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Cauchy Problem with Summable Initial-Value Functions for Parabolic Equations with Translated Potentials

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Abstract: We study the Cauchy problem for differential–difference parabolic equations with potentials undergoing translations with respect to the spatial-independent variable. Such equations are used for the modeling of various phenomena not covered by the classical theory of differential equations (such as nonlinear optics, nonclassical diffusion, multilayer plates and envelopes, and others). From the viewpoint of the pure theory, they are important due to crucially new effects not arising in the case of differential equations and due to the fact that a number of classical methods, tools, and approaches turn out to be inapplicable in the nonlocal theory. The qualitative novelty of our investigation is that the initial-value function is assumed to be summable. Earlier, only the case of bounded (essentially bounded) initial-value functions was investigated. For the prototype problem (the spatial variable is single and the nonlocal term of the equation is single), we construct the integral representation of a solution and show its smoothness in the open half-plane. Further, we find a condition binding the coefficient at the nonlocal potential and the length of its translation such that this condition guarantees the uniform decay (weighted decay) of the constructed solution under the unbounded growth of time. The rate of this decay (weighted decay) is estimated as well.

Keywords: differential–difference operators; parabolic equations; Cauchy problem; summable initial value functions; long-time behavior

MSC: 35R10; 35K15

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1. Introduction

It is well known that, for classical *differential* parabolic equations, qualitative properties of solutions of the Cauchy problem substantially depend on the class of the initial-value functions of the problem. If the initial-value function is bounded (essentially bounded), then the famous Repnikov–Ei'delman stabilization condition is valid (see [1]), i.e., depending on the limit properties of the means of the initial-value function, the solution either has a limit as $t \to \infty$ (and this limit is the same for each value of x) or does not have it at all. If the initial-value function is integrable (summable), then the case qualitatively changes: the said limit always exists, is equal to zero, and is achieved uniformly with respect to x.

This fundamental difference in qualitative properties of solutions is reasonable if we take into account the physics interpretation: only bounded solutions are admitted for the former class of problems (i. e., for problems with $u_0 \in L_\infty(\mathbb{R}^n)$), and only the latter class of problems (i. e., problems with $u_0 \in L_1(\mathbb{R}^n)$) admits finite-energy (nontrivial) solutions.

In this paper, the above phenomenon is studied for differential–difference equations, i. e., equations with translation operators acting (apart from differential operators) on the desired function. Such equations form a special (though quite important) subclass of the class of functional differential equations, i. e., equations with arbitrary non-differential operators acting (apart from differential ones) on the desired function. Those non-differential operators might be integrodifferential ones (see, e. g., [2–8] and references therein), operators of contractions and extensions of the independent variables (see, e.g., [9–13] and references therein), or others (see, e.g., [14,15] and references therein). Although those operators are, in general, bounded (unlike differential ones), they cannot be treated as small perturbations or subordinate terms of the equation: they are *nonlocal terms*, and, as we see in various investigations, the presence of such terms implies the presence of qualitatively new properties of the solutions.

Many researchers all over the world have been dealing with this generalization of classical differential equations (within the contemporary mathematical paradigm, one primarily has to mention the pioneering paper: [16]). The unfailing worldwide interest to this area is mainly caused by the following two circumstances. From the purely theoretical viewpoint, the *nonlocal nature* of differential–difference (and, more generally, functional differential) operators force researchers to develop new tools, methods, and approaches because the standard (although very rich) technique used for differential equations is frequently found to be inapplicable in the functional differential case (in particular, this refers to all maximum-principle methods). On the other hand, functional differential equations arise in various applications not covered by classical differential equations. For instance, parabolic differential–difference equations with potentials undergoing translations, studied in the present paper, arise in models of nonlinear optics (see, e. g., [17–20]).

For the general theory, both aspects are comprehensively covered in [21–24] (also see references therein). The Cauchy problem for parabolic differential–difference equations with bounded initial-value functions is studied in [25]. The case of summable initial-value functions was not studied earlier (up to the knowledge of the authors). The aim of the present paper is to start its investigation.

2. Integral Representations of Solutions

In the sequel, $\|\cdot\|$ denotes the norm in $L_1(-\infty,+\infty)$.

Let a and h be real constants.

On the half-plane $(-\infty,+\infty)\times(0,+\infty)$, define the function

$$\mathcal{E}(x,t)=\int\limits_{-\infty}^{+\infty}e^{-t(\xi^2-a\cos h\xi)}\cos(x\xi-at\sin h\xi)d\xi. \tag{1}$$

From Section 1.1 in [25], it is known that function (1) (the Poissonian kernel) is infinitely differentiable in the whole half-plane and satisfies (in the classical sense) the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+au(x-h,t). \tag{2}$$

For completeness, we provide the procedure to obtain the Poissonian kernel.

Formally applying the Fourier transformation with respect to the variable x to Equation (2), we obtain the equation

$$\frac{d\hat{u}}{dt}=\left(-\xi^2+ae^{-ih\xi}\right)\hat{u}.$$

Note that we obtain an *ordinary differential* equation depending on parameter ξ instead of the *partial differential–difference* equation. The obtained first-order equation is linear and its coefficients are constants. Thus, denoting $u(x,0)$ by $u_0(x)$, one can easily solve the obtained Cauchy problem:

$$\hat{u}(\xi,t)=\widehat{u_0}(\xi)e^{(a\cos h\xi-ia\sin h\xi-\xi^2)t}=e^{(a\cos h\xi-ia\sin h\xi-\xi^2)t}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}u_0(y)e^{-iy\xi}dy.$$

Therefore, to find $u(x,t)$, we have to apply (formally) the inverse Fourier transformation to the last relation:

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{+\infty} e^{(a\cos h\xi - ia\sin h\xi - \xi^2)t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(y) e^{-iy\xi} dy e^{ix\xi} d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_0(y) e^{(a\cos h\xi - ia\sin h\xi - \xi^2)t + i(x-y)\xi} dy d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(y) \int_{-\infty}^{+\infty} e^{(a\cos h\xi - ia\sin h\xi - \xi^2)t + i(x-y)\xi} d\xi dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(x-y) \int_{-\infty}^{+\infty} e^{(a\cos h\xi - ia\sin h\xi - \xi^2)t + iy\xi} d\xi dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(x-y) \int_{-\infty}^{+\infty} e^{(a\cos h\xi - \xi^2)t} \big[\cos(y\xi - at\sin h\xi) - i\sin(y\xi - at\sin h\xi)\big] d\xi dy \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} u_0(x-y) \int_{-\infty}^{+\infty} e^{(a\cos h\xi - \xi^2)t} \cos(y\xi - at\sin h\xi) d\xi dy,
\end{aligned}
$$

which yields the convolution (up to a constant factor) of function (1) with the initial-value function.

Note that all the above operations (such as the change in the order of the integration or the elimination of the odd part of the integrand) are to be justified. Instead, we directly investigate the obtained convolution.

In [25], this is carried out for u_0 from L_∞. Here, we have to perform that for u_0 from L_1.

Thus, let $u_0 \in L_1(-\infty,+\infty)$.

Since the Poissonian kernel satisfies the estimate

$$
\begin{aligned}
|\mathcal{E}(x,t)| &\le \int_{-\infty}^{+\infty} e^{(a\cos h\xi - \xi^2)t} d\xi = \int_{-\infty}^{+\infty} e^{at\cos h\xi} e^{-t\xi^2} d\xi \le e^{|a|t} \int_{-\infty}^{+\infty} e^{-t\xi^2} d\xi \\
&= \frac{e^{|a|t}}{\sqrt{t}} \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta = \frac{e^{|a|t}\sqrt{\pi}}{\sqrt{t}},
\end{aligned} \tag{3}
$$

it follows that the convolution

$$
u(x,t) := \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi \tag{4}
$$

is well defined for each (x,t) from $(-\infty,+\infty) \times (0,+\infty)$ and satisfies the estimate

$$
|u(x,t)| \le \frac{e^{|a|t}}{\sqrt{\pi}\sqrt{t}} \int_{-\infty}^{+\infty} |u_0(\xi)| d\xi = \frac{e^{|a|t}}{\sqrt{\pi t}} \|u_0\|.
$$

Formally differentiate function (4) with respect to the variable x twice inside the integral:

$$
\frac{\partial^2 \mathcal{E}}{\partial x^2} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} \cos(x\xi - at\sin h\xi) d\xi.
$$

Therefore,

$$\left|\frac{\partial^2\mathcal{E}}{\partial x^2}\right| \leq \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\xi^2 e^{at\cos h\xi}e^{-t\xi^2}d\xi \leq \frac{2e^{|a|t}}{\pi}\int\limits_{0}^{+\infty}\xi^2 e^{-t\xi^2}d\xi.$$

The substitution

$$t\xi^2 =: \eta, \xi = \sqrt{\frac{\eta}{t}}, d\xi = \frac{d\eta}{2\sqrt{t\eta}}$$

reduces the last integral to the form

$$\frac{1}{\sqrt{t}}\int\limits_{0}^{+\infty}\frac{\eta}{t}e^{-\eta^2}\frac{d\eta}{2\sqrt{\eta}} = \frac{1}{2t^{\frac{3}{2}}}\int\limits_{0}^{+\infty}\sqrt{\eta}e^{-\eta^2}d\eta,$$

which yields the final estimate

$$\left|\frac{\partial^2\mathcal{E}}{\partial x^2}\right| \leq const\frac{e^{|a|t}}{t^{\frac{3}{2}}}, \tag{5}$$

i.e., the integral obtained by means of the above formal differentiation absolutely converges for each positive t. Hence, this formal differentiation is legible.

Combining estimates (3) and (5) and taking into account that function (1) satisfies Equation (2) in $(-\infty, +\infty) \times (0, +\infty)$, we conclude that

$$\left|\frac{\partial\mathcal{E}}{\partial t}\right| \leq e^{|a|t}\left(\frac{const}{t^{\frac{3}{2}}} + |a|\frac{const}{\sqrt{t}}\right) \tag{6}$$

and, therefore, function (4) can be differentiated with respect to t and twice differentiated with respect to x inside the integral. Hence, function (4) satisfies (in the classical sense) Equation (2) for each positive t and each real x.

Thus, the following assertion is valid.

Theorem 1. *Function* (4) *satisfies the Cauchy problem for Equation* (2) *with the initial-value function* u_0 *in the sense of generalized functions and satisfies Equation* (2) *in the half-plane* $(-\infty, +\infty) \times (0, +\infty)$ *in the classical sense.*

Remark 1. *Here, the fulfillment of the initial-value condition*

$$u\Big|_{t=0} = u_0(x),\ x \in (-\infty, +\infty), \tag{7}$$

is treated in the Gelfand–Shilov sense (see Section 10 in [26]): the distribution $u(x,t)$ *of the variable* x*, depending on the positive parameter* t*, tends to the distribution* $u_0(x)$ *of the variable* x *in the topology of distributions of the variable* x *as the parameter* t *tends to zero from the right (see, e. g., [27]). The fact that Condition* (7) *is satisfied in this case, it is known that it follows from Section 10, Theorem 1 in [26], and the construction of the function* $\mathcal{E}(x,t)$ *(see Section 1.1 in [25]). The only novelty of Theorem 1 is the higher smoothness of the solution outside the initial-value hyperplane.*

3. Decay Rates for Solutions

Estimates (3), (5), and (6) are sufficient to prove the solvability of problems (2) and (7) and to construct integral representations of its solutions. However, they are too rough to investigate their long-time behavior. To obtain more precise estimates, impose the following additional restriction:

$$|a|h^2 \leq 2. \tag{8}$$

Assuming that Condition (8) is satisfied, consider the function $f(\xi) := \xi^2 - a\cos h\xi$. Its derivative is equal to

$$2\xi + ah\sin h\xi = 2\xi\left(1 + \frac{ah}{2}\frac{\sin h\xi}{\xi}\right) = 2\xi\left(1 + \frac{ah^2}{2}\frac{\sin h\xi}{h\xi}\right).$$

Hence, $f'(\xi) \geq 0$ on $(0, +\infty)$, i. e., f is a nondecreasing function on $[0, +\infty)$. Therefore, $f(\xi) \geq f(0) = -a$ on the whole $\mathbb{R}^1$ (the evenness of the function f is taken into account).

Thus, under Condition (8), the following estimate is valid: $\xi^2 - a\cos h\xi \geq -a$, i. e.,

$$-t(\xi^2 - a\cos h\xi) \leq at. \tag{9}$$

3.1. Negative Coefficients at Potentials

Under Condition (8), consider the case where $a < 0$.

Then, $\left|\mathcal{E}(x,t)\right|$ is estimated from above by $2\int\limits_0^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi$. Represent the last integral as the following sum:

$$\int\limits_0^{\sqrt{-2a}} e^{-t(\xi^2 - a\cos h\xi)}d\xi + \int\limits_{\sqrt{-2a}}^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi$$

$$\leq \int\limits_0^{\sqrt{-2a}} e^{at}d\xi + \int\limits_{\sqrt{-2a}}^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi = \sqrt{2|a|}e^{-|a|t} + \int\limits_{\sqrt{2|a|}}^{+\infty} e^{-t(\xi^2 + |a|\cos h\xi)}d\xi$$

$$\leq \sqrt{2|a|}e^{-|a|t} + \int\limits_{\sqrt{2|a|}}^{+\infty} e^{-t(\xi^2 - |a|)}d\xi.$$

On the integration interval of the last integral, we have the inequality $\xi^2 \geq 2|a|$, i. e., $|a| \leq \frac{\xi^2}{2}$, which means that $\xi^2 - |a| \geq \frac{\xi^2}{2}$. Hence, the integral itself does not exceed

$$\int\limits_{\sqrt{2|a|}}^{+\infty} e^{-\frac{t}{2}\xi^2}d\xi \leq \int\limits_0^{+\infty} e^{-\frac{t}{2}\xi^2}d\xi = \frac{1}{2}\sqrt{\frac{2\pi}{t}} = \sqrt{\frac{\pi}{2t}},$$

i. e., we have the estimate

$$\left|\mathcal{E}(x,t)\right| \leq \sqrt{-2a}e^{at} + \sqrt{\frac{\pi}{2t}}.$$

3.2. Positive Coefficients at Potentials

Under Condition (8), consider the case where $a > 0$.

The estimated integral is represented by the sum

$$\int\limits_0^{\sqrt{2a}} e^{-t(\xi^2 - a\cos h\xi)}d\xi + \int\limits_{\sqrt{2a}}^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi$$

$$\leq \int\limits_0^{\sqrt{2a}} e^{at}d\xi + \int\limits_{\sqrt{2a}}^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi = \sqrt{2a}e^{at} + \int\limits_{\sqrt{2a}}^{+\infty} e^{-t(\xi^2 - a\cos h\xi)}d\xi$$

$$\leq \sqrt{2a}e^{at} + \int\limits_{\sqrt{2a}}^{+\infty} e^{-t(\xi^2 - a)}d\xi.$$

In the same way as in the previous case, we prove that $\xi^2 - |a| \geq \frac{\xi^2}{2}$ on the integration interval of the last integral and, therefore, the integral itself does not exceed

$$\int\limits_{\sqrt{2a}}^{+\infty} e^{-\frac{t}{2}\xi^2} d\xi \leq \int\limits_{0}^{+\infty} e^{-\frac{t}{2}\xi^2} d\xi = \frac{1}{2}\sqrt{\frac{2\pi}{t}} = \sqrt{\frac{\pi}{2t}},$$

i. e., the *weight* estimate

$$e^{-(a+\delta)t}\Big|\mathcal{E}(x,t)\Big| \leq \sqrt{2a}e^{-\delta t} + \sqrt{\frac{\pi}{2t}}e^{-(a+\delta)t}$$

holds for each positive δ.

This yields the following assertion.

Theorem 2. *Let Condition (8) be fulfilled. Then, solution (4) tends to zero as $t \to \infty$ uniformly with respect to x from $(-\infty, +\infty)$, provided that $a < 0$ and tends to zero with the weight $e^{-(a+\delta)t}$ as $t \to \infty$ uniformly with respect to x from $(-\infty, +\infty)$, provided that $a > 0$. The rate of the decay is estimated by the inequalities*

$$|u(x,t)| \leq \frac{const\|u_0\|}{\sqrt{t}}$$

and

$$e^{-(a+\delta)t}|u(x,t)| \leq const\|u_0\|e^{-\delta t} \quad \text{for each positive } \delta,$$

respectively.

4. Decay Rates for Derivatives of Solutions

Now, under Condition (8), we estimate the partial derivative of the solution with respect to the spatial variable. The value of the coefficient at the potential is assumed to be negative.

Taking into account that

$$\frac{\partial\mathcal{E}(x,t)}{\partial x} = -\int\limits_{-\infty}^{+\infty} \xi e^{-t(\xi^2 - a\cos h\xi)} \sin{(x\xi - at\sin h\xi)} d\xi,$$

and using the same decomposition scheme for the integral as in Section 3, we conclude that

$$\begin{aligned}\left|\frac{\partial\mathcal{E}(x,t)}{\partial x}\right| &\leq \int\limits_{-\infty}^{+\infty} |\xi| e^{-t(\xi^2 - a\cos h\xi)} d\xi = 2\int\limits_{0}^{+\infty} \xi e^{-t(\xi^2 - a\cos h\xi)} d\xi \\ &= 2\int\limits_{0}^{\sqrt{-2a}} \xi e^{-t(\xi^2 - a\cos h\xi)} d\xi + 2\int\limits_{\sqrt{-2a}}^{+\infty} \xi e^{-t(\xi^2 - a\cos h\xi)} d\xi \\ \leq 2\int\limits_{0}^{\sqrt{2|a|}} \xi e^{-|a|t} d\xi + 2\int\limits_{0}^{+\infty} \xi e^{-\frac{t}{2}\xi^2} d\xi &= 2|a|e^{-|a|t} - \frac{2}{t}\int\limits_{0}^{+\infty} e^{-\frac{t}{2}\xi^2} d\left(-\frac{t}{2}\xi^2\right) \\ &= -2ae^{at} - \frac{2}{t}\lim_{b\to+\infty} e^{-\frac{t}{2}\xi^2}\Big|_0^b = -2ae^{at} + \frac{2}{t}.\end{aligned}$$

Then, for the function $u(x,t)$, we obtain the same decay rate:

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq \frac{1}{\pi}\left|\frac{\partial \mathcal{E}(x,t)}{\partial x}\right| \int_{-\infty}^{+\infty} |u_0(\xi)| d\xi \leq \frac{\|u_0\|}{\pi}\left(-2ae^{at} + \frac{2}{t}\right).$$

Thus,

$$\lim_{t\to+\infty} \frac{\partial u(x,t)}{\partial x} = 0.$$

uniformly with respect to $x \in \mathbb{R}$ with the decay rate of t^{-1}.

Passing to the second partial derivative of u with respect to x, we note that

$$\frac{\partial^2 \mathcal{E}(x,t)}{\partial x^2} = \int_{-\infty}^{+\infty} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} \cos\left(x\xi - at\sin h\xi\right) d\xi$$

and, therefore,

$$\begin{aligned}
\left|\frac{\partial^2 \mathcal{E}(x,t)}{\partial x^2}\right| &\leq \int_{-\infty}^{+\infty} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} d\xi = 2\int_0^{+\infty} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} d\xi \\
&= 2\int_0^{\sqrt{-2a}} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} d\xi + 2\int_{\sqrt{-2a}}^{+\infty} \xi^2 e^{-t(\xi^2 - a\cos h\xi)} d\xi \\
&\leq 2\int_0^{\sqrt{2|a|}} \xi^2 e^{-|a|t} d\xi + 2\int_0^{+\infty} \xi^2 e^{-\frac{t}{2}\xi^2} d\xi \\
&= \frac{2}{3}(2|a|)^{3/2} e^{-|a|t} + 2\left(\frac{2}{t}\right)^{3/2} \int_0^{+\infty} u^2 e^{-u^2} du = \frac{2}{3}(-2a)^{3/2} e^{at} + \frac{C_2}{t^{3/2}}.
\end{aligned}$$

Then, we have the inequality

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq \frac{\|u_0\|}{\pi}\left[\frac{2}{3}(-2a)^{3/2} e^{at} + \frac{C_2}{t^{3/2}}\right].$$

Thus,

$$\lim_{t\to+\infty} \frac{\partial^2 u(x,t)}{\partial x^2} = 0.$$

uniformly with respect to x from $(-\infty, +\infty)$ with the decay rate of $t^{-3/2}$.

Passing to the partial derivative of an arbitrary order m, we see that

$$\frac{\partial^m \mathcal{E}(x,t)}{\partial x^m} = \int_{-\infty}^{+\infty} (-\xi)^m e^{-t(\xi^2 - a\cos h\xi)} \cos\left(x\xi - at\sin h\xi + \frac{\pi m}{2}\right) d\xi$$

and, therefore,

$$\left|\frac{\partial^m \mathcal{E}(x,t)}{\partial x^m}\right| \leq \int\limits_{-\infty}^{+\infty} |(-\xi)^m| e^{-t(\xi^2 - a\cos h\xi)} d\xi = 2\int\limits_{0}^{+\infty} \xi^m e^{-t(\xi^2 - a\cos h\xi)} d\xi$$

$$\leq \frac{2}{m+1}(2|a|)^{(m+1)/2} e^{-|a|t} + 2\left(\frac{2}{t}\right)^{(m+1)/2} \int\limits_{0}^{+\infty} s^m e^{-s^2} ds$$

$$= \frac{2}{m+1}(-2a)^{(m+1)/2} e^{at} + \frac{C_m}{t^{(m+1)/2}}.$$

Then, we have

$$\left|\frac{\partial^m u(x,t)}{\partial x^m}\right| \leq \frac{\|u_0\|}{\pi}\left(\frac{2}{m+1}(-2a)^{(m+1)/2} e^{at} + \frac{C_m}{t^{(m+1)/2}}\right)$$

and, consequently,

$$\lim_{t\to+\infty} \frac{\partial^m u(x,t)}{\partial x^m} = 0$$

uniformly with respect to x from $(-\infty, +\infty)$ with the decay rate of $t^{-(m+1)/2}$. Differentiate the obtained expression with respect to t:

$$\frac{\partial^{m+1}\mathcal{E}(x,t)}{\partial x^m \partial t} = \int\limits_{-\infty}^{+\infty} (-\xi)^m a \sin h\xi e^{-t(\xi^2 - a\cos h\xi)} \sin\left(x\xi - at\sin h\xi + \frac{\pi m}{2}\right) d\xi$$

$$- \int\limits_{-\infty}^{+\infty} (-\xi)^m (\xi^2 - a\cos h\xi) e^{-t(\xi^2 - a\cos h\xi)} \cos\left(x\xi - at\sin h\xi + \frac{\pi m}{2}\right) d\xi$$

$$= \int\limits_{-\infty}^{+\infty} (-1)^{m+1} \xi^{m+2} e^{-t(\xi^2 - a\cos h\xi)} \cos\left(x\xi - at\sin h\xi + \frac{\pi m}{2}\right) d\xi$$

$$+ a\int\limits_{-\infty}^{+\infty} (-\xi)^m e^{-t(\xi^2 - a\cos h\xi)} \cos\left((x-h)\xi - at\sin h\xi + \frac{\pi m}{2}\right) d\xi.$$

Therefore,

$$\left|\frac{\partial^{m+1}\mathcal{E}(x,t)}{\partial x^m \partial t}\right| \leq \int\limits_{-\infty}^{+\infty} |\xi^{m+2}| e^{-t(\xi^2 - a\cos h\xi)} d\xi + |a| \int\limits_{-\infty}^{+\infty} |\xi^m| e^{-t(\xi^2 - a\cos h\xi)} d\xi$$

$$= 2\int\limits_{0}^{+\infty} \xi^{m+2} e^{-t(\xi^2 - a\cos h\xi)} d\xi + |a| \int\limits_{0}^{+\infty} \xi^m e^{-t(\xi^2 - a\cos h\xi)} d\xi$$

$$\leq \frac{2}{m+3}(-2a)^{(m+3)/2} e^{at} + \frac{C_{m+2}}{t^{(m+3)/2}} - a\left[\frac{2}{m+1}(-2a)^{(m+1)/2} e^{at} + \frac{C_m}{t^{(m+1)/2}}\right]$$

$$\leq \widetilde{C} e^{at} + \widetilde{C}_1 \frac{1}{t^{(m+3)/2}} + \widetilde{C}_0 \frac{1}{t^{(m+1)/2}}.$$

We see that no additional differentiation with respect to t changes the least power with respect to t, relating to the decay rate as $t \to +\infty$.

Then, the partial derivative of order $m+n$ of the function $\mathcal{E}(x,t)$ is estimated as follows:

$$\left|\frac{\partial^{m+n}\mathcal{E}(x,t)}{\partial x^m \partial t^n}\right| \leq \widetilde{C}e^{at} + \widetilde{C}_n \frac{1}{t^{(m+2n+1)/2}} + \widetilde{C}_{n-1}\frac{1}{t^{(m+2n-1)/2}}$$
$$\cdots + \widetilde{C}_0 \frac{1}{t^{(m+1)/2}} = \widetilde{C}e^{at} + \sum_{k=0}^{n} \widetilde{C}_k \frac{1}{t^{(m+2k+1)/2}}.$$

This yields the following estimate for the same derivative of the function $u(x,t)$:

$$\left|\frac{\partial^{m+n}u(x,t)}{\partial x^m \partial t^n}\right| \leq \frac{\|u_0\|}{\pi}\left(\widetilde{C}e^{at} + \sum_{k=0}^{n} \widetilde{C}_k \frac{1}{t^{(m+2k+1)/2}}\right).$$

Theorem 3. *Let Condition (8) be fulfilled. Then, the partial derivatives of solution (4) of order $(m+n)$ tend to zero as $t \to \infty$ uniformly with respect to x from $(-\infty,+\infty)$, provided that $a<0$. The rate of the decay is estimated by the inequalities*

$$\left|\frac{\partial^{m+n}u(x,t)}{\partial x^m \partial t^n}\right| \leq \frac{\|u_0\|}{\pi}\left(\widetilde{C}e^{at} + \sum_{k=0}^{n} \widetilde{C}_k \frac{1}{t^{(m+2k+1)/2}}\right).$$

Remark 2. *For the case where the coefficient at the potential is positive, weighted estimates for derivatives of the solution, similar to the weighted estimates for the solution itself, obtained in Section 3, can be obtained in the same way.*

5. Discussion

The solvability of the problem is achieved for all values of a and h; Condition (8) binding these two parameters is additional: it is imposed only to prove the decay (weighted decay) of the solution. No analogs of this phenomenon occur in the classical case of differential equations. The said condition is caused by the nonlocal nature of the investigated problem. Note that it means neither a smallness of the coefficient at the nonlocal term nor a smallness of its translation. Actually, it imposes restrictions on the symbol of the differential–difference operator contained in the investigated equation: since the symbol of the differential–difference operator contained at the right-hand side of Equation (2) is equal to $-\xi^2 + ae^{-ih\xi}$, it follows that Condition (8) guarantees the boundedness of the real part of that symbol by a strictly negative constant (in the case where the coefficient at the potential is negative). This condition is imposed to satisfy estimate (9) used to prove the uniform decay of the solution, and it is worthy to note that restrictions of this type are frequently imposed on real parts of symbols in various investigations of differential–difference equations and operators (see [21–25] and references therein). No effects of this kind arise in the classical case because symbols of differential operators have much simpler structure than symbols of differential–difference operators.

The fact that the adding of potential-type terms to parabolic equations can fundamentally change the nature of their solutions (including the long-time behavior) is known at least from the pioneering work [28]. Further investigations (see, e. g., [29–31] and references therein) demonstrated a great diversity of this phenomenon (depending on the nature, properties, and behavior of the potential-type terms). The prototype case of nonlocal potentials, investigated in the present paper, shows that the nonlocal nature of potentials substantially extends the said diversity.

6. Conclusions

In this paper, we continue the investigation of the Cauchy problem for differential–difference parabolic equations with nonlocal potentials, passing to the case of summable initial-value functions (instead of bounded ones investigated earlier). The prototype case of Equation (2) is considered. We show that the convolution of the initial-value function with function (1), which is the same Poissonian kernel as in the case of bounded initial-value functions, satisfies the investigated problem in the sense of generalized functions and is a classical (infinitely smooth) solution of Equation (2).

For Equation (2), we impose Condition (8). Under this condition, we prove the following (uniform with respect to the spatial-independent variable) estimates of the constructed solution and its derivatives:

- If the coefficient at the potential is negative, then

$$|u(x,t)| \leq \left[\frac{\sqrt{-2a}}{\pi}e^{at} + \frac{1}{\sqrt{2\pi t}}\right]\|u_0\|;$$

- If the coefficient at the potential is positive, then

$$e^{-(a+\delta)t}|u(x,t)| \leq \left[\frac{\sqrt{2a}}{\pi}e^{-\delta t} + \frac{1}{\sqrt{2\pi t}}e^{-(a+\delta)t}\right]\|u_0\|$$

  for each positive δ;
- If the coefficient at the potential is negative, then

$$\left|\frac{\partial^m u(x,t)}{\partial x^m}\right| \leq \frac{\|u_0\|}{\pi}\left(\frac{2}{m+1}(-2a)^{(m+1)/2}e^{at} + \frac{C_m}{t^{(m+1)/2}}\right)$$

  with the decay rate of $\frac{1}{t^{(m+1)/2}}$;
- If the coefficient at the potential is negative, then

$$\left|\frac{\partial^{m+n} u(x,t)}{\partial x^m \partial t^n}\right| \leq \frac{\|u_0\|}{\pi}\left(\widetilde{C}e^{at} + \sum_{k=0}^{n}\widetilde{C}_k\frac{1}{t^{(m+2k+1)/2}}\right)$$

  with the decay rate of $\frac{1}{t^{(m+1)/2}}$, which means that no differentiating with respect to t changes the decay rate.

Thus, the asymptotical properties of solutions fundamentally differ from the case where the initial-value functions are bounded.

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