beta\rangle}, {A_{(2)}}_{\langle\alpha,\beta\rangle}, \ldots, {A_{(n)}}_{\langle\alpha,\beta\rangle}\right)$. Thus, given that f is continuous, the $\langle\alpha,\beta\rangle$-cuts of $\widetilde{B}$ are defined as $B_{\langle\alpha,\beta\rangle} = \left\langle B_\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right], {B^*}_\beta = \left[\underline{B^*}_\beta, \overline{B^*}_\beta\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in [0,1]\right\rangle$, where

$$\underline{B}_\alpha = \inf\left\{y \middle| y = f(x_1, \ldots, x_n), x_i \in {A_{(i)}}_\alpha\right\}, \overline{B}_\alpha = \sup\left\{y \middle| y = f(x_1, \ldots, x_n), x_i \in {A_{(i)}}_\alpha\right\}, \tag{17}$$

$$\underline{B^*}_\beta = \inf\left\{y \middle| y = f(x_1, \ldots, x_n), x_i \in {A^*_{(i)}}_\beta\right\}, \overline{B^*}_\beta = \sup\left\{y \middle| y = f(x_1, \ldots, x_n), x_i \in {A^*_{(i)}}_\beta\right\}. \tag{18}$$

Following [66], when f monotonically increases with respect to $x_i,\ i = 1, 2, \ldots m$ and monotonically decreases in $x_i,\ i = m+1, m+2, \ldots, n$, $m \le n$, $B_\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right]$ is as follows:

$$\begin{aligned} \underline{B}_\alpha &= f\left(\underline{A_{(1)}}_\alpha, \underline{A_{(2)}}_\alpha, \ldots, \underline{A_{(m)}}_\alpha, \overline{A_{(m+1)}}_\alpha, \ldots, \overline{A_{(n)}}_\alpha\right) \text{ and} \\ \overline{B}_\alpha &= f\left(\overline{A_{(1)}}_\alpha, \overline{A_{(2)}}_\alpha, \ldots, \overline{A_{(m)}}_\alpha, \underline{A_{(m+1)}}_\alpha, \ldots, \underline{A_{(n)}}_\alpha\right). \end{aligned} \tag{19}$$

By analogy, the β-cut representation of ${B^*}_\beta = \left[\underline{B^*}_\beta, \overline{B^*}_\beta\right]$ is

$$\begin{aligned} \underline{B^*}_\beta &= f\left(\underline{A^*_{(1)}}_\beta, \underline{A^*_{(2)}}_\beta, \ldots, \underline{A^*_{(m)}}_\beta, \overline{A^*_{(m+1)}}_\beta, \ldots, \overline{A^*_{(n)}}_\beta\right) \text{ and} \\ \overline{B^*}_\beta &= f\left(\overline{A^*_{(1)}}_\beta, \overline{A^*_{(2)}}_\beta, \ldots, \overline{A^*_{(m)}}_\beta, \underline{A^*_{(m+1)}}_\beta, \ldots, \underline{A^*_{(n)}}_\beta\right). \end{aligned} \tag{20}$$

The linear combination of STIFNs is also an STIFN. Therefore, from the STIFNs $\widetilde{A}_{(i)} = \left(A_{(i)}, r_{A_{(i)}}, r^*_{A_{(i)}}\right)$, $\widetilde{B} = (B,\ r_B,\ r^*_B\ )$, where [46]

$$B = \sum_{i=1}^{n} \lambda_i A_{(i)},\ \ r_B = \sum_{i=1}^{n} |\lambda_i| \cdot r_{A_{(i)}},\ \ r^*_B = \sum_{i=1}^{n} |\lambda_i| \cdot r^*_{A_{(i)}}. \tag{21}$$

The evaluation of nonlinear functions using STIFNs does not produce a new STIFN. Despite this limitation, we feel that maintaining a linear shape is relevant. Following the argument in [67] justifying the use of approximating linear fuzzy numbers, complicated forms of IFNs may cause drawbacks in processing imprecise information modeled by these fuzzy structures, and the interpretation of the results becomes more difficult. The same argument, based on the parsimony principle, can be used to maintain the symmetrical structure of the input data.

Thus, we evaluate the approximation to $\widetilde{B} = f\left(\widetilde{A}_{(1)}, \widetilde{A}_{(2)}, \ldots, \widetilde{A}_{(n)}\right)$ with an STIFN $\widetilde{B}^T = (B, r_B, r^*_B)$ when the inputs are $\widetilde{A}_{(i)} = \left(A_{(i)}, r_{A_{(i)}}, r^*_{A_{(i)}}\right), i = 1, 2, \ldots, n$. To do this, we rely on the results of [51] to approximate an LR fuzzy number to the result of a nonlinear function of LR fuzzy numbers, which is based on the linear approximation of α-cuts with a Taylor expansion. In the field of FNs, this methodology produces an STFN that approximates the functions of STFNs, as shown in several actuarial applications [13–15,55].

The extremes of the α-cuts $B_\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right]$ in (19) are approximated by means of a Taylor expansion to the first grade from $\alpha = 1$ to any $\alpha \in [0, 1]$. To do this, we use the gradient $\nabla f(\boldsymbol{A}) = \left(\frac{\partial f}{\partial x_1}(\boldsymbol{A}), \frac{\partial f}{\partial x_2}(\boldsymbol{A}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{A})\right)$, such that $\boldsymbol{A} = (A_1, A_2, \ldots, A_n)$. Therefore, $\underline{B}_\alpha$ can be developed as follows:

$$\begin{aligned} \underline{B}_\alpha \approx \underline{B^T}_\alpha &= f(\boldsymbol{A}) + \left(\sum_{i=1}^m \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r_{A_{(i)}}\right)(\alpha - 1) - \left(\sum_{i=m+1}^n \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r_{A_{(i)}}\right)(\alpha - 1) \\ &= f(\boldsymbol{A}) - \left(\sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r_{A_{(i)}}\right)(1 - \alpha). \end{aligned} \tag{22}$$

Analogously, we develop $\overline{B}_\alpha$ as follows:

$$\begin{aligned} \overline{B}_\alpha \approx \overline{B^T}_\alpha &= f(\boldsymbol{A}) - \left(\sum_{i=1}^m \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r_{A_{(i)}}\right)(\alpha - 1) + \left(\sum_{i=m+1}^n \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r_{A_{(i)}}\right)(\alpha - 1) \\ &= f(\boldsymbol{A}) + \left(\sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r_{A_{(i)}}\right)(1 - \alpha). \end{aligned} \tag{23}$$

Similarly, $B^*{}_\beta = \left[\underline{B^*}_\beta, \overline{B^*}_\beta\right]$, whose exact values are given in (20), can also be determined via Taylor expansion to the first grade from $\beta = 0$ to $\beta \in (0, 1]$. Therefore, for $\underline{B^*}_\beta$, we state

$$\begin{aligned} \underline{B^*}_\beta \approx \underline{B^{T*}}_\beta &= f(\boldsymbol{A}) - \left(\sum_{i=1}^m \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r^*_{A_{(i)}}\right)\beta + \left(\sum_{i=m+1}^n \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r^*_{A_{(i)}}\right)\beta \\ &= f(\boldsymbol{A}) - \left(\sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r^*_{A_{(i)}}\right)\beta. \end{aligned} \tag{24}$$

In the same manner, we expand $\overline{B^*}_\beta$ as follows:

$$\begin{aligned} \overline{B^*}_\beta \approx \overline{B^{T*}}_\beta &= f(\boldsymbol{A}) + \left(\sum_{i=1}^m \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r^*_{A_{(i)}}\right)\beta - \left(\sum_{i=m+1}^n \frac{\partial f}{\partial x_i}(\boldsymbol{A}) \cdot r^*_{A_{(i)}}\right)\beta \\ &= f(\boldsymbol{A}) + \left(\sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r^*_{A_{(i)}}\right)\beta. \end{aligned} \tag{25}$$

Consequently, from (22), (23), (24), and (25), we find that

$$B = f(\boldsymbol{A}), \;\; r_B = \sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r_{A_{(i)}}, \;\; r^*_B = \sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}(\boldsymbol{A})\right| \cdot r^*_{A_{(i)}}. \tag{26}$$

Analogous to [68], we evaluate the relative error measurement in the bounds of $B_{\langle\alpha,\beta\rangle}$, whose exact value can be calculated with (19)–(20), by those of its symmetrical triangular approximation, $B^T_{\langle\alpha,\beta\rangle}$, which must be stated by applying (26) in (11) and (12). Thus, the deviations in $B_\alpha = \left[\underline{B}_\alpha, \overline{B}_\alpha\right]$ are

$$\underline{\varepsilon}_\alpha = \frac{\left|\underline{B}_\alpha - \underline{B^T}_\alpha\right|}{\underline{B}_\alpha}, \overline{\varepsilon}_\alpha = \frac{\left|\overline{B}_\alpha - \overline{B^T}_\alpha\right|}{\overline{B}_\alpha}, \tag{27}$$

and for $B^*{}_\beta = \left[\underline{B^*}_\beta, \overline{B^*}_\beta\right]$,

$$\underline{\varepsilon^*}_\beta = \frac{\left|\underline{B^*}_\beta - \underline{B^{T*}}_\beta\right|}{\underline{B^*}_\beta}, \overline{\varepsilon^*}_\beta = \frac{\left|\overline{B^*}_\beta - \overline{B^{T*}}_\beta\right|}{\overline{B^*}_\beta}. \tag{28}$$

2.3. Intuitionistic Linear Regression with the Minimum Fuzziness Principle and Symmetric Coefficients

Within the fuzzy field, there are a large number of regression methodologies that can be divided into two main groups. In the first type, we can group those based on the minimum fuzziness principle (MFP), and the second includes those based on the minimization of the distance between observations and predictions, such as models that can be labeled fuzzy least-squares models [69].

This dichotomy between the minimum fuzziness principle and distance minimization is also observed in intuitionistic fuzzy regression models. For example, [46] extended the minimum fuzziness principle to an intuitionistic regression setting, and [70] developed a least-squares intuitionistic fuzzy regression methodology. Our paper uses the first approach. Therefore, our model is based on fuzzy regression models with symmetric parameters.

Let us suppose that the equation to be fitted has a dependent factor dependent on m real-value explanatory variables x_i , i = 0, 1, 2, …, m, where $x_0 = 1$ and $x_i \epsilon \mathbb{R}$, $i = 1, 2, \ldots, n$. The outcome is then a linear function of intuitionistic coefficients $\widetilde{A}_{(i)} = \left(A_{(i)}, r_{A_{(i)}}, r^*_{A_{(i)}}\right)$, $i = 0, 1, \ldots, m$ and, thus, an STIFN $\widetilde{Y} = \left(Y, r_Y, r^*_Y\right)$. This is obtained from (21), as follows:

$$Y = \sum_{j=0}^{m} A_{(i)} x_i,\ r_Y = \sum_{i=0}^{n} r_{A_{(i)}} |x_i|,\ r^*_Y = \sum_{i=0}^{n} r^*_{A_{(i)}} |x_i|. \tag{29}$$

Moreover, both the observations of the input variables and the output variable are crisp, which is a common hypothesis in intuitionistic fuzzy regression models. Thus, for the jth observation, the outcome is the crisp number y_j, generated by the crisp income $(1, x_{1j},\ x_{2j}, \ldots, x_{ij}, \ldots, x_{im})$. Therefore, y_j is a possible value of a TIFN $\widetilde{Y}_j = \left(Y_j, r_{Y_j}, r^*_{Y_j}\right)$, whose membership function $\mu_{Y_j}(y_j)$ and nonmembership function $\nu_{Y_j}(y_j)$ in (11)–(12) are determined from (29)

$$Y_j = \sum_{i=0}^{m} A_{(i)} x_{ij},\ r_{Y_j} = \sum_{i=0}^{m} r_{A_{(i)}} |x_{ij}|,\ r^*_{Y_j} = \sum_{i=0}^{m} r^*_{A_{(i)}} |x_{ij}|. \tag{30}$$

The objective is to fit for $\widetilde{A}_{(i)}$, $i = 0, 1, 2, \ldots, m$, an STIFN estimate $\widetilde{a}_{(i)} = \left(a_{(i)}, r_{a_{(i)}}, r^*_{a_{(i)}}\right)$, $i = 0, 1, 2, \ldots, m$ that simultaneously maximizes the membership of the observations in the fitted system and minimizes the uncertainty of that system. Therefore, to find $\widetilde{a}_{(i)}$, the following multiple-objective programming problem must be implemented:

$$\underset{A_{(i)}, r_{A_{(i)}} r^*_{A_{(i)}}, i=0,1,\ldots,n}{\text{minimize}} \left(-\alpha,\ \beta,\ z_1 = \sum_{j=1}^{n} r_{Y_j}\ ,\ z_2 = \sum_{j=1}^{n} r^*_{Y_j}\right),$$

which is subject to

$$\mu_{Y_j}(y_j) \geq\ \alpha,\ \nu_{Y_j}(y_j) \leq \beta,\ j = 1, 2, \ldots, n, r_{A_{(i)}}, r^*_{A_{(i)}} \geq 0,\ i = 0, 1, \ldots,\ m. \tag{31}$$
$$0 \leq \alpha + \beta \leq 1, \alpha, \beta \in [0, 1].$$

To solve (31), we implement the following steps:

Step 1: We state a minimum reachable value $\alpha = g$ and $\beta = 1 - g - h$. Like in possibilistic regression models, $g \in [0, 1)$ scales the total fuzziness of the estimated system. If $g = 0$, the uncertainty of the system is minimal; conversely, the inclusiveness of the observations may be low. On the other hand, a higher g causes all observations to be included with greater intensity, and the predictions of the fitted system are less specific [49].

The value of $h \in [0, 1 - g)$ reflects the level of hesitancy in the system. For $h = 0$, the actual and potential possibility of a particular value are identical; therefore, we have a conventional possibilistic regression. At this step, we decouple (31) as follows:

$$\underset{A_{(i)}, r_{A_{(i)}}, i=0,1,\ldots,n}{\text{minimize}} \; z_1 = \sum_{j=1}^{n} r_{Y_j} \;,$$

subject to

$$\mu_{Y_j}(y_j) \geq g,\; j = 1,\, 2, \ldots,\, n,\; r_{A_{(i)}} \geq 0, i = 0,\, 1, \ldots, m, \tag{32}$$

and

$$\underset{A_{(i)}, r^*_{A_{(i)}}, i=0,1,\ldots,n}{\text{minimize}} \; z_2 = \sum_{j=1}^{n} r^*_{Y_j} \,,$$

subject to

$$\nu_{Y_j}(y_j) \leq 1 - g - h, = 1,\, 2, \ldots,\, n,\; r^*_{A_{(i)}} \geq 0,\; i = 0, 1, \ldots, m. \tag{33}$$

Step 2: We initially state for (32)–(33) that $g = h = 0$. This implies the minimum fuzziness level and no hesitancy. Thus, we adjust a possibilistic regression model and $r_{A_{(i)}} = r^*_{A_{(i)}}$. This leads us to obtain the estimates of $A_{(i)}$ and $r_{A_{(i)}}$, which we denote as $a^{(0)}_{(i)}$ and $r^{(0)}_{a_{(i)}}$, respectively, where $i = 0, 1, \ldots, m$. Thus, we must solve the following:

$$\underset{A_{(i)}, r_{A_{(i)}}, i=0,1,\ldots,n}{\text{minimize}} \; z_1 = z_2 = \sum_{i=0}^{m} r_{A_{(i)}} \sum_{j=1}^{n} |x_{ij}| \;,$$

subject to

$$\sum_{i=0}^{m} A_{(i)} x_{ij} - \sum_{i=0}^{m} r_{A_{(i)}} |x_{ij}| \leq y_j \leq \sum_{i=0}^{m} A_{(i)} x_{ij} + \sum_{i=0}^{m} r_{A_{(i)}} |x_{ij}|,\; j = 1, 2, \ldots, n \tag{34}$$

$$A_{(i)}, r_{A_{(i)}} \geq 0,\; i = 0, 1, \ldots, m.$$

Step 3: To fit the centers in (34), $a^{(0)}_{(i)}$ for $A_{(i)}$ and $r^{(0)}_{a_{(i)}}$ for $r_{A_{(i)}}$ $i = 0, 1, \ldots, m$, the literature proposes two alternatives:

- Alternative 1. The values of $a^{(0)}_{(i)}$ and $r^{(0)}_{a_{(i)}}$ are those that are solved in a unique step (34). In this case, the centers $a^{(0)}_{(i)}$ are those that are obtained in a quantile regression at the median, identical to [71].
- Alternative 2. The value $a^{(0)}_{(i)}$ in the first step is obtained by using ordinary least squares [72]. However, there is no reason why any other method, such as the maximum likelihood or weighted least-squares methods, cannot be used. In the second step, $r^{(0)}_{a_{(i)}}$ is obtained by solving (34) and taking into account that this linear programming problem is as follows, after independently stating $a^{(0)}_{(i)}$:

$$\underset{r_{A_{(i)}}, i=0,1,\ldots,n}{\text{minimize}} \; z_1 = z_2 = \sum_{i=0}^{m} r_{A_{(i)}} \sum_{j=1}^{n} |x_{ij}| \;,$$

subject to

$$-\sum_{i=0}^{n} r_{A_{(i)}}|x_{ij}| \leq y_j - \sum_{i=0}^{n} a_{(i)}^{(0)} x_{ij} \leq \sum_{i=0}^{n} r_{A_{(i)}}|x_{ij}|,\ j = 1,2,\ldots,n \tag{35}$$

$$r_{A_{(i)}} \geq 0,\ i = 0,1,\ldots,m.$$

Step 4: We establish the optimal value of g based on this criterion. This value optimizes what these authors refer to as the credibility of the system [73]. To achieve this, we define the estimation of $\widetilde{Y}_j$, obtained from the parameters adjusted in Step 1 and Step 3 as $\widetilde{y}_j^{(0)} = \left(y_j^{(0)}, r_{y_j}^{(0)}, r_{y_j}^{(0)}\right)$, i.e., $\widetilde{y}_j^{(0)}$ is an STFN where $y_j^{(0)} = \sum_{i=0}^{m} a_{(i)} x_{ij}$, and $r_{y_j}^{(0)} = \sum_{i=0}^{m} r_{a_{(i)}}^{(0)} |x_{ij}|$. So,

$$g = \begin{cases} \frac{1}{2}\left(1 - \frac{\gamma^{(0)}}{\delta^{(0)}}\right) & \gamma^{(0)} < \delta^{(0)} \\ 0 & otherwise \end{cases}, \tag{36}$$

where $\gamma^{(0)} = \sum_{j=1}^{n} \frac{\mu_{y_j^{(0)}}(y_j)}{r_{y_j}^{(0)}}$, $\delta^{(0)} = \sum_{j=1}^{n} \frac{1-\mu_{y_j^{(0)}}(y_j)}{r_{y_j}^{(0)}}$ and then we state that

$$r_{a_{(i)}} = \frac{r_{a_{(i)}}^{(0)}}{1-g}. \tag{37}$$

Step 5: We subsequently proceed to obtain the estimates of $r_{a_{(i)}}^{*}$. To achieve this, the decision maker must determine the degree of hesitancy in the system, where $h \epsilon [0, 1-g)$. In the case where $h = 0$, there is no hesitancy; if $h \to 1-g$, the level of hesitancy tends to be at its maximum. Thus,

$$r_{a_{(i)}}^{*} = \frac{r_{a_{(i)}}^{(0)}}{1-g-h}. \tag{38}$$

3. An Intuitionistic Chain Ladder for Claim Reserving

3.1. Claim Reserving with the Chain-Ladder Method and Stochastic Variability and a Probability–Possibility Transformation

The historical data illustrating the evolution of claims are typically presented in a run-off triangle format, similar to Table 2 [10]. In this table, $C_{i,j}$ represents the accumulated claim cost of insurance contracts originating in the ith development period ($i = 0, 1, \ldots, n$) during the jth claiming period ($j = 0, 1, \ldots, n$). Therefore, the accumulated claims $C_{i,j}$, $i = 1, 2, \ldots, n$; $j = n - i + 1, n - i + 2, \ldots, n$ are unknown and must be fitted.

Table 2. Run-off triangle of accumulated claims.

				Development/Payment Period				
	$i \mid j$	**0**	**1**	**…**	$j = n - i$	**…**	$n - 1$	n
	0	$C_{0,0}$	$C_{0,1}$	…	$C_{0,j}$	…	$C_{0,n-1}$	$C_{0,n}$
	1	$C_{1,0}$	$C_{1,1}$	…	$C_{1,j}$	…	$C_{1,n-1}$	
	⋮	⋮	⋮	⋮	⋮	⋮		
Occurrence/Origin Period	i	$C_{i,0}$	$C_{i,1}$	…	$C_{i,n-i}$	…		
	⋮	⋮	⋮	⋮				
	$n-1$	$C_{n-1,0}$	$C_{n-1,1}$	…				
	n	$C_{n,0}$		…				

An alternative way to present historical data consists of the run-off triangle of incremental claims, in a way similar to Table 3. Table 3 can be obtained from Table 2 by taking into account that $S_{i,j} = C_{i,j} - C_{i,j-1}$, $i = 0, 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-i$, and $S_{i,0} = C_{i,0}$.

Therefore, the incremental claims $S_{i,j}$, $i = 1, 2, \ldots, n$; $j = n - i + 1, n - i + 2, \ldots, n$ are unknown and must be fitted.

Table 3. Run-off triangle of incremental claims.

		Development/Payment Period						
		0	**1**	**…**	$\boldsymbol{j = n - i}$	**…**	$\boldsymbol{n - 1}$	$\boldsymbol{n}$
Occurrence/Origin Period	0	$S_{0,0}$	$S_{0,1}$	…	$S_{0,j}$	…	$S_{0,n-1}$	$S_{0,n}$
	1	$S_{1,0}$	$S_{1,1}$	…	$S_{1,j}$	…	$S_{1,n-1}$	
	⋮	⋮	⋮	⋮	⋮	⋮		
	i	$S_{i,0}$	$S_{i,1}$	…	$S_{i,n-i}$	…		
	⋮	⋮	⋮	⋮				
	$n-1$	$S_{n-1,0}$	$S_{n-1,1}$	…				
	n	$S_{n,0}$		…				

The triangle of accumulated claims (Table 2) is the input of several common methods to fit claim reserves, such as the chain-ladder method or the Bornhuetter–Ferguson method. The key concept of the CL method is the so-called link ratio between development year j and $j + 1$, F_j, which allows us to obtain the cumulative claims of the $(j + 1)$th development period from those of the jth period:

$$C_{i,j+1} = F_j \cdot C_{i,j} \Longrightarrow F_j = \frac{C_{i,j+1}}{C_{i,j}}, \tag{39}$$

where the available observations of F_j are as follows:

$$f_{i,j} = \frac{C_{i,j+1}}{C_{i,j}},\ i = 0, 1, \ldots, n - j - 1 \tag{40}$$

To obtain an average value of F_j, f_j, we consider the widely used CL, which provides an unbiased estimator of F_j [11]. Thus, the average development factor for the jth year is

$$f_j = \frac{\sum_{i=0}^{n-j-1} C_{i,j+1}}{\sum_{i=0}^{n-j-1} C_{i,j}}. \tag{41}$$

The terminal value of accumulated claims for the ith origin year $C_{i,n}$, $i = 1, 2, \ldots, n$ is approximated by $c_{i.n}$, as follows:

$$c_{i.n} = C_{i,n-i} \prod_{j=n-i}^{n-1} f_j, \tag{42}$$

and $c_{i.n}$ is an increasing function of development factors, since the partial derivative $\frac{\partial c_{i,n}}{\partial f_j}$ is

$$\frac{\partial c_{i.n}}{\partial f_j} = C_{i,n-i} \prod_{\substack{k=n-i \\ k \neq j}}^{n-1} f_k. \tag{43}$$

Thus, the reserves are linked with the origin year $i = 1, 2, \ldots, n$, RO_i:

$$RO_i = c_{i.n} - C_{i,n-i} = C_{i,n-i}\left(\prod_{j=n-i}^{n-1} f_j - 1\right). \tag{44}$$

So, the overall provisions, R, are

$$R = \sum_{i=1}^{n} RO_i. \tag{45}$$

The classical chain-ladder method is deterministic. However, this methodology is flexible enough to generate stochastic estimates of fluctuations by applying the SCL method, which is implemented in the following six steps:

1. Obtain the estimates of the observations $C_{i,j}$, $c_{i,j}$, $i = 0, 1, \ldots, n$; $j < n - i$ by using f_j backwards from $C_{i,n-i}$.
2. Calculate an estimate of observed incremental claims (Table 2) by stating $s_{i,j} = c_{i,j} - c_{i,j-1}$, in the case of $s_{i,0} = c_{i,0}$.
3. Calculate the descaled Pearson residuals due to fitting the real incremental claims in Table 2, $S_{i,j}$, with $s_{i,j}$:
$$m_{i,j} = \frac{S_{i,j} - s_{i,j}}{\sqrt{s_{i,j}}}, i = 0, 1, \ldots, n; j \leq n - i.$$
4. Resample $m_{i,j}$, $i = 0, 1, \ldots, n$; $j \leq n - i$. Therefore, we find $m^b_{i,j}$, $i = 0, 1, \ldots, n$; $j \leq n - i$.
5. Calculate the incremental claims $s^b_{i,j} = s_{i,j} + \sqrt{s_{i,j}} m^b_{i,j}$, $i = 0, 1, \ldots, n$; $j \leq n - i$. This implies adjusting a new Table 3.
6. From Table 3, in the above step, we can resample the accumulated claims and construct Table 2. This new table allows us to obtain the development factors (40) and reserves (44) and (45). These six steps can be implemented B times in such a way that predictions of claiming reserves can be obtained as confidence intervals.

Note that Steps 1–6 allow B simulations of loss reserves to be obtained for every origin year $RO_i^{(b_i)}$, $b_i = 1, 2, \ldots, B$ and the whole reserve $R^{(b)}$ and $b = 1, 2, \ldots, B$. Without losing generality, let us suppose that in all the cases, $RO_i^{(b_i)} \leq RO_i^{(b_i+1)}$. Then, the reserve RO_i is contained with a probability $1 - \alpha$ in the interval $RO_{i\alpha}$, such that

$$RO_{i\alpha} = \left[\underline{RO_i}_{\alpha}, \overline{RO_i}_{\alpha}\right] = \left[RO_i^{(round[B \cdot \frac{\alpha}{2}])}, RO_i^{(round[B \cdot (1 - \frac{\alpha}{2})])}\right], \tag{46}$$

which can be interpreted as the α-cuts of a possibilistic estimate of the reserves of the ith year $\ddot{RO}_i$.

Therefore, we can estimate a confidence interval for the overall reserves in two ways. A conservative estimate, R'_α, is $R'_\alpha = \sum_{i=1}^{n} RO_{i\alpha}$, such that R'_α can be considered the α-cuts of the possibility distribution $\ddot{R}\prime$:

$$R'_\alpha = \left[\underline{R'}_\alpha, \overline{R\prime}_\alpha\right] = \left[\sum_{i=1}^{n} \underline{RO_i}_\alpha, \sum_{i=1}^{n} \overline{RO_i}_\alpha\right] \tag{47}$$

A more specific approximation of the overall reserves implies inducing a confidence interval with a probability level $1 - \alpha$, R_α from $R^{(b)} \leq R^{(b+1)}$, $b = 1, 2, \ldots, B$ by calculating the following:

$$R_\alpha = \left[\underline{R}_\alpha, \overline{R}_\alpha\right] = \left[R^{(round[B \cdot \frac{\alpha}{2}])}, R^{(round[B \cdot (1 - \frac{\alpha}{2})])}\right]. \tag{48}$$

Therefore, from the probabilistic confidence interval R_α, we can induce a possibility distribution function $\ddot{R}$ by considering Definition 5.

3.2. An Intuitionistic Fuzzy Chain-Ladder Method

3.2.1. Fitting Symmetrical Intuitionistic Triangular Fuzzy Development Factors

Let us express relation (39), in which, from a known accumulated claim amount in the jth development period, we must obtain the accumulated quantity in the $(j+1)$th development period, which is uncertain because the development factor is an STIFN. So,

$$\widetilde{C}_{i,j+1} = \widetilde{F}_{(j)} \cdot C_{i,j} \tag{49}$$

where $\widetilde{C}_{i,j+1} = \left(C_{i,j+1}, r_{C_{i,j+1}}, r^*_{C_{i,j+1}}\right)$ and $\widetilde{F}_{(j)} = \left(F_{(j)}, r_{F_{(j)}}, r^*_{F_{(j)}}\right)$. Therefore, from (21), $C_{i,j+1} = F_{(j)} \cdot C_{i,j}$, $r_{C_{i,j+1}} = C_{i,j} \cdot r_{F_{(j)}}$ and $r^*_{C_{i,j+1}} = C_{i,j} \cdot r^*_{F_{(j)}}$.

To fit $\widetilde{F}_{(j)}$ by means of $\widetilde{f}_{(j)} = \left(f_{(j)}, r_{f_{(j)}}, r^*_{f_{(j)}}\right)$, we consider the data in Table 2. The couples (y,x) are defined as $\left(C_{i,j+1}, C_{i,j}\right), i = 0, 1, \ldots, n-j-1$. Therefore, $\widetilde{f}_{(j)} = \left(f_{(j)}, r_{f_{(j)}}, r^*_{f_{(j)}}\right)$ is first fitted, where $f^{(0)}_{(j)} = f_{(j)}$ and $r^{(0)}_{f_{(j)}} = r_{f_{(j)}} = r^*_{f_{(j)}}$ are the optimum values of the arguments in the programming problem (32)–(33) for $g = h = 0$. Therefore, we must solve the version of (34) for relation (49):

$$\underset{F_{(j)}, r_{F_{(j)}}}{\text{minimize}} z_1 = z_2 = r_{F_{(j)}} \sum_{i=0}^{n-j-1} C_{i,j}\ ,$$

which is subject to

$$F_{(j)} C_{i,j} - C_{i,j} r_{F_{(j)}} \le C_{i,j+1} \le F_{(j)} C_{i,j} + C_{i,j} r_{F_{(j)}},\ i = 0, 1, \ldots, n-j-1. \tag{50}$$

$$r_{F_{(j)}} \ge 0.$$

By dividing the inclusion constraints in (50) by $C_{i,j}$, $i = 0, 1, \ldots, n-j-1$, the independent terms turn into (40), $f_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}$. Likewise, the cost function of (50) has only one argument. Therefore, the linear pro-gramming problem becomes

$$\underset{F_{(j)}, r_{F_{(j)}}}{\text{minimize}} z_1 = z_2 = r_{F_{(j)}},$$

which is subject to

$$F_{(j)} - r_{F_{(j)}} \le f_{i,j} \le F_{(j)} + r_{F_{(j)}},\ i = 0, 1, \ldots, n-j-1. \tag{51}$$

$$r_{F_{(j)}} \ge 0.$$

To solve (51), we can follow Alternatives 1 and 2 in Section 2.3. By using Alternative 1, the solution of that linear programming problem allows us to obtain $f^{(0)}_{(j)}$ as the result of the quantile regression at the median and, simultaneously, $r^{(0)}_{f_{(j)}}$. Models (51) can be implemented by Alternative 2 in Section 2.3 by prefixing $f^{(0)}_{(j)}$ with the CL formula (41). In this case, the linear programming problem (51) becomes

$$\underset{r_{F_{(j)}}}{\text{minimize}} z_1 = z_2 = r_{F_{(j)}}\ ,$$

which is subject to

$$-r_{F_{(j)}} \le f_{i,j} - f^{(0)}_{(j)} \le r_{F_{(j)}},\ i = 0, 1, \ldots, n-j-1. \tag{52}$$

$$r_{F_{(j)}} \ge 0,$$

and so,

$$r^{(0)}_{f_{(j)}} = \max_{i=0,1,\ldots,n-i-1} \left| f_{i,j} - f^{(0)}_{(j)} \right|. \tag{53}$$

Then, the empirical estimates of $\widetilde{F}_{(j)} = \left(F_{(j)}, r_{F_{(j)}}, r^*_{F_{(j)}}\right)$, $\widetilde{f}_{(j)} = \left(f_{(j)}, r_{f_{(j)}}, r^*_{f_{(j)}}\right)$, $j = 0, 1, \ldots, n-1$, are obtained as follows:

i. Considering that $f_{(j)} = f^{(0)}_{(j)}$,
ii. $r_{f_{(j)}}$ is obtained by calculating g in Step 4 of Section 2.3 with (36) and (37).
iii. Finally, $r^*_{f_{(j)}}$ is adjusted in Step 5 of Section 2.3 by subjectively stating the degree of system hesitancy, h, and using (38).

The structure of the data in Table 2 leads us to obtain the last development factor $\widetilde{f}_{(n-1)}$ with only the pair $(C_{0,n}, C_{0,n-1})$. Therefore, it is easy to verify that $f_{(j)} = f^{(0)}_{(j)} = \frac{C_{0,n}}{C_{0,n-1}}$, but this approach also leads to the unrealistic conclusion that it is a certain parameter, i.e., $r_{f_{(j)}} = r^*_{f_{(j)}} = 0$. Mack [11], in his stochastic free-distribution modeling of reserves over the CL model, addresses this issue based on the intuition that the absolute uncertainty of the development factors tends to decrease over time, as does the expected value of these factors. Thus, the standard deviation of the development factor F_{n-1} is estimated as the minimum of the standard deviation of F_{n-3} and F_{n-2} and the ratio between the variance of F_{n-3} and the standard deviation of F_{n-2}. Taking this idea into consideration and considering that k times the standard deviation of random quantities can be interpreted as the radius of an equivalent STFN [54,56], $\widetilde{f}_{(n-1)} = \left(f_{(n-1)}, r_{f_{(n-1)}}, r^*_{f_{(n-1)}}\right)$, where

$$f_{(n-1)} = \frac{C_{0,n}}{C_{0,n-1}}, \tag{54}$$

and

$$r_{f_{(n-1)}} = \min\left\{ \frac{{r_{f_{(n-3)}}}^2}{r_{f_{(n-2)}}}, r_{f_{(n-3)}}, r_{f_{(n-2)}} \right\},\ r^*_{f_{(n-1)}} = \min\left\{ \frac{{r^*_{f_{(n-3)}}}^2}{r^*_{f_{(n-2)}}}, r^*_{f_{(n-3)}}, r^*_{f_{(n-2)}} \right\}. \tag{55}$$

3.2.2. Fitting Reserves with Symmetric Triangular Intuitionistic Fuzzy Development Factors

To state the reserves, we must estimate the terminal value of the claims in every origin year $i = 1, 2, \ldots, n$, $\widetilde{c}_{i,n}$, which can be expressed through its $\langle \alpha, \beta \rangle$-cuts as follows:

$$c_{i,n\,\langle\alpha,\beta\rangle} = \left\langle c_{i,n\,\alpha} = \left[\underline{c_{i,n}}_{\alpha}, \overline{c_{i,n}}_{\alpha}\right], c^*_{i,n\,\beta} = \left[\underline{c^*_{i,n}}_{\beta}, \overline{c^*_{i,n}}_{\beta}\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in (0,1) \right\rangle. \tag{56}$$

Specifically, $c_{i,n\,\langle\alpha,\beta\rangle}$ is obtained from $C_{i,n-i}$ and the $\langle \alpha, \beta \rangle$-cuts of $\widetilde{f}_{(j)}$, $f_{(j)\,\langle\alpha,\beta\rangle}$, $j = n-i, n-i+1, \ldots, n-1$ by adapting (42) to an intuitionistic setting:

$$c_{i,n\,\langle\alpha,\beta\rangle} = C_{i,n-i} \prod_{j=n-i}^{n-1} f_{(j)\,\langle\alpha,\beta\rangle}, \tag{57}$$

and thus, $c_{i,n\,\alpha} = \left[\underline{c_{i,n}}_{\alpha}, \overline{c_{i,n}}_{\alpha}\right]$ is obtained considering that (42) is an increasing function of development factors:

$$\begin{aligned} c_{i,n\,\alpha} &= \left[C_{i,n-i} \prod_{j=n-i}^{n-1} \underline{f_{(j)}}_{\alpha}, C_{i,n-i} \prod_{j=n-i}^{n-1} \overline{f_{(j)}}_{\alpha} \right] \\ &= \left[C_{i,n-i} \prod_{j=n-i}^{n-1} \left(f_{(j)} - r_{f_{(j)}} (1-\alpha) \right), C_{i,n-i} \prod_{j=n-i}^{n-1} \left(f_{(j)} + r_{f_{(j)}} (1-\alpha) \right) \right]. \end{aligned} \tag{58}$$

Similarly, $c^*_{i,n\beta} = \left[\underline{c^*_{i,n}}_\beta, \overline{c^*_{i,n\beta}}\right]$ is calculated as follows:

$$\begin{aligned} c^*_{i,n\beta} &= \left[C_{i,n-i}\prod_{j=n-i}^{n-1}\underline{f^*_{(j)}}_\beta, C_{i,n-i}\prod_{j=n-i}^{n-1}\overline{f^*_{(j)\beta}}\right] \\ &= \left[C_{i,n-i}\prod_{j=n-i}^{n-1}\left(f_{(j)} - r^*_{f_{(j)}}\beta\right), C_{i,n-i}\prod_{j=n-i}^{n-1}\left(f_{(j)} + r^*_{f_{(j)}}\beta\right)\right]. \end{aligned} \tag{59}$$

Note that $\widetilde{c}_{i,n}$ is not an STIFN. However, by using derivatives (43) and (22)–(25), we can approximate $\widetilde{c}_{i,n} \approx \widetilde{c}^T_{i,n} = \left(c_{i,n}, , r_{c_{i,n}}, r^*_{c_{i,n}}\right)$, where the center is

$$c_{i,n} = C_{i,n-i}\prod_{j=n-i}^{n-1} f_{(j)}, \tag{60}$$

and the radii are

$$r_{c_{i,n}} = C_{i,n-i}\sum_{j=n-i}^{n-1}\left(\prod_{\substack{k=n-i\\k\neq j}}^{n-1} f_k\right) r_{f_{(j)}}, r^*_{c_{i,n}} = C_{i,n-i}\sum_{j=n-i}^{n-1}\left(\prod_{\substack{k=n-i\\k\neq j}}^{n-1} f_k\right) r^*_{f_{(j)}}. \tag{61}$$

Therefore, by using (44), we can obtain the intuitionistic reserves for the ith origin year $\widetilde{RO}_i$ through $RO_{i\langle\alpha,\beta\rangle} = \left\langle RO_{i\alpha} = \left[\underline{RO_i}_\alpha, \overline{RO_{i\alpha}}\right], RO^*_{i\beta} = \left[\underline{RO^*_i}_\beta, \overline{RO^*_{i\beta}}\right], 0 \le \alpha+\beta \le 1, \alpha,\beta \in (0,1)\right\rangle$ by calculating:

$$RO_{i\langle\alpha,\beta\rangle} = c_{i,n\langle\alpha,\beta\rangle} - C_{i,n-i}. \tag{62}$$

Then,

$$\begin{aligned} RO_{i\alpha} &= \left[C_{i,n-i}\left(\prod_{j=n-i}^{n-1}\left(f_{(j)} - r_{f_{(j)}}(1-\alpha)\right) - 1\right), C_{i,n-i}\left(\prod_{j=n-i}^{n-1}\left(f_{(j)} + r_{f_{(j)}}(1-\alpha)\right) - 1\right)\right], \\ RO^*_{i\beta} &= \left[C_{i,n-i}\left(\prod_{j=n-i}^{n-1}\left(f_{(j)} - r^*_{f_{(j)}}(1-\alpha)\right) - 1\right), C_{i,n-i}\left(\prod_{j=n-i}^{n-1}\left(f_{(j)} + r^*_{f_{(j)}}(1-\alpha)\right) - 1\right)\right]. \end{aligned} \tag{63}$$

The intuitionistic fuzzy estimate of reserves of the ith year is not an STIFN. However, an STIFN approximate $\widetilde{RO}_i \approx \widetilde{RO}_i^T = \left(RO_i, r_{RO_i}, r^*_{RO_i}\right)$ is obtained by the following:

$$\widetilde{RO}_i^T = c^T_{i,n} - C_{i,n-i} = \left(c_{i,n} - C_{i,n-i}, r_{c_{i,n}}, r^*_{c_{i,n}}\right), \tag{64}$$

and so, considering (60) and (61),

$$\begin{aligned} RO_i &= \widetilde{c}^T_{i,n} - C_{i,n-i} = C_{i,n-i}\left(\prod_{j=n-i}^{n-1} f_j - 1\right) \\ r_{RO_i} &= C_{i,n-i}\sum_{j=n-i}^{n-1}\left(\prod_{\substack{k=n-i\\k\neq j}}^{n-1} f_k\right) r_{f_{(j)}}, r^*_{RO_i} = C_{i,n-i}\sum_{j=n-i}^{n-1}\left(\prod_{\substack{k=n-i\\k\neq j}}^{n-1} f_k\right) r^*_{f_{(j)}}. \end{aligned} \tag{65}$$

Similarly, an intuitionistic fuzzy estimate of the overall reserve $\tilde{R}$ is obtained with (45) through $R_{\langle\alpha,\beta\rangle} = \left\langle R_\alpha = \left[\underline{R}_\alpha, \overline{R}_\alpha\right], R^*_\beta = \left[\underline{R^*}_\beta, \overline{R^*}_\beta\right], 0 \le \alpha + \beta \le 1, \alpha, \beta \in (0,1)\right\rangle$. By implementing $R_{\langle\alpha,\beta\rangle} = \sum_{i=1}^{n} RO_{i\langle\alpha,\beta\rangle}$. Then,

$$R_\alpha = \left[\sum_{i=1}^{n} \underline{RO_i}_\alpha, \sum_{i=1}^{n} \overline{RO}_{i\alpha}\right], R^*_\beta = \left[\sum_{i=1}^{n} \underline{RO^*_i}_\beta, \sum_{i=1}^{n} \overline{RO^*_i}_\beta\right]. \tag{66}$$

Therefore, an STIFN approximate to $\tilde{R} \approx \tilde{R}^T = \left(R, r_R, r^*_R\right)$ is obtained simply as follows:

$$\tilde{R}^T = \sum_{i=1}^{n} \overset{\sim}{RO}_i^T = \left(\sum_{i=1}^{n} RO_i, \sum_{i=1}^{n} r_{RO_i}, \sum_{i=1}^{n} r^*_{RO_i}\right). \tag{67}$$

4. Empirical Application

4.1. Estimating Loss Reserves with Deterministic and Stochastic Chain-Ladder Method

Below, we present an empirical application based on the run-off triangle of accumulated claims shown in Table 4. These data were utilized in [74,75]. Table 4 also illustrates the development factors found using (41). Thus, we observe that a crisp development factor f_0 = 1.899 is estimated, indicating that the accumulated claims from development years zero to one increase on average by 89.90% for all origin years. Similarly, we can interpret the estimates of the development factors f_1, f_2, f_3, and f_4.

Table 5 presents the individual reserves obtained for each of the origin years i = 1, 2, . . ., 5 and the total reserves with a deterministic CL. Thus, we can observe that as the origin year increases, the reserve to be allocated increases, as claims from more development years are pending. It can be noted that in both Tables 4 and 5, we obtain the expected values of the link ratios and reserves, but we do not have any estimation of their variability. This analysis is carried out in Tables 6 and 7, where reserves are estimated using the SCL method, and the obtained possibilistic confidence intervals are interpreted as possibility distributions, using Definition 5 of Section 2.

Table 6 displays a table of incremental claims analogous to Table 3 that is deduced from Table 4. Table 4 also shows the theoretical table of incremental claims that are deduced from the development factors of the chain-ladder method. The difference between the observed and theoretical tables of incremental claims through descaled Pearson residuals allows the implementation of the SCL method, described in Section 3.1, to fit the variability of reserves by origin year and total reserves.

Table 7 presents the results obtained with $B = 5000$ bootstrapping resamples. The confidence intervals were calculated with Equations (46)–(48). The upper endpoints of the confidence intervals obtained for probability levels α = 0, 0.01, 0.05, and 0.1 are the 100%, 99.5%, 97.5%, and 95% estimated percentiles for the reserves, respectively. These quantiles are commonly used to estimate extreme claim scenarios.

Within the total reserves, we distinguished two confidence intervals: R'_α (47) and R_α (48). The former arises from adding the confidence intervals associated with the reserves of each origin year. Thus, as shown in Table 7, for a confidence level of 100% $(\alpha = 0)$, we obtained the overall reserves R'_0 = [63.68, 97.77] + [526.48, 628.62] + [1507.29, 1677.46] + [2720.66, 2997.42] + [4641.72, 5105.40] = [9459.82, 10,506.67]. The most prudent reserve value would be 10506.67 since it arises from the sum of the estimates of the value that accumulates 100% probability of the reserves from each origin year. In contrast, R_α arises from the application of (45) in each of the B = 5000 simulations, making it a narrower confidence interval. Table 7 shows that R_0 = [9533.03, 10,481.08], so the prudent value for the reserve based on this confidence interval is 10,481.08.

Table 7 also shows that the reserves of each origin year, and the total reserves can be estimated through a possibility distribution by gathering and fitting successive confidence intervals (46)–(48) from $\alpha = \varepsilon$ ($\varepsilon \approx 0$) to $\alpha = 1$. By applying Definition 5 and Remark 3, we can adjust the reserves of the fifth period to a possibility distribution $\ddot{RO}_5$ whose core is 4826.23 and support [4641.72, 5105.40]. Likewise, Table 6 also shows that we can obtain a possible estimate of overall reserves $\ddot{R\prime} = \sum_{i=1}^{5} \ddot{RO}_i$ whose center is 9899.31, supporting [9459.82, 10,506.67].

Table 4. Run-off triangle of accumulated claims used in this paper.

$i \mid j$	0	1	2	3	4	5
0	1001	1855	2423	2988	3335	3403
1	1113	2103	2774	3422	3844	
2	1265	2433	3233	3977		
3	1490	2873	3883			
4	1725	3261				
5	1889					
f_j	1.899	1.329	1.232	1.120	1.020	

Source: Faculty and Institute of Actuaries [74].

Table 5. Deterministic loss reserves obtained by using the chain-ladder method.

RO_1	RO_2	RO_3	RO_4	RO_5	R
78.38	567.93	1584.67	2842.10	4826.23	9899.31

Table 6. Run-off triangles of observed incremental claims and theoretical incremental claims with chain-ladder development factors (41).

Observed Incremental Claims							Theoretical Incremental Claims						
$i \mid j$	0	1	2	3	4	5	$i \mid j$	0	1	2	3	4	5
0	1001	854	568	565	347	68	0	957.27	861.02	598.44	561.04	357.24	68
1	1113	990	671	648	422		1	1103.37	992.43	689.78	646.66	411.76	
2	1265	1168	800	744			2	1278.49	1149.95	799.26	749.30		
3	1490	1383	1010				3	1538.06	1383.41	961.53			
4	1725	1536					4	1716.81	1544.19				
5	1889						5	1889					

Source: Own elaboration from the Faculty and Institute of Actuaries [74].

Table 7. Estimates of reserves with bootstrapping confidence intervals and chain-ladder development factors.

α	$RO_{1\alpha}$	$RO_{2\alpha}$	$RO_{3\alpha}$	$RO_{4\alpha}$
1	[78.38, 78.38]	[567.93, 567.93]	[1584.67, 1584.67]	[2842.10, 2842.10]
0.25	[76.94, 79.83]	[564.14, 571.63]	[1577.02, 1592.89]	[2830.92, 2852.36]
0.5	[75.08, 81.62]	[560.41, 576.65]	[1569.64, 1602.17]	[2820.37, 2866.04]
0.75	[71.78, 87.40]	[554.77, 583.26]	[1558.79, 1613.60]	[2804.07, 2883.77]
0.1	[69.09, 91.11]	[548.46, 589.15]	[1547.12, 1625.22]	[2787.26, 2901.24]
0.05	[68.06, 92.87]	[544.98, 593.35]	[1539.65, 1633.24]	[2775.83, 2915.09]
0.01	[66.74, 94.64]	[537.02, 602.96]	[1522.32, 1651.93]	[2751.60, 2942.85]
0	[63.68, 97.77]	[526.48, 628.62]	[1507.29, 1677.46]	[2720.66, 2997.42]

α	$RO_{5\alpha}$	R'_α	R_α
1	[4826.23, 4826.23]	[9899.31, 9899.31]	[9899.31, 9899.31]
0.25	[4808.51, 4838.96]	[9857.53, 9935.67]	[9866.55, 9930.80]
0.5	[4792.94, 4859.91]	[9818.44, 9986.38]	[9834.38, 9972.34]
0.75	[4768.92, 4887.95]	[9758.33, 10,055.97]	[9786.68, 10,028.49]
0.1	[4739.83, 4918.17]	[9691.76, 10,124.88]	[9733.84, 10,078.13]
0.05	[4721.56, 4937.74]	[9650.07, 10,172.29]	[9702.70, 10,107.32]
0.01	[4682.60, 4994.47]	[9560.29, 10,286.87]	[9644.19, 10,193.22]
0	[4641.72, 5105.40]	[9459.82, 10,506.67]	[9533.03, 10,481.08]

Note: (a) $1-\alpha$ represents the confidence level of the probabilistic confidence interval, which can be interpreted as the α-cut of the equivalent fuzzy number; (b) R'_α is the overall reserve calculated by summing the confidence intervals $\sum_{i=1}^{5} RO_{i_\alpha}$ and R_α, which are the confidence intervals of the reserves, by applying bootstrapping.

4.2. Estimating Loss Reserves with a Symmetric Triangular Intuitionistic Fuzzy Chain Ladder

Next, we put into work the methodology developed in Section 3.2, which allows us to estimate the claim reserves with STIFNs, with the data of the run-off triangle in Table 4. We also compare the results obtained with those of the bootstrap estimates using the SCL method in Table 6, which we reinterpret as α-cuts of possibility distributions. Therefore, to obtain the estimates of $\widetilde{F}_{(0)} = \left(F_{(0)}, r_{F_{(0)}}, r^*_{F_{(0)}}\right)$ and $\widetilde{f}_{(0)} = \left(f_{(0)}, r_{f_{(0)}}, r^*_{f_{(0)}}\right)$, we solve the linear programming problem (51), whose constraints are built up with the link ratios of each origin year $i = 1, 2, \ldots, 5$, as shown in Table 8:

$$\underset{F_{(0)}, r_{F_{(0)}}}{\text{minimize}} z_1 = z_2 = r_{F_{(0)}},$$

which is subject to

$$\begin{gathered} F_{(0)} - r_{F_{(0)}} \leq 1.853 \leq F_{(0)} + r_{F_{(0)}}, \\ F_{(0)} - r_{F_{(0)}} \leq 1.889 \leq F_{(0)} + r_{F_{(0)}}, \\ F_{(0)} - r_{F_{(0)}} \leq 1.923 \leq F_{(0)} + r_{F_{(0)}}, \\ F_{(0)} - r_{F_{(0)}} \leq 1.928 \leq F_{(0)} + r_{F_{(0)}}, \\ F_{(0)} - r_{F_{(0)}} \leq 1.890 \leq F_{(0)} + r_{F_{(0)}}, \\ r_{F_{(0)}} \geq 0. \end{gathered}$$

Table 8. Run-off triangle of individual link ratios, $f_{i,j}$, $i = 0, 1, \ldots, 4$; $j = 0, 1, \ldots, n - i - 1$.

$i \mid j$	0	1	2	3	4	5
0	1.853	1.306	1.233	1.116	1.020	
1	1.889	1.319	1.234	1.123		
2	1.923	1.329	1.230			
3	1.928	1.352				
4	1.890					
5						

Table 9 shows the STIFNs adjusted to development factors for $j = 0, 1, 2, 3, 4$. Thus, if the estimate $F_{(0)}$ is not prefixed with (41), we obtain $f^{(0)}_{(0)} = 1.891$ and $r^{(0)}_{f_{(0)}} = 0.038$, and (36) and (37) allow us to obtain an optimum uncertainty level for membership functions of development factor $g = 0.14$. Thus, from (38), $r_{f_{(0)}} = 0.044$. The degree of system hesitancy, h, must be estimated subjectively by the decision maker. This may be linked, for example, to the perceived reliability of the data or the predictability of the insurance environment. The calculations in this numerical application are performed with $h = 0.1$, so we obtain $r^*_{f_{(0)}} = 0.049$.

In Section 3.2, we also state that $f^{(0)}_{(0)}$ can be predefined with the deterministic CL shown in Table 4. Therefore, $f^{(0)}_{(0)} = 1.899$, and by using (51), $r^{(0)}_{f_{(j)}} = 0.046$. Equations (36) and (37) allow us to obtain an optimum uncertainty degree $g = 0$. Therefore, from (38) $r_{f_{(0)}} = 0.046$ and by using the hesitancy level $h = 0.1$, $r^*_{f_{(0)}} = 0.051$.

Note that the spreads of $\widetilde{f}_{(4)}$, $r_{f_{(4)}}$ and $r^*_{f_{(4)}}$ cannot be obtained from the sample in Table 4 since only one individual link ratio exists. To fit these spreads, we use (55) and then set the following:

$$r_{f_{(4)}} = \min\left\{\frac{0.0035^2}{0.0067}; 0.0035; 0.0067\right\} = 0.0035$$

$$r^*_{f_{(4)}} = \min\left\{ \frac{0.0042^2}{0.0081}; 0.0042; 0.0081 \right\} = 0.0042.$$

Table 9. Symmetric triangular intuitionistic fuzzy number estimation of development factors with $h = 0.1$.

Parameters of Intuitionistic Fuzzy Regression (Alternative 1)					
j	**0**	**1**	**2**	**3**	**4**
$f_{(j)}$	1.891	1.329	1.232	1.120	1.020
$r^{(0)}_{f_{(j)}}$	0.038	0.023	0.002	0.004	---
g	0.140	0.179	0.457	0.500	---
$r_{f_{(j)}}$	0.044	0.028	0.003	0.007	0.003
$r^*_{f_{(j)}}$	0.049	0.031	0.004	0.009	0.004
Parameters of Intuitionistic Fuzzy Regression (Alternative 2)					
j	**0**	**1**	**2**	**3**	**4**
$f_{(j)}$	1.8995	1.3291	1.2321	1.1200	1.0204
$r^{(0)}_{f_{(j)}}$	0.0463	0.0229	0.0020	0.0038	---
g	0.0000	0.0000	0.4179	0.4274	---
$r_{f_{(j)}}$	0.0463	0.0229	0.0035	0.0067	0.0035
$r^*_{f_{(j)}}$	0.0515	0.0255	0.0042	0.0081	0.0042

Table 10 shows the estimates of the overall loss reserves of the intuitionistic claim reserves calculated with the two alternatives proposed in Section 3.2, $\tilde{R}$. Thus, first, we compute the "exact" $\langle\alpha, \beta\rangle$-cuts of both methods. This involves using (58) and (59) to determine the terminal accumulated claims; (63) to find $\tilde{RO}_i$; and $i = 1, 2, \ldots, 5$ and (66) to determine the total value. Table 10 also shows the STIFN approximation of the total reserve, $\tilde{R}^T$, which is obtained using the sequential use of (60), (61), (65), and (67). Table 10 also shows the errors calculated with (27) and (28). Their values suggest that the symmetric triangular approximation is almost perfect. Note that the maximum error lies in the β-cuts of the nonmembership function at $\beta = 1$ and does not exceed 0.15% in any case.

The results of Table 7, which come from bootstrapping resamples, can be interpreted as α-level sets of possibility distributions. Therefore, they can be compared with fuzzy intuitionistic estimates, which are constructed through two possibility distributions. In other words, the probabilistic intervals obtained with bootstrapping and the $\langle\alpha, \beta\rangle$-cuts can be interpreted by the actuary in a similar manner. Thus, according to Table 7, the value of reserves that includes 100% of their possible values could be given as 10,506.67 if we sum the 100th percentile of the reserves associated with all origin years, and 10,481.08 if we consider the 100th percentile of the bootstrap simulations of overall reserves. These results are similar and comparable to those obtained with the membership function of the overall reserves obtained in Table 10. We can observe in the α-cuts of the reserves, R^T_α, that if they are calculated with Alternative 1, their prudent estimate can range between 10,391.15 (at the 0.25-cut) and 10,565.45 (at the 0-cut). The conclusions we can draw from the fit obtained with Alternative 2 are similar, as the upper end of the 0-cut is 10,563.24 and that of the 0.25-cut is 10,397.25.

The β-cuts of the nonmembership functions complement the information provided by the α-cuts of the membership functions, introducing the existence of bipolarity. Thus, in Table 10, Alternative 1 for estimating the development factors offers an upper bound at the 0-cut for the loss reserves of $\overline{R^T}_0 = 10{,}565.45$ and an upper limit of the 1-cut of the nonmembership function, $\overline{R^{T*}}_1 = 10{,}688.71$. Thus, the use of IFNs in reserve estimation allows us to first obtain an estimation of the most extreme possible scenario (10,565.45), whose adjustment does not use subjective information at any time but rather uses only run-off triangle data. That is, the meaning of the estimation is analogous to that obtained with stochastic simulation or what we would obtain with the use of possibilistic regression. However, the use of IFNs also allows us to obtain an estimation of the scenario that we could classify as potentially more extreme through the higher value of the 1-cut of the nonmembership function. The quantification of this scenario requires the participation of the decision maker, who must indicate a perceived degree of hesitancy, which in this numerical application was $h = 0.1$.

Table 10. $\langle\alpha, \beta\rangle$-cuts of the intuitionistic fuzzy estimates of overall reserves with the two methodologies proposed in this paper.

		Alternative 1				**Alternative 2**			
α	β	$\underline{R}_\alpha$	$\overline{R}_\alpha$	$\underline{R^*}_\beta$	$\overline{R^*}_\beta$	$\underline{R}_\alpha$	$\overline{R}_\alpha$	$\underline{R^*}_\beta$	$\overline{R^*}_\beta$
1	0	9868.25	9868.25	9868.25	9868.25	9899.31	9899.31	9899.31	9899.31
0.75	0.25	9694.52	10,043.12	9663.92	10,074.16	9733.86	10,065.82	9708.72	10,091.30
0.5	0.5	9521.94	10,219.16	9461.17	10,281.66	9569.46	10,233.40	9519.52	10,284.69
0.25	0.75	9350.50	10,396.36	9260.00	10,490.77	9406.10	10,402.04	9331.70	10,479.50
0	1	9180.19	10,574.72	9060.38	10,701.50	9243.80	10,571.76	9145.26	10,675.72
		$\tilde{R}^T = (9868.25,\ 697.21,\ 820.46)$				$\tilde{R}^T = (9899.31,\ 663.93,\ 765.15)$			
α	β	$\underline{R^T}_\alpha$	$\overline{R^T}_\alpha$	$\underline{R^{T*}}_\beta$	$\overline{R^{T*}}_\beta$	$\underline{R^T}_\alpha$	$\overline{R^T}_\alpha$	$\underline{R^{T*}}_\beta$	$\overline{R^{T*}}_\beta$
1	0	9868.25	9868.25	9868.25	9868.25	9899.31	9899.31	9899.31	9899.31
0.75	0.25	9693.94	10,042.55	9663.13	10,073.36	9733.33	10,065.29	9708.02	10,090.60
0.5	0.5	9519.64	10,216.85	9458.01	10,278.48	9567.35	10,231.27	9516.73	10,281.88
0.25	0.75	9345.34	10,391.15	9252.90	10,483.59	9401.36	10,397.25	9325.45	10,473.17
0	1	9171.04	10,565.45	9047.78	10,688.71	9235.38	10,563.24	9134.16	10,664.46
α	β	$\underline{\varepsilon}_\alpha$	$\overline{\varepsilon}_\alpha$	$\underline{\varepsilon^*}_\beta$	$\overline{\varepsilon^*}_\beta$	$\underline{\varepsilon}_\alpha$	$\overline{\varepsilon}_\alpha$	$\underline{\varepsilon^*}_\beta$	$\overline{\varepsilon^*}_\beta$
1	0	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.75	0.25	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
0.5	0.5	0.02%	0.02%	0.03%	0.03%	0.02%	0.02%	0.03%	0.03%
0.25	0.75	0.06%	0.05%	0.08%	0.07%	0.05%	0.05%	0.07%	0.06%
0	1	0.10%	0.09%	0.14%	0.12%	0.09%	0.08%	0.12%	0.11%

Note: The errors $\overline{\varepsilon^*}_\beta$, $\overline{\varepsilon}_\alpha$, and $\underline{\varepsilon^*}_\beta$, $\overline{\varepsilon^*}_\beta$ are obtained with (27) and (28).

Using other types of modeling for the underlying link ratios in the run-off triangle, such as LR or adaptive functions, allows us to obtain the same <0,1>-cut and <1,0>-cut as our method if the IFN estimates have the same centers and radii. However, the rest of the $\langle\alpha, \beta\rangle$-cuts, which can be assimilated to structured simulations of the variables involved in the analysis, would change, varying their amplitude. In the case of using adaptive functions, the linear functions used in this paper can be considered as a baseline, with an order of $m = 1$. From this baseline, $m < 1$ implies a dilation of the $\langle\alpha, \beta\rangle$-levels, and thus, they will incorporate more uncertainty. In contrast, $m > 1$ indicates a contraction of the results compared to those obtained with STIFNs. Thus, the $\langle\alpha, \beta\rangle$-cuts will have a smaller width.

Table 11 shows the estimated reserves associated with the five origin years and the overall reserves through the STIFNs. The use of this type of IFN can be very useful for applying the actuarial judgement required to set a definite crisp value for the loss reserves. In the case of total reserves, if we take Alternative 2 from Section 3.2 as a reference for decision making, the most reliable value is 9899.31, which coincides with (40). Possible deviations of up to 663.93 are estimated, and deviations exceeding 765.15 are considered not possible. Regarding deviations between 663.93 and 765.15, there is hesitancy about their feasibility. When considering only the data in the run-off triangle, the conclusion must be that they are not possible. On the other hand, they are possible based on the degree of hesitancy perceived by the decision maker.

Table 11. Symmetrical triangular intuitionistic fuzzy estimates of reserves.

	Alternative 1	Alternative 2
$\tilde{RO}_1^T$	(78.38, 12.28, 15.05)	(78.38, 13.34, 16.11)
$\tilde{RO}_2^T$	(567.93, 43.40, 53.90)	(567.93, 42.66, 51.62)
$\tilde{RO}_3^T$	(1584.67, 66.38, 82.21)	(1584.67, 66.72, 80.69)
$\tilde{RO}_3^T$	(2842.10, 200.87, 236.11)	(2842.10, 179.74, 207.03)
$\tilde{RO}_4^T$	(4795.16, 374.28, 433.19)	(4826.23, 361.47, 409.70)
$\tilde{R}$	(9868.25, 697.21, 820.46)	(9899.31, 663.93, 765.15)

5. Conclusions and Further Research

The determination of insurance loss reserves must be prudent, necessitating the quantification of their expected value and potential deviations from that value. To ascertain the most plausible value, a statistical method such as the chain-ladder (CL) method is utilized to estimate the expected claim evolution. Subsequently, it is necessary to estimate possible deviations from these values with greater reliability. The contributions of this work include providing tools for estimating and interpreting such values using fuzzy set theory and intuitionistic fuzzy set theory.

The first contribution of this work is to show that the information obtained through stochastic models such as bootstrapping and the use of conventional fuzzy numbers are similar. In fact, we can reinterpret the value and variability of reserves obtained with the stochastic CL (SCL) methodology with possibility distributions. Therefore, both instruments capture epistemic uncertainty.

The second and main contribution of our work is the generalization of developments in claim reserving with fuzzy numbers to the use of intuitionistic fuzzy numbers (IFNs). This tool allows the introduction of bipolar information about possible reserve variability into the estimation, i.e., both "positive" information about feasible parameter values and negative information about those that cannot be taken in any case.

This work assumes that the parameters governing the evolution of claims are symmetrical and triangular IFNs (STIFNs). Special attention is given to the approximation of each IFN to be of the same nature as the results that arise from its functional handling. Linear shapes often provide effective resolution in practical applications of fuzzy set theory. Moreover, symmetry often allows for a good balance between parsimony and comprehensiveness in capturing available information and facilitates interpretability of the results by end-users who may not necessarily have knowledge of fuzzy logic. The value of loss reserves when the development factors are estimated using the STIFN technique can be easily approximated through the most likely scenario, obtained with conventional chain-ladder methodology, and by evaluating the deviations from this value with the gradient function of the terminal value of claims from each origin year in the spreads of the membership and nonmembership functions of the link ratios.

The results provided by the proposed method can be very useful in actuarial practice since they can be interpreted very intuitively by the person responsible for establishing reserves, as there is no need for knowledge of fuzzy set theory. While the center of an STIFN quantifies, in a very synthetic way, the most reliable value of reserves, the two spreads provide an approximation of the maximum deviations from this value, the maximum achievable deviation, and the first not-achievable deviation. On the other hand, representing reserves through $\langle\alpha, \beta\rangle$-cuts allows for the structuring of simulations on their appropriate value in multiple scenarios, which can be of great help to decision makers.

Certainly, the limitation of using STIFNs is that they do not account for asymmetry in the link ratios, and similarly, they do not allow for the introduction of more refined calibration of possibility distributions, such as adaptive membership functions. This latter issue implies that introducing nuances, such as concentration and dilation, is not possible.

However, the proposed scheme can be adapted to accommodate more sophisticated forms of membership and nonmembership functions.

Our extension of intuitionistic regression can be applied in other financial and actuarial contexts where possibilistic regression has already been used, such as, for example, estimating the implied moments of options [76,77]. A natural extension of this work would involve introducing intuitionistic uncertainty into the analysis of non-life insurance claims, expanding the results obtained with fuzzy numbers to calculate discounted reserves [78], the discounted values of non-life insurance liabilities [79], or the terminal values of an insurance company [80,81].

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