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Article

Generalized Matrix Spectral Factorization with Symmetry and Construction of Quasi-Tight Framelets over Algebraic Number Fields

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Abstract: The rational field $\mathbb{Q}$ is highly desired in many applications. Algorithms using the rational number field $\mathbb{Q}$ algebraic number fields use only integer arithmetics and are easy to implement. Therefore, studying and designing systems and expansions with coefficients in $\mathbb{Q}$ or algebraic number fields is particularly interesting. This paper discusses constructing quasi-tight framelets with symmetry over an algebraic field. Compared to tight framelets, quasi-tight framelets have very similar structures but much more flexibility in construction. Several recent papers have explored the structure of quasi-tight framelets. The construction of symmetric quasi-tight framelets directly applies the generalized spectral factorization of 2×2 matrices of Laurent polynomials with specific symmetry structures. We adequately formulate the latter problem and establish the necessary and sufficient conditions for such a factorization over a general subfield $\mathbb{F}$ of $\mathbb{C}$, including algebraic number fields as particular cases. Our proofs of the main results are constructive and thus serve as a guideline for construction. We provide several examples to demonstrate our main results.

Keywords: generalized matrix spectral factorization; quasi-tight framelets; framelet filter banks; symmetric framelets

MSC: 42C40; 42C15; 41A15; 65D07

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1. Introduction

1.1. Backgrounds

In many science and engineering applications, people often digitalize data and use particular devices to process and analyze them. Quite commonly, data are stored in integers or rational numbers for machine processes, such as an 8-bit grayscale image or audio in a digital recorder. Therefore, the rational field $\mathbb{Q}$, or any of its finite extensions (also known as algebraic number fields), are desired in many scientific computing applications. Implementing algorithms using algebraic number fields is quite efficient since only integer arithmetics are involved. Consequently, studying and designing systems and expansions over algebraic number fields is of great interest.

Over the past decades, wavelets and framelets have been extensively studied and applied in numerous applications, such as signal processing and numerical algorithms. Several excellent properties, such as sparse multi-scale representations and fast discrete transforms, make wavelets and framelets perform well in the applications above. As a generalization of wavelets, framelets not only preserve almost all the good properties of wavelets but also offer the new feature of redundancy, which gives framelets robustness under data corruption or quantization. Over the past few years, framelets with attractive properties, such as symmetry and high-order vanishing moments, have been well-investigated (see e.g., [1–23]). One of the fascinating topics is to design symmetric (or anti-symmetric) framelets (or framelet filter banks) whose coefficients belong to $\mathbb{Q}$ or an algebraic number field. The properties and examples of such framelets have been extensively explored in papers such as [2,8,11,15,16,24–26].

Motivated by the work described above, in this paper, we focus on (dyadic) quasi-tight framelets with coefficients over an algebraic number field. The notion of a quasi-tight framelet generalizes a tight framelet. It is well-known that a tight framelet filter bank is often derived from a refinement filter/mask that satisfies the *sub-QMF* condition. However, many refinement filters fail to meet the sub-QMF condition, and we cannot obtain a tight framelet filter bank in such cases. Therefore, we hope to obtain something similar to a tight framelet filter bank from an arbitrary refinement filter, thus motivating the introduction of a quasi-tight framelet filter bank. The notion of quasi-tight framelets was first introduced in [25] and was then investigated in several subsequent recent papers, such as [2,25–27]. Existing studies have demonstrated the advantages of quasi-tight framelets over tight framelets. On the one hand, quasi-tight and tight framelets behave almost identically. On the other hand, compared to tight framelets, quasi-tight framelets have much more flexibility and are much easier to construct. Therefore, it is natural and necessary to consider quasi-tight framelets. The main goal of this paper is to provide a characterization and, more importantly, algorithms for constructing quasi-tight framelets with symmetry and coefficients in an algebraic number field.

1.2. Generalized Matrix Spectral Factorization with Symmetry

The construction of a symmetric quasi-tight framelet filter bank is closely related to *generalized matrix spectral factorization* with symmetry. To properly formulate this problem, we must first introduce some notations and concepts.

Throughout this paper, $\mathbb{F} \subseteq \mathbb{C}$ is a field that satisfies

$$\overline{x} \in \mathbb{F} \quad \text{whenever} \quad x \in \mathbb{F}, \tag{1}$$

where $\overline{x}$ denotes the complex conjugate of x. The condition (1) is mild and normal when constructing wavelet and framelet filter banks. The most commonly used fields $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ and algebraic number fields, such as $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(i), \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})$, all satisfy (1). A *Laurent polynomial* with coefficients from a field $\mathbb{F}$ takes the form

$$\mathsf{u}(z) = \sum_{k\in\mathbb{Z}} u[k]z^k, \quad z \in \mathbb{C}\backslash\{0\},$$

where its coefficient sequence $u = \{u[k]\}_{k\in\mathbb{Z}}$ has only finitely many non-zero terms (in such cases, we say that u is *finitely supported*) and $u[k] \in \mathbb{F}$ for all $k \in \mathbb{Z}$. Denote $\mathbb{F}[z, z^{-1}]$, the set (ring) of all Laurent polynomials with coefficients in $\mathbb{F}$. Throughout this paper, we use the sans serif font style letters to denote (matrices of) Laurent polynomials and the normal/regular font style letters to denote the corresponding coefficient sequence of the Laurent polynomial.

We say that $\mathsf{u} \in \mathbb{F}[z, z^{-1}]$ has *symmetry* if

$$\mathsf{u}(z) = \epsilon z^c \mathsf{u}(z^{-1}), \tag{2}$$

for some $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$. It is trivial to see that (2) is equivalent to

$$u[k] = \epsilon u[c-k], \qquad \forall\, k \in \mathbb{Z}. \tag{3}$$

If $\mathsf{u} \in \mathbb{F}[z, z^{-1}]$ is not the zero polynomial and has symmetry, we define

$$\mathsf{Su}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}(z^{-1})}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{4}$$

The operator S is known as a *symmetry operator* in wavelet and framelet analysis. Denote $\mathrm{Sym}_{\mathbb{F}}$, the set of Laurent polynomials with symmetry and coefficients in $\mathbb{F}$. It is easy to see that (2) holds if, and only if, $\mathsf{Su}(z) = \epsilon z^c$ is a monomial. For $\mathsf{u} \in \mathrm{Sym}_{\mathbb{F}}$, we call the monomial $\mathsf{Su}(z)$ the *symmetry type* of u. For convenience, if $\mathsf{u} = 0$ is the zero polynomial, then u *has any symmetry type*.

Let $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P[k]z^k$ be a $t \times r$ matrix of Laurent polynomials, where $P[k] \in \mathbb{F}^{t\times r}$ for all $k \in \mathbb{Z}$, we define the *Hermitian conjugate* of P via

$$\mathsf{P}^{\mathsf{H}}(z) := \overline{\mathsf{P}(\overline{z^{-1}})}^{\mathsf{T}} = \sum_{k\in\mathbb{Z}} \overline{P[k]}^{\mathsf{T}} z^{-k}, \quad z \in \mathbb{T} := \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}.$$

When $\mathsf{P}(z)$ is a 1×1 matrix, that is, $\mathsf{P}(z)$ is a Laurent polynomial, then the Hermitian conjugate $\mathsf{P}^{\mathsf{H}}(z)$ is just the conjugate of $\mathsf{P}(z)$ for all $z \in \mathbb{T}$. In this case, we denote $\mathsf{P}^*(z) := \mathsf{P}^{\mathsf{H}}(z)$. Note that if $\mathbb{F} \subseteq \mathbb{R}$, then $\mathsf{P}^{\mathsf{H}}(z) = \mathsf{P}(z^{-1})^{\mathsf{T}}$. For an $n \times n$ matrix P of Laurent polynomials, we say that P is *Hermitian* if $\mathsf{P}^{\mathsf{H}} = \mathsf{P}$. If all entries of $\mathsf{P} \in (\mathbb{F}[z, z^{-1}])^{t\times r}$ have symmetry, then we can define its *symmetry type* SP. For a $t \times r$ matrix $\mathsf{P} := \mathsf{P}(z)$ whose entries are Laurent polynomials in $\mathrm{Sym}_{\mathbb{F}}$, define the $t \times r$ matrix of monomials SP through $[\mathsf{SP}(z)]_{j,k} := \mathsf{S}[\mathsf{P}_{j,k}](z)$ for all $1 \leqslant j \leqslant t$ and $1 \leqslant k \leqslant r$.

Next, we discuss how the symmetry property behaves under matrix operations. Let P, Q, and R be $t \times r$, $t \times r$, and $r \times s$ matrices of Laurent polynomials in $\mathrm{Sym}_{\mathbb{F}}$. We have the following definitions:

- If $\mathsf{SP} = \mathsf{SQ}$, then $\mathsf{P} \pm \mathsf{Q}$ have symmetry with $\mathsf{S}[\mathsf{P} + \mathsf{Q}] = \mathsf{S}[\mathsf{P} - \mathsf{Q}] = \mathsf{SP} = \mathsf{SQ}$.
- We say that the symmetry type of P is *compatible* or P has *compatible symmetry* if

$$\mathsf{SP}(z) = \mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)\mathsf{S}\boldsymbol{\eta}_2(z) \tag{5}$$

  holds for some $1 \times r$ and $1 \times s$ row vectors of Laurent polynomials $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ with symmetry. For convenience, denote $\mathrm{Sym}_{\mathbb{F}^{r\times s}}$, the set of all $r \times s$ matrices $\mathsf{P} := \mathsf{P}(z)$ with entries in $\mathrm{Sym}_{\mathbb{F}}$ and have compatible symmetry.
- We say that *the multiplication* PR *is compatible* if

$$\mathsf{SP}(z) = \mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)\mathsf{S}\boldsymbol{\eta}_2(z), \quad \mathsf{SR}(z) = \mathsf{S}\boldsymbol{\eta}_2^{\mathsf{H}}(z)\mathsf{S}\boldsymbol{\eta}_3(z),$$

  for some $1 \times r$, $1 \times s$, and $1 \times t$ row vectors $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, and $\boldsymbol{\eta}_3$ of Laurent polynomials in $\mathrm{Sym}_{\mathbb{F}}$. It is obvious that in this case, PR has compatible symmetry with

$$\mathsf{S}[\mathsf{PR}](z) = \mathsf{SP}(z)\mathsf{SR}(z) = \mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)\mathsf{S}\boldsymbol{\eta}_3(z).$$

We are ready to formulate the generalized matrix spectral factorization problem with symmetry. Let $\mathsf{M}(z) \in \mathrm{Sym}_{\mathbb{F}^{r\times r}}$ be such that $\mathsf{M} = \mathsf{M}^{\mathsf{H}}$ and $\mathsf{SM} = \mathsf{S}\boldsymbol{\eta}^{\mathsf{H}}\mathsf{S}\boldsymbol{\eta}$ for some $1 \times r$ vector $\boldsymbol{\eta}$ of Laurent polynomials with symmetry. By $\mathrm{Diag}(x_1, \ldots, x_r)$, we mean the $r \times r$ diagonal matrix whose diagonal elements are (ordered) $x_1, \ldots, x_r$. A *generalized spectral factorization of* M *with symmetry over the field* $\mathbb{F}$ is of the form

$$\mathsf{M}(z) = \mathsf{U}(z)\,\mathsf{D}\,\mathrm{Diag}(\epsilon_1, \ldots, \epsilon_s)\,\mathsf{D}^{\mathsf{H}}\,\mathsf{U}^{\mathsf{H}}(z), \tag{6}$$

where $\mathsf{U} \in \mathrm{Sym}_{\mathbb{F}^{r\times s}}$, $\mathsf{D} = \mathrm{Diag}(c_1, \ldots, c_s)$ for some $c_1, \ldots, c_s \in \mathbb{C}$ with $|c_1|^2, \ldots, |c_s|^2 \in \mathbb{F}$, $\epsilon_1, \ldots, \epsilon_s \in \{-1, 1\}$, and all multiplications in (6) are compatible. When the underlying field $\mathbb{F} = \mathbb{C}$, then we may choose $\mathsf{D} := I_s$, which is the $s \times s$ identity matrix and thus (6) reduces to

$$\mathsf{M}(z) = \mathsf{U}(z)\mathrm{Diag}(\epsilon_1, \ldots, \epsilon_s)\mathsf{U}^{\mathsf{H}}(z).$$

In particular, we are interested in the case when $r = s = 2$ because it is related to constructing (dyadic) quasi-tight or tight framelet filter banks with two generators that we will discuss later. In this case, the generalized spectral factorization becomes

$$\mathsf{M}(z) = \mathsf{U}(z)\mathsf{D}\mathrm{Diag}(\epsilon_1, \epsilon_2)\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z), \tag{7}$$

for some $\mathsf{U} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ and $\mathsf{D} := \mathrm{Diag}(c_1, c_2)$.

1.3. Related Work

When $\epsilon_1 = \epsilon_2 = 1$ in (7), the factorization is known as a *spectral factorization* of M. The spectral factorization problem and the construction of tight framelets have been well-investigated in several pieces of work. For instance, refs. [6,9] studied the case when $\mathbb{F} = \mathbb{C}$; ref. [15] addressed the case when $\mathbb{F} = \mathbb{Q}$ and specifically studied how to construct tight framelets with rational coefficients; ref. [16] investigated the case when $\mathbb{F} \subseteq \mathbb{C}$ is any field that satisfies (1). If we drop the symmetry requirements of U in (7) and let $\epsilon_1 = \epsilon_2 = 1$, then we have the well-known matrix Fejér–Riesz lemma (see [28]) in the literature.

On the other hand, the case when $\epsilon_1 \neq \epsilon_2$ in (7) is much more complicated and different from the case $\epsilon_1 = \epsilon_2$, as we no longer have the positive semi-definite property $\mathsf{M}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Some recent related papers, such as [2,26], investigated (7) with or without symmetry for the case when $\mathbb{F} = \mathbb{C}$ and $\epsilon_1 \neq \epsilon_2$. Our goal is to complete the picture of the generalized spectral factorization problem by resolving the unsolved general case when $\mathbb{F} \subseteq \mathbb{C}$ is an arbitrary field that satisfies (1).

1.4. Our Contributions and Paper Structure

Here, we summarize our contributions. First, we completely solve the generalized spectral factorization of 2×2 matrices with symmetry over a general subfield $\mathbb{F}$ of $\mathbb{C}$ that is closed under complex conjugation. In particular, we are interested in Hermitian matrices $\mathsf{M} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with symmetry type $\mathsf{SM} = \begin{bmatrix} 1 & \mathsf{SM}_{1,2} \\ \mathsf{SM}_{1,2}^* & 1 \end{bmatrix}$, as these matrices are particularly concerned with framelet constructions. We establish the necessary and sufficient conditions for such a matrix M to admit a factorization as in (7) with $\epsilon_1 \neq \epsilon_2$. Next, we apply the theory of generalized spectral factorization to construct symmetric quasi-tight framelets with two generators. We consider how to derive the high-pass filters with symmetry and coefficients in an algebraic number field from a refinement filter a that fails to satisfy the sub-QMF condition. Moreover, our construction guarantees that the high-pass filters achieve the highest possible order of vanishing moments.

The paper is organized as follows: In Section 2, we establish the first main result Theorem 1 on the generalized spectral factorization of 2×2 matrices with symmetry over a general field $\mathbb{F}$ that satisfies (1). Our proof is constructive and thus can serve as a factorization algorithm. Next, in Section 3, we briefly review the basic concepts of framelets and demonstrate the connection between constructing quasi-tight framelets and the generalized spectral factorization. Then, we apply the results from Section 2 to deduce our second main result Theorem 3 on constructing quasi-tight framelets with symmetry and coefficients over the general field $\mathbb{F}$. Finally, Section 4 provides several examples of quasi-tight framelets with symmetry.

2. Generalized Spectral Factorization with Symmetry over Algebraic Number Fields

In this section, we establish the first main result of the paper on generalized matrix spectral factorization of Laurent polynomial matrices with symmetry over a field $\mathbb{F}$ that satisfies (1).

For simplicity, define

$$\mathrm{HS}_{\mathbb{F},2} := \left\{ \mathsf{M} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}} : \mathsf{M} = \mathsf{M}^{\mathsf{H}} \text{ and } \mathsf{SM} = \begin{bmatrix} 1 & \mathsf{SM}_{1,2} \\ \mathsf{SM}_{1,2}^* & 1 \end{bmatrix} \right\}.$$

Any $\mathsf{M} \in \mathrm{HS}_{\mathbb{F},2}$ is Hermitian and has compatible symmetry. Here is the main theorem on the generalized spectral factorization with symmetry over $\mathbb{F}$:

Theorem 1. *Let $\mathsf{M} \in \mathrm{HS}_{\mathbb{F},2}$ be such that $\mathsf{SM}_{1,2}(z) = \epsilon z^c$ for some $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$. Suppose $\gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{1,2}^*, \mathsf{M}_{2,2}) = 1$, then the following two statements are equivalent:*

(1) $\det(\mathsf{M}(z)) = -|\mathsf{C}|^2 \mathsf{d}(z)\mathsf{d}^*(z)$ *for some* $\mathsf{C} \in \mathbb{C}$ *with* $|\mathsf{C}|^2 \in \mathbb{F}$ *and some* $\mathsf{d} \in \mathrm{Sym}_{\mathbb{F}}$;

(2) *there exist* $\mathsf{U} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ *and* $\mathsf{D} := Diag(c_1, c_2)$ *for some* $c_1, c_2 \in \mathbb{C}$ *with* $|c_1|^2, |c_2|^2 \in \mathbb{F}$ *such that*

$$\mathsf{M}(z) = \mathsf{U}(z)\mathsf{D}Diag(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z), \tag{8}$$

and the symmetry type of U *satisfies*

$$\frac{\mathrm{S}\mathsf{U}_{1,1}(z)}{\mathrm{S}\mathsf{U}_{2,1}(z)} = \frac{\mathrm{S}\mathsf{U}_{1,2}(z)}{\mathrm{S}\mathsf{U}_{2,2}(z)} = \epsilon z^c; \tag{9}$$

Proof. (2) $\Rightarrow$ (1): Suppose (2) holds, then take $\mathsf{d} := \det(\mathsf{U})$ and $C := c_1c_2$, we have item (1).

(1) $\Rightarrow$ (2): Suppose (1) holds, then a desired matrix factorization of M that satisfies both (8) and (9) can be obtained by performing the following steps:

- **Step 1.** Construct $\mathring{\mathsf{M}} \in \mathrm{HS}_{\mathbb{F},2}$ and $\mathsf{U}_1 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ such that
  (i) $\gcd(\mathring{\mathsf{M}}_{1,1}, \mathring{\mathsf{M}}_{1,2}) = \gcd(\mathring{\mathsf{M}}_{1,2}, \mathring{\mathsf{M}}_{2,2}) = 1$;
  (ii) $\det(\mathring{\mathsf{M}}(z)) = -|\mathsf{C}|^2\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^*(z)$ for some $\mathring{\mathsf{d}} \in \mathrm{Sym}_{\mathbb{F}}$;
  (iii) $\mathsf{M}(z) = \mathsf{U}_1(z)\mathring{\mathsf{M}}(z)\mathsf{U}_1^{\mathsf{H}}(z)$ and all multiplications are compatible.
- **Step 2.** Construct $\mathsf{B} \in \mathrm{HS}_{\mathbb{F},2}$ and $\mathsf{U}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ such that
  (i) $\det(\mathsf{B}(z)) = -|\mathsf{C}|^2$;
  (ii) $\mathring{\mathsf{M}}(z) = \mathsf{U}_2(z)\mathsf{B}(z)\mathsf{U}_2^{\mathsf{H}}(z)$ and all multiplications are compatible.
- **Step 3.** Construct $\mathsf{U}_3 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ and $\mathsf{D} := \mathrm{Diag}(c_1, c_2)$ with $c_1, c_2 \in \mathbb{C}$ and $|c_1|^2, |c_2|^2 \in \mathbb{F}$ such that $\mathsf{B} = \mathsf{U}_3\mathsf{D}\mathrm{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}_3^{\mathsf{H}}$ and all multiplications are compatible.

The justifications of Steps 1–3 are long and technical, so we postpone them to later subsections. Once we have finished the above three steps, define $\mathsf{U} := \mathsf{U}_1\mathsf{U}_2\mathsf{U}_3$ and let D be the same as in Step 3; all claims of the item (2) follow immediately. The proof is complete. □

Remark 1. *To obtain a generalized factorization, we need to shrink the* length *(see the definition in Section 2.1) of* $\det(\mathsf{M}(z))$. *More specifically, we need to find a matrix* $\mathsf{B} \in \mathrm{HS}_{\mathbb{F},2}$ *whose determinant is a constant and* $\mathsf{W} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ *so that* $\mathsf{M}(z) = \mathsf{W}(z)\mathsf{B}(z)\mathsf{W}^{\mathsf{H}}(z)$ *with all multiplications being compatible. For the case when* $\mathbb{F} = \mathbb{C}$, *which has been studied in [2], the approach of [2] to achieve this relies on a matrix normal form with compatible symmetry (see [Theorem 3.9 [2]]) and splitting* $\det(\mathsf{M}(z))$ *into a product of linear factors over* $\mathbb{C}$ *(see [Lemma 3.12 [2]]). Some linear factors of* $\det(\mathsf{M}(z))$ *are used to construct a desired matrix* $\mathsf{W} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ *that help shrink the length of* $\det(\mathsf{M}(z))$. *However, we cannot adopt the same approach to the general case (i.e., when* $\mathbb{F} \subseteq \mathbb{C}$ *is an arbitrary subfield). When* $\mathsf{M} \in \mathrm{HS}_{\mathbb{F},2}$ *such that* $\mathbb{F} \subseteq \mathbb{C}$ *is an arbitrary subfield and* $z - z_0$ *is a linear factor of* $\det(\mathsf{M}(z))$, *then it may happen that* $z_0 \notin \mathbb{F}$. *As a result, to ensure that all entries of* $\mathsf{W}(z)$ *have coefficients in* $\mathbb{F}$, *we cannot use the* $z - z_0$ *if it is not in* $\mathbb{F}[z, z^{-1}]$. *When working on the proof of Theorem 1, everything (e.g., factorizing polynomials) must be performed over* $\mathbb{F}$, *not* $\mathbb{C}$, *which is why difficulties arise. Consequently, we provide a new approach that consists of steps 1 and 2 in our proof, in which we show some new techniques.*

When factorizing the matrix B *in Step 3, several steps involve taking square roots of numbers to obtain the matrix* U_3 *as required. For the case when* $\mathbb{F} = \mathbb{C}$, *there is no problem because* $\mathbb{C}$ *is closed under taking square roots. But for a general subfield* $\mathbb{F}$ *of* $\mathbb{C}$, *we must be careful with this as we have to ensure that all entries of* U_3 *are in* $\mathbb{F}$. *Therefore, though we can borrow some ideas from [2] to justify Step 3, our proof will have new elements and tools to overcome the difficulties of taking square roots.*

2.1. On the Symmetry Property of Laurent Polynomials

Before we justify steps 1–3, let us review some important facts about the symmetry property of Laurent polynomials.

For $\mathsf{u}(z) = \sum_{k\in\mathbb{Z}} u[k]z^k \in \mathbb{F}[z, z^{-1}]\backslash\{0\}$, define

- the *lower degree* of u by
$$\mathrm{ldeg}(\mathsf{u}) := \min\{k \in \mathbb{Z} : u[k] \neq 0\};$$
- the *degree* of u by
$$\deg(\mathsf{u}) := \max\{k \in \mathbb{Z} : u[k] \neq 0\};$$
- the *length* of u by
$$\mathrm{len}(\mathsf{u}) := \deg(\mathsf{u}) - \mathrm{ldeg}(\mathsf{u}).$$

If $\mathsf{u} = 0$, then we define $\mathrm{len}(0) := -\infty$.

The following proposition can be verified by direct computation:

Proposition 1. *Suppose* $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *such that* $\mathsf{Su}(z) = \epsilon_u z^{c_u}$ *and* $\mathsf{Sv}(z) = \epsilon_v z^{c_v}$ *for some* $\epsilon_u, \epsilon_v \in \{-1, 1\}$ *and* $c_u, c_v \in \mathbb{Z}$. *Then,*

(1) *if* $\mathsf{Su} = \mathsf{Sv}$, *then* $\mathsf{u} \pm \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *with* $\mathsf{S}[\mathsf{u} \pm \mathsf{v}](z) = \mathsf{Su} = \epsilon_u z^{c_u}$;
(2) $\mathsf{uv} \in \mathrm{Sym}_{\mathbb{F}}$ *with* $\mathsf{S}[\mathsf{uv}](z) = \epsilon_u \epsilon_v z^{c_u + c_v}$;
(3) *if* v *divides* u, *then* $\mathsf{u}/\mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *with* $\mathsf{S}[\mathsf{u}/\mathsf{v}](z) = \epsilon_u \epsilon_v z^{c_u - c_v}$;
(4) $\mathsf{u}^* \in \mathrm{Sym}_{\mathbb{F}}$ *with* $\mathsf{S}[\mathsf{u}^*](z) = [\mathsf{Su}]^*(z) = \epsilon_u z^{-c_u}$.

If in addition $\mathsf{u} \neq 0$, *then* $\mathsf{S}[\mathsf{u}^{\mathsf{H}}] = \mathsf{Su}^{-1}$, $c_u = \mathrm{ldeg}(\mathsf{u}) + \deg(\mathsf{u})$, *and* $\mathrm{prt}(\mathrm{len}(\mathsf{u})) = \mathrm{prt}(c_u)$, *where* $\mathrm{prt}(k)$ *is defined for every* $k \in \mathbb{Z}$ *as*

$$\mathrm{prt}(k) := \begin{cases} 0, & \text{if } k \in 2\mathbb{Z}, \\ 1, & \text{if } k \notin 2\mathbb{Z}. \end{cases} \tag{10}$$

To characterize the symmetry property of a Laurent polynomial, we need to analyze the multiplicities of its roots. Denote $\mathsf{Z}(\mathsf{u}, z_0)$ the multiplicity of the root of u at z_0. The following result is well-known (see e.g., [Proposition 2.2 [6]] and [Lemma 3.2 [2]]).

Lemma 1. *Let* $\mathsf{u} \in \mathbb{F}[z, z^{-1}]$.

(1) $\mathsf{u} \in \mathrm{Sym}_{\mathbb{F}}$ *if, and only if,* $\mathsf{Z}(\mathsf{u}, z_0) = \mathsf{Z}(\mathsf{u}, z_0^{-1})$ *for all* $z_0 \in \mathbb{C}\backslash\{0\}$.
(2) *If* $\mathsf{u} \in \mathrm{Sym}_{\mathbb{F}}$ *is a non-zero Laurent polynomial with symmetry type* $\mathsf{Su}(z) = \epsilon z^c$ *for some* $\epsilon \in \{\pm 1\}$ *and* $c \in \mathbb{Z}$, *then* $\epsilon = (-1)^{\mathsf{Z}(\mathsf{u},1)}$ *and* $\mathrm{prt}(c) = \mathrm{prt}(\mathsf{Z}(\mathsf{u}, 1) + \mathsf{Z}(\mathsf{u}, -1))$.

Next, we discuss some properties of matrices of Laurent polynomials with compatible symmetry. Suppose $\mathsf{P} \in \mathrm{Sym}_{\mathbb{F}^{r \times s}}$ that satisfies (5) for some $1 \times r$ and $1 \times s$ row vectors of Laurent polynomials $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ with symmetry. We can tell the symmetry relations between the rows and columns of $\mathsf{P}(z)$ from $\mathsf{S}\boldsymbol{\eta}_1(z)$ and $\mathsf{S}\boldsymbol{\eta}_2(z)$:

$$\frac{\mathsf{SP}_{j,l}(z)}{\mathsf{SP}_{k,l}(z)} = \frac{[\mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)]_j}{[\mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)]_k}, \quad \frac{\mathsf{SP}_{l,j}(z)}{\mathsf{SP}_{l,k}(z)} = \frac{[\mathsf{S}\boldsymbol{\eta}_2(z)]_j}{[\mathsf{S}\boldsymbol{\eta}_2(z)]_k}, \quad \forall l = 1, \ldots, s, \quad \forall j, k = 1, \ldots, r. \tag{11}$$

For a matrix $\mathsf{P} \in \mathrm{Sym}_{\mathbb{F}^{n \times n}}$, we can prove by induction that $\det(\mathsf{P}) \in \mathrm{Sym}_{\mathbb{F}}$ with

$$\mathsf{S}[\det(\mathsf{P})](z) = \prod_{j=1}^{n} \Big([\mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)]_j [\mathsf{S}\boldsymbol{\eta}_2(z)]_j \Big) = \prod_{j=1}^{n} \mathsf{SP}_{j,j}(z). \tag{12}$$

If in addition P is *strongly invertible*, that is, $\det(\mathsf{P}(z)) = cz^k$ for some $c \in \mathbb{F} \setminus \{0\}$ and $k \in \mathbb{Z}$, then $\mathsf{P}^{-1} \in \mathrm{Sym}_{\mathbb{F}^{n \times n}}$. In this case, as $\mathsf{P}^{-1} = \det(\mathsf{P})^{-1} \,\mathrm{adj}(\mathsf{P})$, where $\mathrm{adj}(\mathsf{P})$ is the adjugate matrix of P, and

$$[\mathsf{S}[\mathrm{adj}(\mathsf{P})](z)]_{j,k} = \prod_{1 \leqslant l \leqslant n, l \neq k} \; \prod_{1 \leqslant m \leqslant n, m \neq j} [\mathsf{S}\boldsymbol{\eta}_1^{\mathsf{H}}(z)]_l [\mathsf{S}\boldsymbol{\eta}_2(z)]_m = \mathsf{S}[\det(\mathsf{P})](z) [\mathsf{S}\boldsymbol{\eta}_1(z)]_k [\mathsf{S}\boldsymbol{\eta}_2^{\mathsf{H}}(z)]_j,$$

we see that $\mathsf{P}^{-1} \in \mathrm{Sym}_{\mathbb{F}^{n\times n}}$ and it has symmetry type

$$\mathsf{S}[\mathsf{P}^{-1}](z) = \mathsf{S}\boldsymbol{\eta}_2^{\mathsf{H}}(z)\mathsf{S}\boldsymbol{\eta}_1(z). \tag{13}$$

From the above discussion, if $\mathsf{P} \in \mathrm{Sym}_{\mathbb{F}^{n\times n}}$ is strongly invertible, then for $\mathsf{Q} \in \mathrm{Sym}_{\mathbb{F}^{n\times m}}$ and $\mathsf{R} \in \mathrm{Sym}_{\mathbb{F}^{m\times n}}$, if the multiplication $\mathsf{PQ} =: \mathsf{M}$ (resp. $\mathsf{RP} =: \mathsf{B}$) is compatible, then so is the multiplication $\mathsf{P}^{-1}\mathsf{M}$ (resp. BP^{-1}).

2.2. Justification of Step 1 in the Proof of Theorem 1

To find the desired matrices $\mathring{\mathsf{M}} \in \mathrm{HS}_{\mathbb{F},2}$ and $\mathsf{U}_1 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ in Step 1, we need to take out all the common divisors of the entries of M. We first introduce the following lemma which generalizes [Lemma 3.15 [2]]:

Lemma 2. *Let $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ be such that $\mathsf{v}\mathsf{v}^*$ divides $\mathsf{u}\mathsf{u}^*$. Then, there exists $\mathsf{s} \in \mathrm{Sym}_{\mathbb{F}}$ such that*

$$\mathsf{s}(z)\mathsf{s}^*(z) = \frac{\mathsf{u}(z)\mathsf{u}^*(z)}{\mathsf{v}(z)\mathsf{v}^*(z)}. \tag{14}$$

Moreover, for any $\mathsf{s} \in \mathrm{Sym}_{\mathbb{F}}$ that satisfies (14), we have $\mathsf{Ss}(z) = z^{2k}\frac{\mathsf{Su}(z)}{\mathsf{Sv}(z)}$ for some $k \in \mathbb{Z}$.

Proof. Define $\mathsf{q}(z) := \frac{\mathsf{u}(z)\mathsf{u}^*(z)}{\mathsf{v}(z)\mathsf{v}^*(z)}$. By the assumptions on u and v, we have $\mathsf{q} \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{Sq} = 1$. Suppose $\mathsf{w} \in \mathbb{F}[z, z^{-1}]$ is an irreducible factor of q that does not have symmetry. By the symmetry of $\mathsf{q}(z)$, we see that $\mathsf{w}(z^{-1})$ is another irreducible factor of $\mathsf{q}(z)$ and is coprime to w. Thus, $\mathsf{w}_0(z) := \mathsf{w}(z)\mathsf{w}((z)^{-1})$ divides $\mathsf{q}(z)$. Moreover, w divides u or u^*. By the symmetry of u, we conclude that w_0 divides u or u^* and thus $\mathsf{w}_0\mathsf{w}_0^*$ divides $\mathsf{u}\mathsf{u}^*$. By the choice of w_0, we see that it does not divide v. Consequently, $\mathsf{w}_0\mathsf{w}_0^*$ divides q.

From the above discussion, we can write

$$\mathsf{q}(z) = |\beta|^2 \prod_{j=1}^{t} \mathsf{w}_j(z)\mathsf{w}_j^*(z), \tag{15}$$

where $\beta \in \mathbb{F}$ and $\mathsf{w}_1, \ldots, \mathsf{w}_t \in \mathrm{Sym}_{\mathbb{F}}$. By letting $\mathsf{s} := \beta\prod_{j\in\Lambda}\mathsf{w}_j$, we have $\mathsf{M} \in \mathrm{Sym}_{\mathbb{F}}$ and (14) holds.

Now, let $\mathsf{s} \in \mathrm{Sym}_{\mathbb{F}}$ be arbitrary such that (14) holds. Denote $\mathsf{Ss}(z) = \epsilon_s z^{c_s}$, $\mathsf{Su}(z) = \epsilon_u z^{c_u}$ and $\mathsf{Sv}(z) = \epsilon_v z^{c_v}$ for some $\epsilon_s, \epsilon_u, \epsilon_v \in \{\pm 1\}$ and $c_s, c_u, c_v \in \mathbb{Z}$. By Lemma 1, we have $\epsilon_s = (-1)^{\mathsf{Z}(\mathsf{d},1)} = (-1)^{\mathsf{Z}(\mathsf{u},1)-\mathsf{Z}(\mathsf{v},1)} = \epsilon_u/\epsilon_v$. Moreover, by Proposition 1, we obtain

$$\mathrm{prt}(c_s) = \mathrm{prt}(\mathrm{len}(\mathsf{s})) = \mathrm{prt}(\mathrm{len}(\mathsf{u}) - \mathrm{len}(\mathsf{v})) = \mathrm{prt}(c_u - c_v).$$

Therefore, we can find $k \in \mathbb{Z}$ such that $\mathsf{Ss}(z) = z^{2k}\frac{\mathsf{Su}(z)}{\mathsf{Sv}(z)}$. □

Remark 2. *[Lemma 3.15 [2]] is a special case of Lemma 2 with $\mathbb{F} = \mathbb{C}$. The original proof of [Lemma 3.15 [2]] relies on [Theorem 2.9 [6]] and is over-complicated from our point of view. Hence, we presented the above simpler self-contained proof.*

Next, we have the following lemma that allows us to further take out the common factors of entries of M. For $\mathsf{p}, \mathsf{q} \in \mathbb{F}[z, z^{-1}]$, by $\gcd(\mathsf{p}, \mathsf{q})$, we mean the greatest common divisor of p and q in $\mathbb{F}[z, z^{-1}]$.

Lemma 3. *Let $\mathsf{M} \in \mathrm{HS}_{\mathbb{F},2}$ be such that $\det(\mathsf{M}(z)) = -|\mathsf{C}|^2\mathsf{d}(z)\mathsf{d}^*(z)$ for some $\mathsf{d} \in \mathrm{Sym}_{\mathbb{F}}$ and some $C \in \mathbb{C}$ with $|\mathsf{C}|^2 \in \mathbb{F}$. Define $\mathsf{h}_1 := \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}\mathsf{M}_{1,2}^*)$ and $\mathsf{h}_2 := \gcd(\mathsf{M}_{2,2}, \mathsf{M}_{1,2}\mathsf{M}_{1,2}^*)$. If $\gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{1,2}^*, \mathsf{M}_{2,2}) = 1$, then there exist $\mathsf{r}_1, \mathsf{r}_2 \in \mathrm{Sym}_{\mathbb{F}}$ that both divide $\mathsf{M}_{1,2}$ and satisfy*

$$\mathsf{h}_1(z) = \alpha_1 z^{n_1} \mathsf{r}_1(z)\mathsf{r}_1^*(z) \quad \textit{and} \quad \mathsf{h}_2(z) = \alpha_2 z^{n_2} \mathsf{r}_2(z)\mathsf{r}_2^*(z), \tag{16}$$

for some $\alpha_1, \alpha_2 \in \mathbb{F}$ *and* $n_1, n_2 \in \mathbb{Z}$. *Moreover,* $\gcd(\mathsf{r}_1, \mathsf{r}_2) = \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{2,2})$.

Proof. Define $\mathsf{q} := \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{2,2})$. As all entries of M have symmetry, so is q. Moreover, we have $\gcd(\mathsf{q}, \mathsf{q}^*) = \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{1,2}^*, \mathsf{M}_{2,2}) = 1$. Thus by letting $\tilde{\mathsf{M}} := \mathrm{Diag}(\mathsf{q}^{-*}, 1)\mathsf{M}\mathrm{Diag}(\mathsf{q}^{-1}, 1)$, we see that $\tilde{\mathsf{M}} \in \mathrm{HS}_{\mathbb{F},2}$, $\gcd(\tilde{\mathsf{M}}_{1,1}, \tilde{\mathsf{M}}_{1,2}, \tilde{\mathsf{M}}_{2,2}) = 1$ and $\det(\tilde{\mathsf{M}}) = -|C|^2\tilde{\mathsf{d}}\tilde{\mathsf{d}}^*$, where $\tilde{\mathsf{d}} := \frac{\mathsf{d}}{\mathsf{q}} \in \mathrm{Sym}_{\mathbb{F}}$. Observe that $\mathsf{q}\mathsf{q}^*$ divides h_1 and h_2, so $\tilde{\mathsf{h}}_1 := \frac{\mathsf{h}_1}{\mathsf{q}\mathsf{q}^*}$ and $\tilde{\mathsf{h}}_2 := \frac{\mathsf{h}_2}{\mathsf{q}\mathsf{q}^*}$ are well-defined Laurent polynomials in $\mathbb{F}[z, z^{-1}]$.

We prove that

$$\tilde{\mathsf{h}}_1 = \alpha_1 z^{n_1} \tilde{\mathsf{r}}_1(z)\tilde{\mathsf{r}}_1^*(z),$$

for some $\alpha_1 \in \mathbb{F}$, $n_1 \in \mathbb{Z}$ and $\tilde{\mathsf{r}}_1 \in \mathrm{Sym}_{\mathbb{F}}$ that divides $\tilde{\mathsf{M}}_{1,2}$. First, note that if $\tilde{\mathsf{M}}_{2,2} = 0$, then $\gcd(\tilde{\mathsf{M}}_{1,1}, \tilde{\mathsf{M}}_{1,2}) = 1$, so $\tilde{\mathsf{h}}_1 \equiv 1$ and the claim clearly holds.

Now, assume $\tilde{\mathsf{M}}_{2,2} \neq 0$. We first prove that $\gcd(\tilde{\mathsf{h}}_1, \tilde{\mathsf{M}}_{2,2}) = 1$. Let $\mathsf{p} \in \mathbb{F}[z, z^{-1}]$ be an irreducible element such that p divides both $\tilde{\mathsf{h}}_1$ and $\tilde{\mathsf{M}}_{2,2}$. By the choice of $\tilde{\mathsf{h}}_1$, we see that p divides $\tilde{\mathsf{M}}_{1,1}$ and $\tilde{\mathsf{M}}_{1,2}^*\tilde{\mathsf{M}}_{1,2}$. Because p is irreducible, we must have either p or p^* divides $\tilde{\mathsf{M}}_{1,2}$. Moreover, as $\tilde{\mathsf{M}}_{1,1}$ and $\tilde{\mathsf{M}}_{2,2}$ are Hermitian, both must be divisible by p^*. Consequently, p or p^* divides all $\tilde{\mathsf{M}}_{1,1}, \tilde{\mathsf{M}}_{1,2}$ and $\tilde{\mathsf{M}}_{2,2}$. By the assumption $\gcd(\tilde{\mathsf{M}}_{1,1}, \tilde{\mathsf{M}}_{1,2}, \tilde{\mathsf{M}}_{2,2}) = 1$, up to a monomial, we must have $\mathsf{p} = 1$. Next, we show that if $\mathsf{h} \in \mathbb{F}[z, z^{-1}]$ is irreducible and divides $\tilde{\mathsf{h}}_1$, then $\mathsf{h}\mathsf{h}^*$ must divide $\tilde{\mathsf{h}}_1$. Indeed, if h is an irreducible factor of $\tilde{\mathsf{h}}_1$, then by using a similar argument of the proof of the previous claim, either h or h^* divides $\tilde{\mathsf{M}}_{1,2}$ and thus $\mathsf{h}\mathsf{h}^*$ divides $\tilde{\mathsf{M}}_{1,2}\tilde{\mathsf{M}}_{1,2}^*$. On the other hand, as $-|C|^2\tilde{\mathsf{d}}\tilde{\mathsf{d}}^* = \det(\tilde{\mathsf{M}}) = \tilde{\mathsf{M}}_{1,1}\tilde{\mathsf{M}}_{2,2} - \tilde{\mathsf{M}}_{1,2}\tilde{\mathsf{M}}_{1,2}^*$, we see that h divides $\tilde{\mathsf{d}}\tilde{\mathsf{d}}^*$. Using the irreducibility of h, we see that $\mathsf{h}\mathsf{h}^*$ divides $\tilde{\mathsf{d}}\tilde{\mathsf{d}}^*$. Now, using the fact that $\gcd(\tilde{\mathsf{h}}_1, \tilde{\mathsf{M}}_{2,2}) = 1$ and $\tilde{\mathsf{M}}_{1,1} = \frac{\tilde{\mathsf{M}}_{1,2}\tilde{\mathsf{M}}_{1,2}^* - |C|^2\tilde{\mathsf{d}}\tilde{\mathsf{d}}^*}{\tilde{\mathsf{M}}_{2,2}}$, we conclude that $\mathsf{h}\mathsf{h}^*$ divides $\tilde{\mathsf{M}}_{1,1}$. Consequently, we have $\mathsf{h}\mathsf{h}^*$ divides $\tilde{\mathsf{h}}_1$.

We now analyze the symmetry of the factors of $\tilde{\mathsf{h}}_1$. Let $\mathsf{w} \in \mathbb{F}[z, z^{-1}]$ be an irreducible factor of $\tilde{\mathsf{h}}_1$ that has no symmetry. Then, $\mathsf{w}(z^{-1})$ must be relatively prime to w. By the definition of $\tilde{\mathsf{h}}_1$, we see that w must divide $\tilde{\mathsf{d}}$ or $\tilde{\mathsf{d}}^*$. Noting that $\tilde{\mathsf{d}}$ has symmetry, we must have $\mathsf{v}(z) := \mathsf{w}(z)\mathsf{w}(z^{-1})$ divides $\tilde{\mathsf{d}}$ or $\tilde{\mathsf{d}}^*$ and thus $\mathsf{v}\mathsf{v}^*$ divides $\det(\tilde{\mathsf{M}})$. Similarly, we have $\mathsf{v}\mathsf{v}^*$ divides $\tilde{\mathsf{M}}_{1,2}\tilde{\mathsf{M}}_{1,2}^*$. As $\gcd(\tilde{\mathsf{h}}_1, \tilde{\mathsf{M}}_{2,2}) = 1$, we conclude that $\mathsf{v}\mathsf{v}^*$ divides $\tilde{\mathsf{M}}_{1,1}$ and thus $\mathsf{v}\mathsf{v}^*$ divides $\tilde{\mathsf{h}}_1$. Consequently, we can write

$$\tilde{\mathsf{h}}_1(z) = c_1 z^{n_1} \prod_{j=1}^{t} \mathsf{w}_j(z)\mathsf{w}_j^*(z), \tag{17}$$

such that $c_1 \in \mathbb{C}$, $n_1 \in \mathbb{Z}$, and all $\mathsf{w}_1, \ldots \mathsf{w}_t \in \mathrm{Sym}_{\mathbb{F}}$. By letting $\tilde{\mathsf{r}}_1(z) := \prod_{j=1}^{t} \mathsf{w}_j(z)$, we have $\tilde{\mathsf{r}}_1 \in \mathrm{Sym}_{\mathbb{F}}$ and $\tilde{\mathsf{h}}_1(z) = c_1 z^{n_1} \tilde{\mathsf{r}}_1(z)\tilde{\mathsf{r}}_1^*(z)$. Observe that each polynomial w_j in (17) either divides $\tilde{\mathsf{M}}_{1,2}$ or $\tilde{\mathsf{M}}_{1,2}^*$. Therefore, without loss of generality, we may assume that all w_j's divide $\tilde{\mathsf{M}}_{1,2}$. As a result, we have $\tilde{\mathsf{r}}_1 | \tilde{\mathsf{M}}_{1,2}$.

Similarly, we can show that there exists $\tilde{\mathsf{r}}_2 \in \mathrm{Sym}_{\mathbb{F}}$ that divides $\tilde{\mathsf{M}}_{1,2}$ and satisfies $\tilde{\mathsf{h}}_2(z) = c_2 z^{n_2} \tilde{\mathsf{r}}_2(z)\tilde{\mathsf{r}}_2^*(z)$ for some $c_2 z^{n_2} \in \mathbb{F}[z, z^{-1}]$. Furthermore, by the choices of $\tilde{\mathsf{h}}_1, \tilde{\mathsf{h}}_2$ and the fact that $\gcd(\tilde{\mathsf{M}}_{1,1}, \tilde{\mathsf{M}}_{1,2}, \tilde{\mathsf{M}}_{2,2}) = 1$, we must have $\gcd(\tilde{\mathsf{r}}_1, \tilde{\mathsf{r}}_2) = 1$.

Finally, by letting $\mathsf{r}_1 := \mathsf{q}\tilde{\mathsf{r}}_1$ and $\mathsf{r}_2 := \mathsf{q}\tilde{\mathsf{r}}_2$, all claims of the lemma hold and the proof is complete. □

Remark 3. *Let* $\mathsf{M} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ *be such that* $\det(\mathsf{M}(z)) = -|C|^2\mathsf{d}(z)\mathsf{d}^*(z)$ *for some* $\mathsf{d} \in \mathrm{Sym}_{\mathbb{F}}$ *and* $C \in \mathbb{C}$ *with* $|C|^2 \in \mathbb{F}$. *Suppose* $\gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{1,2}^*, \mathsf{M}_{2,2}) = 1$, *then by Lemma 3, we can define the following sets for convenience:*

$$G_1(\mathsf{M}) := \{\mathsf{r}_1 \in \mathrm{Sym}_{\mathbb{F}} : \mathsf{r}_1 \textit{ divides } \mathsf{M}_{1,2} \textit{ and } \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}\mathsf{M}_{1,2}^*) = \alpha_1 z^{n_1} \mathsf{r}_1 \mathsf{r}_1^* \\ \textit{for some } \alpha_1 \in \mathbb{C}, n_1 \in \mathbb{Z}\}, \tag{18}$$

$$G_2(\mathsf{M}) := \{\mathsf{r}_2 \in \mathrm{Sym}_{\mathbb{F}} : \mathsf{r}_2 \textit{ divides } \mathsf{M}_{1,2} \textit{ and } \gcd(\mathsf{M}_{2,2}, \mathsf{M}_{1,2}\mathsf{M}_{1,2}^*) = \alpha_2 z^{n_2} \mathsf{r}_2 \mathsf{r}_2^* \\ \textit{for some } \alpha_2 \in \mathbb{C}, n_2 \in \mathbb{Z}\}. \tag{19}$$

For any $\mathsf{r}_1 \in G_1(\mathsf{M})$ *and* $\mathsf{r}_2 \in G_2(\mathsf{M})$, *we have* $\gcd(\mathsf{r}_1, \mathsf{r}_2) = \gcd(\mathsf{M}_{1,1}, \mathsf{M}_{1,2}, \mathsf{M}_{2,2})$.

Now, we have all the tools to justify Step 1.

Justification of Step 1 in the Proof of Theorem 1. As M satisfies all assumptions of Lemma 3, we can define $G_1(\mathsf{M})$ as in (18) and $G_2(\mathsf{M})$ as in (19). Choose $\mathsf{r}_1 \in G_1(\mathsf{M})$ and $\mathsf{r}_2 \in G_2(\mathsf{M})$. Define

$$\mathsf{U}_1 := \mathrm{Diag}(\mathsf{r}_1, \mathsf{r}_2), \quad \mathring{\mathsf{M}} := \mathsf{U}_1^{-1}\mathsf{M}\mathsf{U}_1^{-\mathsf{H}}.$$

It is trivial that $\mathring{\mathsf{M}} \in \mathrm{HS}_{\mathbb{F},2}$ satisfies item (i) in Step 1.

Next, we have $\mathsf{M} = \mathsf{U}_1\mathring{\mathsf{M}}\mathsf{U}_1^{\mathsf{H}}$,

$$\mathsf{S}\mathring{\mathsf{M}}(z) = \begin{bmatrix} 1 \\ \mathsf{S}\mathsf{M}_{1,2}^*(z)\mathsf{S}\mathsf{r}_1(z)\mathsf{S}\mathsf{r}_2^*(z) \end{bmatrix} \begin{bmatrix} 1 & \mathsf{S}\mathsf{M}_{1,2}(z)\mathsf{S}\mathsf{r}_1^*(z)\mathsf{S}\mathsf{r}_2(z) \end{bmatrix},$$

and

$$\mathsf{S}\mathsf{U}_1(z) = \begin{bmatrix} \mathsf{S}\mathsf{r}_1(z) \\ \mathsf{S}\mathsf{M}_{1,2}^*(z)\mathsf{S}\mathsf{r}_1(z) \end{bmatrix} \begin{bmatrix} 1 & \mathsf{S}\mathsf{M}_{1,2}(z)\mathsf{S}\mathsf{r}_1^*(z)\mathsf{S}\mathsf{r}_2(z) \end{bmatrix}.$$

Thus, it is easy to see that item (iii) in Step 1 holds.

Finally, since

$$\det(\mathring{\mathsf{M}}(z)) = \frac{\det(\mathsf{M}(z))}{\det(\mathsf{U}_1(z))\det(\mathsf{U}_1(z))^*} = \frac{-|C|^2\mathsf{d}(z)\mathsf{d}^*(z)}{\det(\mathsf{U}_1(z))\det(\mathsf{U}_1(z))^*},$$

then by Lemma 2, there exists $\mathring{\mathsf{d}} \in \mathrm{Sym}_{\mathbb{F}}$ such that $\det(\mathring{\mathsf{M}}) = -|C|^2\mathring{\mathsf{d}}\mathring{\mathsf{d}}^*$. This justifies item (ii) in Step 1. □

2.3. *Justification of Step 2 in the Proof of Theorem 1*

To justify Step 2, we need the following Euclidean algorithm compatible with the symmetry property. We have the following lemma, which is a straightforward generalization of [Theorem 3.9 [2]].

Lemma 4 (Extended Euclidean Algorithm for Laurent polynomials in $\mathrm{Sym}_{\mathbb{F}}$)**.** *Let* $\mathsf{a}, \mathsf{b} \in \mathrm{Sym}_{\mathbb{F}}$ *and define* $\mathsf{r} := \gcd(\mathsf{a}, \mathsf{b})$. *There exist* $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *such that*

$$\mathsf{a}(z)\mathsf{u}(z) + \mathsf{b}(z)\mathsf{v}(z) = \mathsf{r}(z). \tag{20}$$

Furthermore, $\mathsf{S}\mathsf{a}(z)\mathsf{S}\mathsf{u}(z) = \mathsf{S}\mathsf{b}(z)\mathsf{S}\mathsf{v}(z) = \mathsf{S}\mathsf{r}(z)$ *and* $\gcd(\mathsf{u}, \mathsf{v}) = 1$.

Justification of Step 2 in the Proof of Theorem 1. Let $\mathring{\mathsf{M}} \in \mathrm{HS}_{\mathbb{F},2}$ and $\mathring{\mathsf{d}} \in \mathrm{Sym}_{\mathbb{F}}$ be the same as in Step 1. By Lemma 4, there exist $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ such that

$$\mathsf{u}(z)\mathring{\mathsf{M}}_{1,2}(z) + \mathsf{v}(z)\mathring{\mathsf{M}}_{2,2}(z) = 1, \quad \mathsf{S}\mathsf{u}(z)\mathsf{S}\mathring{\mathsf{M}}_{1,2}(z) = \mathsf{S}\mathsf{v}(z)\mathsf{S}\mathring{\mathsf{M}}_{2,2}(z) = 1, \quad \gcd(\mathsf{u}, \mathsf{v}) = 1. \tag{21}$$

Define

$$\mathsf{V}_1 := \begin{bmatrix} \mathring{\mathsf{M}}_{2,2} & -\mathring{\mathsf{M}}_{1,2} \\ \mathsf{u} & \mathsf{v} \end{bmatrix}, \quad \mathring{\mathsf{B}} := \mathsf{V}_1\mathring{\mathsf{M}}\mathsf{V}_1^{\mathsf{H}}.$$

Then, $\mathsf{V}_1 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$, $\det(\mathsf{V}_1) = 1$ and thus $\det(\mathring{\mathsf{B}}) = \det(\mathring{\mathsf{M}})$. Furthermore, direct calculation shows that

$$\mathring{\mathsf{B}} = \begin{bmatrix} \det(\mathring{\mathsf{M}})\mathring{\mathsf{M}}_{2,2} & \det(\mathring{\mathsf{M}})\mathsf{u}^* \\ \det(\mathring{\mathsf{M}})\mathsf{u} & \mathsf{u}\mathsf{u}^*\mathring{\mathsf{M}}_{1,1} + \mathsf{v}\mathsf{u}^*\mathring{\mathsf{M}}_{1,2}^* + \mathsf{v}^*\mathsf{u}\mathring{\mathsf{M}}_{1,2} + \mathsf{v}\mathsf{v}^*\mathring{\mathsf{M}}_{2,2} \end{bmatrix}.$$

By letting

$$\mathsf{V}_2 := \mathrm{Diag}(\mathring{\mathsf{d}}, 1), \quad \mathsf{B} := \begin{bmatrix} -|C|^2\mathring{\mathsf{M}}_{2,2} & -|C|^2\mathring{\mathsf{d}}^*\mathsf{u}^* \\ -|C|^2\mathring{\mathsf{d}}\mathsf{u} & \mathring{\mathsf{B}}_{2,2} \end{bmatrix}, \quad \mathsf{U}_2 := \mathsf{V}_1^{-1}\mathsf{V}_2,$$

we have $\mathring{\mathsf{M}} = \mathsf{U}_2\mathsf{B}\mathsf{U}_2^{\mathsf{H}}$ and $\det(\mathsf{B}) = -|C|^2$. Moreover, the symmetry relations in (21) yields $\mathsf{S}\mathsf{u} = \mathsf{S}\mathring{\mathsf{M}}_{1,2}^*$ and $\mathsf{S}\mathsf{v} = 1$. By calculation, we have

$$\mathsf{S}\mathsf{B}(z) = \begin{bmatrix} 1 & \mathsf{S}\mathring{\mathsf{d}}^*(z)\mathsf{S}\mathsf{u}^*(z) \\ \mathsf{S}\mathring{\mathsf{d}}(z)\mathsf{S}\mathsf{u}(z) & 1 \end{bmatrix} = \begin{bmatrix} \mathsf{S}\mathring{\mathsf{d}}^*(z) \\ \mathsf{S}\mathsf{u}(z) \end{bmatrix} \begin{bmatrix} \mathsf{S}\mathring{\mathsf{d}}(z) & \mathsf{S}\mathsf{u}^*(z) \end{bmatrix},$$

$$\mathsf{S}\mathsf{U}_2(z) = \begin{bmatrix} \mathsf{S}\mathring{\mathsf{d}}(z) & \mathsf{S}\mathsf{u}^*(z) \\ \mathsf{S}\mathring{\mathsf{d}}(z)\mathsf{S}\mathsf{u}(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathsf{S}\mathsf{u}(z) \end{bmatrix} \begin{bmatrix} \mathsf{S}\mathring{\mathsf{d}}(z) & \mathsf{S}\mathsf{u}^*(z) \end{bmatrix}.$$

Therefore, $\mathsf{U}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$, $\mathsf{B} \in \mathrm{HS}_{\mathbb{F},2}$, and the matrix multiplications in $\mathring{\mathsf{M}} = \mathsf{U}_2\mathsf{B}\mathsf{U}_2^*$ are compatible. This completes the justification of Step 2. □

2.4. Justification of Step 3 in the Proof of Theorem 1

Justifying Step 3 is the most technical part of the proof of Theorem 1. Let B be the same as in Step 2. The symmetry type of the entry $\mathsf{B}_{1,2}$ is critical in justifying Step 3. Denote $\mathsf{S}\mathsf{B}_{1,2}(z) := \epsilon_b z^{c_b}$ for some $\epsilon_b \in \{-1, 1\}$ and $c_b \in \mathbb{Z}$. Consider the following two cases:

- **Case 1.** $\epsilon_b = 1$ or $\epsilon_b = -1$ and $c_b \in 2\mathbb{Z} + 1$;
- **Case 2.** $\epsilon_b = -1$ and $c_b \in 2\mathbb{Z}$.

First, we work on the justification for case 1, which relies heavily on the long division with symmetry. We have two auxiliary lemmas.

Lemma 5. *Let* $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *be such that* $\mathrm{len}(\mathsf{u}) > \mathrm{len}(\mathsf{v})$ *and* $\mathsf{v} \neq 0$. *Then, there exist* $\mathsf{q}, \mathsf{r} \in \mathrm{Sym}_{\mathbb{F}}$ *such that* $\mathsf{u} = \mathsf{v}\mathsf{q} + \mathsf{r}$, $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{u})$ *and* $\mathsf{S}\mathsf{u} = \mathsf{S}\mathsf{r} = \mathsf{S}\mathsf{v}\mathsf{S}\mathsf{q}$.

Proof. Define $M_u := \deg(\mathsf{u})$, $m_u := \mathrm{ldeg}(\mathsf{u})$, $M_v := \deg(\mathsf{v})$ and $m_v := \mathrm{ldeg}(\mathsf{v})$, so we write

$$\mathsf{u}(z) = \sum_{k=m_u}^{M_u} u[k]z^k, \quad \mathsf{v}(z) = \sum_{n=m_v}^{M_v} v[n]z^n,$$

where $u[k], v[n] \in \mathbb{F}$ are coefficients for all $k = m_u, \ldots, M_u$ and $n = m_v, \ldots, M_v$. Denote $\mathsf{S}\mathsf{u}(z) = \epsilon_u z^{c_u}$ and $\mathsf{S}\mathsf{v}(z) = \epsilon_v z^{c_v}$ for some $\epsilon_u, \epsilon_v \in \{-1, 1\}$ and $c_u, c_v \in \mathbb{Z}$. It follows that

$$u[m_u] = \epsilon_u u[M_u], \quad M_u + m_u = c_u, \quad v[m_v] = \epsilon_v v[M_v], \quad M_v + m_v = c_v.$$

Define

$$\mathsf{q}(z) := \frac{u[M_u]}{v[M_v]} z^{M_u - M_v} + \frac{u[m_u]}{v[m_v]} z^{m_u - m_v}.$$

Direct calculation yields

$$\begin{aligned} \mathsf{q}(z^{-1}) &= \frac{u[M_u]}{v[M_v]} z^{M_v - M_u} + \frac{u[m_u]}{v[m_v]} z^{m_v - m_u} \\ &= \epsilon_u \epsilon_v \frac{u[m_u]}{v[m_v]} z^{m_u - m_v - (c_u - c_v)} + \epsilon_u \epsilon_v \frac{u[M_u]}{v[M_v]} z^{M_u - M_v - (c_u - c_v)}. \end{aligned}$$

Therefore,

$$\mathsf{Sq}(z) = \frac{\mathsf{q}(z)}{\mathsf{q}(z^{-1})} = \epsilon_u \epsilon_v z^{c_u - c_v} = \frac{\mathsf{Su}(z)}{\mathsf{Sv}(z)},$$

and thus $\mathsf{q} \in \mathrm{Sym}_{\mathbb{F}}$. Moreover, by letting $\mathsf{r} := \mathsf{u} - \mathsf{vq}$, it is trivial that $\mathsf{r} \in \mathrm{Sym}_{\mathbb{F}}$ and $\mathsf{Su} = \mathsf{Sr} = \mathsf{SvSq}$.

Next, we show that $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{u})$. As $\mathrm{len}(\mathsf{u}) > \mathrm{len}(\mathsf{v})$, we have $M_u - M_v > m_u - m_v$. So $\mathrm{ldeg}(\mathsf{q}) = m_u - m_v$ and $\deg(\mathsf{q}) = M_u - M_v$, which implies $\mathrm{ldeg}(\mathsf{vq}) = m_u$ and $\deg(\mathsf{vq}) = M_u$. Hence, $\mathrm{len}(\mathsf{r}) \leqslant \mathrm{len}(\mathsf{u} - \mathsf{vq}) \leqslant M_u - m_u = \mathrm{len}(\mathsf{u})$. Furthermore, observe that the coefficient of the z^{M_u} term in vq is $u(M_u)$. This implies $\deg(\mathsf{r}) = \deg(\mathsf{u} - \mathsf{vq}) < M_u$ and thus $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{u})$. This completes the proof. □

Lemma 6. *Let* $\mathsf{u}, \mathsf{v} \in \mathrm{Sym}_{\mathbb{F}}$ *be such that* $\mathsf{v} \neq 0$. *Then, there exist* $\mathsf{q}, \mathsf{r} \in \mathrm{Sym}_{\mathbb{F}}$ *such that* $\mathrm{len}(\mathsf{r}) \leqslant \mathrm{len}(\mathsf{v})$,

$$\mathsf{u} = \mathsf{vq} + \mathsf{r} \textit{ and } \mathsf{Su} = \mathsf{Sr} = \mathsf{SvSq}. \tag{22}$$

Moreover, if $\mathsf{Su}(z)\mathsf{Sv}(z) \neq -z^{2k}$ *for all* $k \in \mathbb{Z}$, *then* r *can be chosen such that* $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{v})$.

Proof. The case when $\mathrm{len}(\mathsf{u}) < \mathrm{len}(\mathsf{v})$ is trivial, we just choose $\mathsf{q} := 0$ and $\mathsf{r} := \mathsf{u}$. So, we assume that $\mathrm{len}(\mathsf{u}) \geqslant \mathrm{len}(\mathsf{v})$ and the rest of the proof will be for this case.

We construct $\mathsf{w}, \mathsf{q}_1 \in \mathrm{Sym}_{\mathbb{F}}$ such that

$$\mathsf{u} = \mathsf{vq}_1 + \mathsf{w}, \quad \mathrm{len}(\mathsf{w}) \leqslant \mathrm{len}(\mathsf{v}) \text{ and } \mathsf{Su} = \mathsf{Sw} = \mathsf{SvSq}_1. \tag{23}$$

Here is how such w and q_1 are constructed:

- If $\mathrm{len}(\mathsf{u}) = \mathrm{len}(\mathsf{v})$, simply choose $\mathsf{w} = \mathsf{u}$ and $\mathsf{q}_1 = 0$.
- If $\mathrm{len}(\mathsf{u}) > \mathrm{len}(\mathsf{v})$, set $\mathsf{u}^{(0)} := \mathsf{u}$. Starting from $l = 0$, whenever $\mathrm{len}(\mathsf{u}^{(l)}) > \mathrm{len}(\mathsf{v})$, apply Lemma 5 to find $\mathsf{u}^{(l+1)}, \mathsf{q}^{(l)} \in \mathrm{Sym}_{\mathbb{F}}$ such that $\mathsf{u}^{(l+1)} = \mathsf{u}^{(l)} - \mathsf{vq}^{(l)}$, $\mathsf{Su}^{(l+1)} = \mathsf{Su}^{(l)} = \mathsf{SvSq}^{(l)}$ and $\mathrm{len}(\mathsf{u}^{(l+1)}) < \mathrm{len}(\mathsf{u}^{(l)})$. As $\mathrm{len}(\mathsf{u})$ is finite, this iterative process must stop at some point. In particular, at some $l = N$, we must have $\mathrm{len}(\mathsf{u}^{(N)}) \leqslant \mathrm{len}(\mathsf{v}) < \mathrm{len}(\mathsf{u}^{(N-1)})$. Define

$$\mathsf{q}_1 := \sum_{l=0}^{N-1} \mathsf{q}^{(l)}, \quad \mathsf{w} := \mathsf{u}^{(N)}. \tag{24}$$

Clearly $\mathrm{len}(\mathsf{w}) \leqslant \mathrm{len}(\mathsf{v})$. Moreover, by the definition of w and q_1 above, together with the fact that

$$\mathsf{Su}^{(l+1)} = \mathsf{Su}, \quad \mathsf{Sq}^{(l)} = \frac{\mathsf{Su}^{(l+1)}}{\mathsf{Sv}} = \frac{\mathsf{Su}}{\mathsf{Sv}}, \quad \forall l = 0, \ldots, N-1,$$

it is easy to see that $\mathsf{w}, \mathsf{q}_1 \in \mathrm{Sym}_{\mathbb{F}}$ and (22) holds.

If $\mathsf{Su}(z)\mathsf{Sv}(z) = -z^{2k}$ for some $k \in \mathbb{Z}$, then let $\mathsf{r} := \mathsf{w}$ and $\mathsf{q} := \mathsf{q}_1$, we have $\mathsf{r}, \mathsf{q} \in \mathrm{Sym}_{\mathbb{F}}$, $\mathrm{len}(\mathsf{r}) \leqslant \mathrm{len}(\mathsf{v})$ and (22) holds.

Suppose $\mathsf{Su}(z)\mathsf{Sv}(z) \neq -z^{2k}$ for all $k \in \mathbb{Z}$. Denote $\mathsf{Su}(z) = \epsilon_u z^{c_u}$ and $\mathsf{Sv}(z) = \epsilon_v z^{c_v}$ for some $\epsilon_u, \epsilon_v \in \{-1, 1\}$ and $c_u, c_v \in \mathbb{Z}$. We consider the following two cases:

- $\mathsf{Su}(z)\mathsf{Sv}(z) = \epsilon z^{2k+1}$ for some $\epsilon \in \{-1, 1\}$ and $k \in \mathbb{Z}$: In this case, we must have $\mathrm{prt}(c_u) \neq \mathrm{prt}(c_v)$. Then, by Proposition 1, we have $\mathrm{len}(\mathsf{u}) \neq \mathrm{len}(\mathsf{v})$ and thus $\mathrm{len}(\mathsf{u}) > \mathrm{len}(\mathsf{v})$. Define $\mathsf{r} := \mathsf{w}$ and $\mathsf{q} := \mathsf{q}_1$, where $\mathsf{q}_1, \mathsf{w} \in \mathrm{Sym}_{\mathbb{F}}$ are defined as in (24). We see that (22) holds. Moreover, we have $\mathsf{Sr}(z) = \mathsf{Su}(z) = \epsilon_u z^{c_u}$ and thus it follows from Proposition 1 that

$$\mathrm{prt}(\mathrm{len}(\mathsf{r})) = \mathrm{prt}(c_u) \neq \mathrm{prt}(\mathrm{len}(c_v)) = \mathrm{prt}(\mathrm{len}(\mathsf{v})).$$

Therefore, $\mathrm{len}(\mathsf{r}) \neq \mathrm{len}(\mathsf{v})$ and thus $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{v})$.

- $\mathsf{Su}(z)\mathsf{Sv}(z) = z^{2k}$ for some $k \in \mathbb{Z}$: In this case, we have $\epsilon_u = \epsilon_v$. Let $\mathsf{w}, \mathsf{q}_1 \in \mathrm{Sym}_{\mathbb{F}}$ be such that (23) holds. If $\mathrm{len}(\mathsf{w}) < \mathrm{len}(\mathsf{v})$, then simply let $\mathsf{r} := \mathsf{w}$ and $\mathsf{q} := \mathsf{q}_1$, we see that $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{v})$ and (22) holds. Otherwise, if $\mathrm{len}(\mathsf{w}) = \mathrm{len}(\mathsf{v})$, define

$$M_w := \deg(\mathsf{w}), \quad m_w := \mathrm{ldeg}(\mathsf{w}), \quad M_v := \deg(\mathsf{v}), \quad m_v := \mathrm{ldeg}(\mathsf{v}).$$

Then, it is trivial that $M_w - M_v = m_w - m_v$. Write

$$\mathsf{w}(z) = \sum_{k=m_w}^{M_w} w[k]z^k, \quad \mathsf{v}(z) = \sum_{n=m_v}^{M_v} v[n]z^n,$$

for some coefficients $w[k], v[n] \in \mathbb{F}$ for all $k = m_w, \ldots, M_w$ and $n = m_v, \ldots, M_v$. Define

$$\mathsf{q}_2(z) := \frac{w[M_w]}{v[M_v]} z^{M_w - M_v} = \frac{w[M_w]}{v[M_v]} z^{m_w - m_v}, \quad \mathsf{r} := \mathsf{w} - \mathsf{v}\mathsf{q}_2. \tag{25}$$

It is trivial that $\deg(\mathsf{v}\mathsf{q}_2) = M_w$, $\mathrm{ldeg}(\mathsf{v}\mathsf{q}_2) = m_w$ and the coefficient of the z^{M_w} term in $\mathsf{v}\mathsf{q}_2$ is $w(M_w)$. Therefore, we have $\deg(\mathsf{r}) < M_w$, $\deg(\mathsf{r}) \geqslant m_w$ and thus $\mathrm{len}(\mathsf{r}) < \mathrm{len}(\mathsf{w}) = \mathrm{len}(\mathsf{v})$. On the other hand, using Proposition 1, direct calculation yields

$$\begin{aligned}\mathsf{Sv}(z)\mathsf{Sq}_2(z) &= [\epsilon_v z^{c_v}][z^{2M_w - 2M_v}] = \epsilon_v z^{(c_v - M_v) - M_v} z^{2M_w} \\ &= \epsilon_v z^{m_v - M_v} z^{2M_w} = \epsilon_w z^{m_w - M_w} z^{2M_w} = \mathsf{Sw}(z).\end{aligned}$$

By letting $\mathsf{q} := \mathsf{q}_1 + \mathsf{q}_2$, it is now straightforward to verify that $\mathsf{r}, \mathsf{q} \in \mathrm{Sym}_{\mathbb{F}}$ and (22) holds.

The proof is now complete. □

Justification of Step 3 in the Proof of Theorem 1, Case 1. There are two major steps in the constructions of U_3 and D that are required:

(SS1) Find $\mathsf{W}_1 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ such that W_1 is strongly invertible, $\tilde{\mathsf{B}} := \mathsf{W}_1 \mathsf{B} \mathsf{W}_1^{\mathsf{H}}$ has at least one zero entry, and all multiplications are compatible.

(SS2) Factorize $\tilde{\mathsf{B}}$:

$$\tilde{\mathsf{B}} = \mathsf{W}_2 \mathrm{Diag}(|c_1|^2, -|c_2|^2)\mathsf{W}_2, \tag{26}$$

such that $\mathsf{W}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$, $c_1, c_2 \in \mathbb{C}$ satisfy $|c_1|^2, |c_2|^2 \in \mathbb{F}$, and all matrix multiplications are compatible. By letting $\mathsf{U}_3 := \mathsf{W}_1^{-1}\mathsf{W}_2$ and $\mathsf{D} := \mathrm{Diag}(c_1, c_2)$, all claims of Step 3 hold.

We first justify (SS1). The case when B has one zero entry is trivial; choose $\mathsf{W}_1 := I_2$. So, we only need to justify the case when B has no zero entry. Set $\mathsf{B}^{(0)} := \mathsf{B}$. In this case, as $\det(\mathsf{B})$ is a constant and $\det(\mathsf{B}) = \mathsf{B}_{1,1}\mathsf{B}_{2,2} - \mathsf{B}_{1,2}\mathsf{B}_{1,2}^*$, we must have $\mathrm{ldeg}(\mathsf{B}_{1,1}\mathsf{B}_{2,2}) = \mathrm{ldeg}(\mathsf{B}_{1,2}\mathsf{B}_{1,2}^*)$ and $\deg(\mathsf{B}_{1,1}\mathsf{B}_{2,2}) = \deg(\mathsf{B}_{1,2}\mathsf{B}_{1,2}^*)$. Therefore,

$$\mathrm{len}(\mathsf{B}_{1,1}) + \mathrm{len}(\mathsf{B}_{2,2}) = \mathrm{len}(\mathsf{B}_{1,1}\mathsf{B}_{2,2}) = \mathrm{len}(\mathsf{B}_{1,2}\mathsf{B}_{1,2}^*) = 2\,\mathrm{len}(\mathsf{B}_{1,2}).$$

As a result, either $\mathrm{len}(\mathsf{B}_{1,1}) \leqslant \mathrm{len}(\mathsf{B}_{1,2})$ or $\mathrm{len}(\mathsf{B}_{2,2}) \leqslant \mathrm{len}(\mathsf{B}_{1,2})$.

We now claim that there exists $\mathsf{W}_1^{(0)} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ such that $\mathsf{W}_1^{(0)}$ is strongly invertible, all multiplications in $\mathsf{B}^{(1)} := \mathsf{W}_1^{(0)}\mathsf{B}^{(0)}\mathsf{W}_1^{(0)\mathsf{H}}$ are compatible, and $\mathrm{len}(\mathsf{B}_{1,2}^{(1)}) < \mathrm{len}(\mathsf{B}_{1,2}^{(0)})$. Let $j \in \{1, 2\}$ and suppose $\mathrm{len}(\mathsf{B}_{j,j}) \leqslant \mathrm{len}(\mathsf{B}_{1,2})$. As $\mathsf{SB}_{j,j} = 1$, we have $\mathsf{SB}_{j,j}\mathsf{SB}_{1,2} = \epsilon_b z^{c_b}$. By the assumptions that $\epsilon_b = 1$ or $\epsilon_b = -1$ and $c_b \in 2\mathbb{Z} + 1$, we can apply Lemma 6 to find

$\mathsf{r}^{(0)}, \mathsf{q}^{(0)} \in \mathrm{Sym}_{\mathbb{F}}$ such that $\mathrm{len}(\mathsf{r}^{(0)}) < \mathrm{len}(\mathsf{B}_{j,j})$, $\mathsf{B}_{1,2} = \mathsf{B}_{j,j}\mathsf{q}^{(0)} + \mathsf{r}^{(0)}$ and $\mathsf{SB}_{1,2} = \mathsf{Sr}^{(0)} = \mathsf{SB}_{j,j}\mathsf{Sq}^{(0)} = \mathsf{Sq}^{(0)}$. Define

$$\mathsf{W}_1^{(0)}(z) := \begin{cases} \begin{bmatrix} 1 & 0 \\ -\mathsf{q}^{(0)*}(z) & 1 \end{bmatrix}, & \text{if } j = 1, \\ \\ \begin{bmatrix} 1 & -\mathsf{q}^{(0)}(z) \\ 0 & 1 \end{bmatrix}, & \text{if } j = 2. \end{cases}$$

In either case, W_1^0 is strongly invertible and $\mathsf{W}_1^{(0)} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with

$$\mathsf{SW}_1^{(0)}(z) = \begin{bmatrix} 1 \\ \mathsf{Sq}^{(0)*}(z) \end{bmatrix} \begin{bmatrix} 1 & \mathsf{Sq}^{(0)}(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathsf{Sr}^{(0)*}(z) \end{bmatrix} \begin{bmatrix} 1 & \mathsf{Sr}^{(0)}(z) \end{bmatrix},$$

for some $\mathsf{B}_{1,1}^{(1)}, \mathsf{B}_{2,2}^{(1)} \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{SB}_{1,1}^{(1)}(z) = \mathsf{SB}_{2,2}^{(1)}(z) = 1$. Furthermore, we have

$$\mathsf{B}^{(1)}(z) := \mathsf{W}_1^{(0)}(z)\mathsf{B}^{(0)}(z)\mathsf{W}_1^{(0)\mathsf{H}}(z) = \begin{bmatrix} \mathsf{B}_{1,1}^{(1)}(z) & \mathsf{r}^{(0)}(z) \\ \mathsf{r}^{(0)*}(z) & \mathsf{B}_{2,2}^{(1)}(z) \end{bmatrix}. \tag{27}$$

Therefore, $\mathrm{len}(\mathsf{B}_{1,2}^{(1)}) < \mathrm{len}(\mathsf{B}_{1,2}^{(0)})$, $\mathsf{B}^{(1)} \in \mathrm{HS}_{\mathbb{F},2}$ has symmetry type $\mathsf{SB}^{(1)} = \mathsf{SW}_1^{(0)}$ and thus all multiplications in (27) are compatible. This proves the claim.

If the matrix $\mathsf{B}^{(1)}$ has one zero entry, then we just let $\mathsf{W}_1 := \mathsf{W}_1^{(0)}$, $\tilde{\mathsf{B}} := \mathsf{B}^{(1)}$ and go to step 2. Otherwise, apply the process in the above claim. To be specific, for $j \in \mathbb{N}$, if $\mathsf{B}^{(j)}$ has no zero entry, then construct $\mathsf{W}_1^{(j)} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ such that $\mathsf{W}_1^{(j)}$ is strongly invertible, all multiplications in $\mathsf{B}^{(j+1)} := \mathsf{W}_1^{(j)}\mathsf{B}^{(j)}\mathsf{W}_1{}^{(j)\mathsf{H}}$ are compatible, and $\mathrm{len}(\mathsf{B}_{1,2}^{(j+1)}) < \mathrm{len}(\mathsf{B}_{1,2}^{(j)})$. By performing this iterative process, we obtain a sequence of matrices $\mathsf{B}^{(0)}, \mathsf{B}^{(1)}, \mathsf{B}^{(2)}, \ldots$ such that $\mathrm{len}(\mathsf{B}_{1,2}^{(0)}) > \mathrm{len}(\mathsf{B}_{1,2}^{(1)}) > \mathrm{len}(\mathsf{B}_{1,2}^{(2)}) > \ldots$. Hence, the sequence must have finite length; that is, the iterative process terminates at some $j = N$, and $\mathsf{B}^{(N)}$ has one zero entry. By letting

$$\mathsf{W}_1 := \mathsf{W}_1^{(N-1)} \ldots \mathsf{W}_1^{(1)}\mathsf{W}_1^{(0)}, \quad \tilde{\mathsf{B}} := \mathsf{W}_1\mathsf{B}\mathsf{W}_1,$$

we have $\mathsf{W}_1 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ and is strongly invertible, $\tilde{\mathsf{B}} = \mathsf{B}^{(N)} \in \mathrm{HS}_{\mathbb{F},2}$ and has one zero entry, and all multiplications above are compatible. This justifies (SS1).

Once we have obtained $\tilde{\mathsf{B}} \in \mathrm{HS}_{\mathbb{F},2}$ that has one zero entry, we move on to (SS2). Here, we consider three cases and define W_2 for each.

- If $\tilde{\mathsf{B}}_{1,1} = 0$, define $\mathsf{W}_2 := \begin{bmatrix} \tilde{\mathsf{B}}_{1,2} & \tilde{\mathsf{B}}_{1,2} \\ (\tilde{\mathsf{B}}_{2,2}+1)/2 & (\tilde{\mathsf{B}}_{2,2}-1)/2 \end{bmatrix}$. Then, $\mathsf{SW}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with $\mathsf{SW}_2 = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}_{1,2}^* \end{bmatrix} \begin{bmatrix} \mathsf{S}\tilde{\mathsf{B}}_{1,2} & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}$ and (26) holds with $c_1 = c_2 = 1$.
- If $\tilde{\mathsf{B}}_{1,2} = 0$, then $\tilde{\mathsf{B}}_{1,2}^* = 0$ and thus $\tilde{\mathsf{B}}$ is a diagonal matrix. Note that $\det(\tilde{\mathsf{B}})$ is a non-positive constant and $\tilde{\mathsf{B}} = \tilde{\mathsf{B}}^{\mathsf{H}}$. It follows that $\tilde{\mathsf{B}}_{1,1} = \lambda_1$ and $\tilde{\mathsf{B}}_{2,2} = \lambda_2$ for some constants $\lambda_1, \lambda_2 \in \mathbb{F} \cap \mathbb{R}$ with $\lambda_1\lambda_2 = \det(\tilde{\mathsf{B}}) \leqslant 0$. Then,
  - if $\lambda_1 \leqslant 0$ and $\lambda_2 \geqslant 0$, define $\mathsf{W}_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $\mathsf{W}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with $\mathsf{SW}_2 = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}_{1,2}^* \end{bmatrix} \begin{bmatrix} \mathsf{S}\tilde{\mathsf{B}}_{1,2} & 1 \end{bmatrix}$ and (26) holds with $c_1 = \sqrt{\lambda_2}$ and $c_2 = \sqrt{-\lambda_1}$.
  - if $\lambda_1 \geqslant 0$ and $\lambda_2 \leqslant 0$, define $\mathsf{W}_2 := I_2$, we have $\mathsf{W}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with $\mathsf{SW}_2 = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}_{1,2}^* \end{bmatrix} \begin{bmatrix} 1 & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}$ and (26) holds with $c_1 = \sqrt{\lambda_1}$, $c_2 = \sqrt{-\lambda_2}$..

- If $\tilde{\mathsf{B}}_{2,2} = 0$, define $\mathsf{W}_2 := \begin{bmatrix} (\tilde{\mathsf{B}}_{1,1}+1)/2 & (\tilde{\mathsf{B}}_{1,1}-1)/2 \\ \tilde{\mathsf{B}}^*_{1,2} & \tilde{\mathsf{B}}^*_{1,2} \end{bmatrix}$. Then, $\mathsf{W}_2\mathrm{Diag}(1,-1)\mathsf{W}_2^{\mathsf{H}} = \tilde{\mathsf{B}}$ and $\mathsf{S}\mathsf{W}_2 \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with $\mathsf{S}\mathsf{W}_2 = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}^*_{1,2} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$ and (26) holds with $c_1 = c_2 = 1$.

Define $\mathsf{U}_3 := \mathsf{W}_1^{-1}\mathsf{W}_2$ and $\mathsf{D} := \mathrm{Diag}(c_1, c_2)$. It is trivial that $\mathsf{B} = \mathsf{U}_3\mathsf{D}\mathrm{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}_3^{\mathsf{H}}$ holds. Moreover, as the multiplications in $\tilde{\mathsf{B}} = \mathsf{W}_1\mathsf{B}\mathsf{W}_1^{\mathsf{H}}$ are compatible and $\mathsf{S}\tilde{\mathsf{B}} = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}^*_{1,2} \end{bmatrix} \begin{bmatrix} 1 & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}$, we see that the symmetry type of W_1 can be written as $\mathsf{S}\mathsf{W}_1 = \begin{bmatrix} 1 \\ \mathsf{S}\tilde{\mathsf{B}}^*_{1,2} \end{bmatrix} \mathsf{S}\boldsymbol{\eta}_1$ for some 1×2 vector $\boldsymbol{\eta}_1$ of Laurent polynomials in $\mathrm{Sym}_{\mathbb{F}}$. Therefore, by (13), we have $\mathsf{S}[\mathsf{W}_1^{-1}] = \mathsf{S}\boldsymbol{\eta}_1^* \begin{bmatrix} 1 & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}$, which implies that $\mathsf{W} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ with symmetry type

$$\mathsf{S}\mathsf{U}_3 = \mathsf{S}\boldsymbol{\eta}_1^*\mathsf{S}\boldsymbol{\eta}_2 \text{ for some } \mathsf{S}\boldsymbol{\eta}_2 \in \left\{ \begin{bmatrix} \mathsf{S}\tilde{\mathsf{B}}_{1,2} & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}, \begin{bmatrix} \mathsf{S}\tilde{\mathsf{B}}_{1,2} & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathsf{S}\tilde{\mathsf{B}}_{1,2} \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right\}.$$

Finally, observe that any 2×2 constant diagonal matrix has symmetry type $\mathsf{S}\boldsymbol{\eta}_2^*\mathsf{S}\boldsymbol{\eta}_2$ for any $\mathsf{S}\boldsymbol{\eta}_2$ in the set above. This shows that all multiplications in $\mathsf{B} = \mathsf{U}_3\mathsf{D}\mathrm{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}_3^{\mathsf{H}}$ are compatible and finishes the justification of (SS2). □

For case 2, the proof is much more tedious than case 1.

Justification of Step 3 in the Proof of Theorem 1, Case 2. First, consider the case when $\det(\mathsf{B}) = 0$. Since $\mathsf{S}\mathsf{B}_{1,2}(z) = -z^{c_b}$ with $c_b \in 2\mathbb{Z}$, it follows from Lemma 1 that $\mathrm{prt}(\mathsf{Z}(\mathsf{B}_{1,2}, 1)), \mathrm{prt}(\mathsf{Z}(\mathsf{B}_{1,2}, -1)) \in 2\mathbb{Z}+1$. This means $\mathsf{B}_{1,2}$ has a factor $(1-z)$. Furthermore, because $\det(\mathsf{B}) = \mathsf{B}_{1,1}\mathsf{B}_{2,2} - \mathsf{B}_{1,2}\mathsf{B}^*_{1,2}$ and $\mathsf{S}\mathsf{B}_{1,1} = \mathsf{S}\mathsf{B}_{2,2} = 1$, we see that $(1-z)(1-z^{-1})$ must divide $\mathsf{B}_{1,1}$ or $\mathsf{B}_{2,2}$. Therefore, there exists $\mathring{\mathsf{B}} \in \mathrm{HS}_{\mathbb{F},2}$ such that $\mathsf{B} = \mathsf{W}_1\mathring{\mathsf{B}}\mathsf{W}_1^{\mathsf{H}}$, where $\mathsf{W}_1 = \mathrm{Diag}(1-z, 1)$ or $\mathrm{Diag}(1, 1-z^{-1})$. Furthermore, $\det(\mathring{\mathsf{B}}) = 0$ and $\mathsf{S}\mathring{\mathsf{B}}_{1,2}(z) = z^{c_b-1}$. Therefore, $\mathring{\mathsf{B}}$ satisfies all assumptions of case 1, and all claims follow.

Now assume that $\det(\mathsf{B}) = -|\mathsf{C}|^2 \neq 0$. Observe that $\mathsf{B}_{1,1} \neq 0$, as otherwise $-|\mathsf{C}|^2 = \det(\mathsf{B}) = -\mathsf{B}_{1,2}\mathsf{B}^*_{1,2}$ and thus $\mathsf{B}_{1,2}$ is a non-zero monomial, which can never have symmetry type $-z^{c_b}$ with $c_b \in 2\mathbb{Z}$. Now, we see from $\mathsf{S}\mathsf{B}_{1,1} = 1$ that $-\mathrm{ldeg}(\mathsf{B}_{1,1}) = \deg(\mathsf{B}_{1,1}) := m$ for some $m \in \mathbb{N}_0$. Let $\mathsf{Q}_1(z) := \sum_{j=0}^{m} Q_1[j]z^j$ and $\mathsf{Q}_2(z) := \sum_{j=0}^{m} Q_2[j]z^j$ be two Laurent polynomials in $\mathbb{F}[z, z^{-1}]$, where $Q_1[0], \ldots, Q_1[m], Q_2[0], \ldots, Q_2[m] \in \mathbb{F}$ are some unknown parameters. Define $k := \frac{c_b}{2}$. Using long division, there are $\mathsf{Q}_3, \mathsf{Q}_4 \in \mathbb{F}[z, z^{-1}]$ such that

$$\mathsf{B}^*_{1,2}(z)\mathsf{Q}_1(z) + z^{m-k}\mathsf{Q}^*_2(z) = \mathsf{B}_{1,1}(z)\mathsf{Q}_3(z) + \mathsf{Q}_4(z), \tag{28}$$

and $-m \leqslant \mathrm{ldeg}(\mathsf{Q}_4) \leqslant \deg(\mathsf{Q}_4) \leqslant m$. By setting $\mathsf{Q}_4 = 0$, (28) induces a system of $2m+1$ homogeneous linear equations with $2m+2$ variables. Therefore, the set $S := \{(Q_1[0], \ldots, Q_1[m], Q_2[0], \ldots, Q_2[m]) : (28) \text{ holds with } \mathsf{Q}_4 = 0\} \subseteq \mathbb{F}^{2m+2}$ has a non-zero element, say $(q_1[0], \ldots, q_1[m], q_2[0], \ldots, q_2[m])$. By letting

$$\tilde{\mathsf{W}}_{1,1}(z) := \sum_{j=0}^{m} q_1[j]z^j, \quad \tilde{\mathsf{W}}_{1,2}(z) := \sum_{j=0}^{m} q_2[j]z^j,$$

we have

$$\mathsf{B}^*_{1,2}(z)\tilde{\mathsf{W}}_{1,1}(z) + z^{m-k}\tilde{\mathsf{W}}^*_{1,2}(z) = \mathsf{B}_{1,1}(z)\tilde{\mathsf{W}}_{2,1}(z), \tag{29}$$

for some $\tilde{\mathsf{W}}_{2,1}(z) \in \mathbb{F}[z, z^{-1}]$. By $\mathsf{S}\mathsf{B}_{1,1}(z) = 1$ and $\mathsf{S}\mathsf{B}^*_{1,2}(z) = -z^{-2k}$, we deduce from (29) that

$$-z^m\mathsf{B}^*_{1,2}(z)\tilde{\mathsf{W}}_{1,1}(z^{-1}) + z^{-k}\tilde{\mathsf{W}}^*_{1,2}(z^{-1}) = z^{m-2k}\mathsf{B}_{1,1}(z)\tilde{\mathsf{W}}_{2,1}(z^{-1}). \tag{30}$$

Define

$$\mathring{\mathsf{W}}_{1,1}(z) := \begin{cases} \tilde{\mathsf{W}}_{1,1}(z), & \text{if } \mathsf{S}\tilde{\mathsf{W}}_{1,1}(z) = z^m \text{ and } \mathsf{S}\tilde{\mathsf{W}}_{1,2}(z) = -z^m, \\ \dfrac{\tilde{\mathsf{W}}_{1,1}(z) - z^m\tilde{\mathsf{W}}_{1,1}(z^{-1})}{2}, & \text{otherwise,} \end{cases}$$

$$\mathring{W}_{1,2}(z) := \begin{cases} \tilde{W}_{1,2}(z), & \text{if } S\tilde{W}_{1,1}(z) = z^m \text{ and } S\tilde{W}_{1,2}(z) = -z^m, \\ \dfrac{\tilde{W}_{1,2}(z) + z^m \tilde{W}_{1,2}(z^{-1})}{2}, & \text{otherwise,} \end{cases}$$

$$\mathring{W}_{2,1}(z) := \begin{cases} \tilde{W}_{2,1}(z), & \text{if } S\tilde{W}_{1,1}(z) = z^m \text{ and } S\tilde{W}_{1,2}(z) = -z^m, \\ \dfrac{\tilde{W}_{2,1}(z) + z^{m-2k} \tilde{W}_{2,1}(z^{-1})}{2} & \text{otherwise.} \end{cases}$$

Then, $\mathring{W}_{1,1}, \mathring{W}_{1,2}, \mathring{W}_{2,1} \in \mathrm{Sym}_{\mathbb{F}}$ with

$$S\mathring{W}_{1,1}(z) = -\epsilon_w z^m, \quad S\mathring{W}_{1,2}(z) = \epsilon_w z^m, \quad S\mathring{W}_{2,1}(z) = \epsilon_w z^{m-2k},$$

for some $\epsilon_w \in \{-1, 1\}$. Moreover, $\mathring{W}_{1,1}$ and $\mathring{W}_{1,2}$ are not both 0 and satisfy

$$B^*_{1,2}(z)\mathring{W}_{1,1}(z) + z^{m-k}\mathring{W}^*_{1,2}(z) = B_{1,1}(z)\mathring{W}_{2,1}(z). \tag{31}$$

It follows that

$$\begin{aligned} & B_{1,2}(z)[B^*_{1,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}\mathring{W}^*_{1,1}(z)] \\ =& [B_{1,1}(z)B_{2,2}(z) + |C|^2]\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}\mathring{W}^*_{1,1}(z) \\ =& B_{1,1}(z)B_{2,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}[z^{k-m}\mathring{W}_{1,2}(z) + B_{1,2}(z)\mathring{W}^*_{1,1}(z)] \\ =& B_{1,1}(z)B_{2,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k} B^*_{1,1}(z)\mathring{W}^*_{2,1}(z) \\ =& B_{1,1}(z)[B_{2,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}\mathring{W}^*_{2,1}(z)]. \end{aligned}$$

Since $\det(B)$ is a non-zero constant, we have $\gcd(B_{1,1}, B_{1,2}) = 1$. Thus, we conclude from the above calculation that $B_{1,1}(z)$ divides $B^*_{1,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}\mathring{W}^*_{1,1}(z)$. This means there exists $\mathring{W}_{2,2} \in \mathrm{Sym}_{\mathbb{F}}$ with $S\mathring{W}_{2,2}(z) = -\epsilon_w z^{m-2k}$ such that

$$B^*_{1,2}(z)\mathring{W}_{1,2}(z) + |C|^2 z^{m-k}\mathring{W}^*_{1,1}(z) = B_{1,1}(z)\mathring{W}_{2,2}(z). \tag{32}$$

Combining (31) and (32), we obtain the following technical identity:

$$\begin{bmatrix} \mathring{W}_{2,2}(z) & -\mathring{W}_{1,2}(z) \\ -\mathring{W}_{2,1}(z) & \mathring{W}_{1,1}(z) \end{bmatrix} \begin{bmatrix} B_{1,1}(z) \\ B^*_{1,2}(z) \end{bmatrix} = z^{m-k} \begin{bmatrix} |C|^2\mathring{W}^*_{1,1}(z) \\ -\mathring{W}^*_{1,2}(z) \end{bmatrix}. \tag{33}$$

With (33), we can now try to write all entries of B in terms of $\mathring{W}_{1,1}, \mathring{W}_{1,2}, \mathring{W}_{2,1}$ and $\mathring{W}_{2,2}$:

- $B_{1,1}$ and $B_{1,2}$: By left multiplying $\begin{bmatrix} \mathring{W}^*_{1,2} & |C|^2\mathring{W}^*_{1,1} \end{bmatrix}$ to both sides of (33), we have

$$[|C|^2\mathring{W}^*_{1,1}\mathring{W}_{1,1} - \mathring{W}^*_{1,2}\mathring{W}_{1,2}]B^*_{1,2} = [|C|^2\mathring{W}^*_{1,1}\mathring{W}_{2,1} - \mathring{W}^*_{1,2}\mathring{W}_{2,2}]B_{1,1}. \tag{34}$$

  As $\gcd(B_{1,1}, B_{1,2}) = 1$, we have $B_{1,1}$ divides $|C|^2\mathring{W}^*_{1,1}\mathring{W}_{1,1} - \mathring{W}^*_{1,2}\mathring{W}_{1,2}$. Recall that $0 \leqslant \mathrm{ldeg}(\tilde{W}_{1,1}) \leqslant \deg(\tilde{W}_{1,1}) \leqslant m$, it follows from the definition of $\mathring{W}_{1,1}$ that $0 \leqslant \mathrm{ldeg}(\mathring{W}_{1,1}) \leqslant \deg(\mathring{W}_{1,1}) \leqslant m$. Similarly $0 \leqslant \mathrm{ldeg}(\mathring{W}_{1,2}) \leqslant \deg(\mathring{W}_{1,2}) \leqslant m$. Hence,

$$-m \leqslant \mathrm{ldeg}(|C|^2\mathring{W}_{1,1}\mathring{W}^*_{1,1} - \mathring{W}_{1,2}\mathring{W}^*_{1,2}) \leqslant \deg(|C|^2\mathring{W}_{1,1}\mathring{W}^*_{1,1} - \mathring{W}_{1,2}\mathring{W}^*_{1,2}) \leqslant m.$$

  Moreover, it is easy to see from the symmetry types of $\mathring{W}_{1,1}$ and $\mathring{W}_{1,2}$ that $Z(\mathring{W}_{1,1}, 1) \neq Z(\mathring{W}_{1,2}, 1)$. Therefore, $Z(\mathring{W}_{1,1}\mathring{W}^*_{1,1}, 1) \neq Z(\mathring{W}_{1,2}\mathring{W}^*_{1,2}, 1)$ and $|C|^2\mathring{W}_{1,1}\mathring{W}^*_{1,1} - \mathring{W}_{1,2}\mathring{W}^*_{1,21} \neq 0$. Consequently, we must have

$$B_{1,1}(z) = \alpha[|C|^2\mathring{W}_{1,1}(z)\mathring{W}^*_{1,1}(z) - \mathring{W}_{1,2}(z)\mathring{W}^*_{1,2}(z)] \tag{35}$$

for some $\alpha \in \mathbb{F} \cap \mathbb{R}$ with $\alpha \neq 0$, and it follows from (34) that

$$\mathsf{B}_{1,2}(z) = \alpha[|\mathsf{C}|^2 \mathring{\mathsf{W}}_{1,1}(z)\mathring{\mathsf{W}}_{2,1}^*(z) - \mathring{\mathsf{W}}_{1,2}(z)\mathring{\mathsf{W}}_{2,2}^*(z)]. \tag{36}$$

- By left multiplying $\begin{bmatrix}\mathring{\mathsf{W}}_{1,1} & \mathring{\mathsf{W}}_{1,2}\end{bmatrix}$ to both sides of (33), direct calculation and (33) together yield

$$\det(\mathring{\mathsf{W}}(z))\mathsf{B}_{1,1}(z) = \alpha^{-1} z^{m-k} \mathsf{B}_{1,1}(z), \tag{37}$$

where

$$\mathring{\mathsf{W}}(z) := \begin{bmatrix}\mathring{\mathsf{W}}_{1,1}(z) & \mathring{\mathsf{W}}_{1,2}(z) \\ \mathring{\mathsf{W}}_{2,1}(z) & \mathring{\mathsf{W}}_{2,2}(z)\end{bmatrix}.$$

Thus, $\det(\mathring{\mathsf{W}}(z)) = \alpha^{-1} z^{m-k}$.

Next, by left multiplying $\begin{bmatrix}\mathring{\mathsf{W}}_{2,2}^* & |\mathsf{C}|^2\mathring{\mathsf{W}}_{2,1}^*\end{bmatrix}$ to both sides of (33) and using (36), we have

$$\begin{aligned}&[\mathring{\mathsf{W}}_{2,2}^*(z)\mathring{\mathsf{W}}_{2,2}(z) - |\mathsf{C}|^2\mathring{\mathsf{W}}_{2,1}^*(z)\mathring{\mathsf{W}}_{2,1}(z)]\mathsf{B}_{1,1}(z) + \alpha^{-1}\mathsf{B}_{1,2}^*(z)\mathsf{B}_{1,2}(z) \\ =&|\mathsf{C}|^2 z^{m-k}\det(\mathring{\mathsf{W}}(z))^* = -\alpha^{-1}\det(\mathsf{B}(z)).\end{aligned}$$

It follows that

$$\mathsf{B}_{2,2}(z) = \alpha[|\mathsf{C}|^2 \mathring{\mathsf{W}}_{2,1}^*(z)\mathring{\mathsf{W}}_{2,1}(z) - \mathring{\mathsf{W}}_{2,2}^*(z)\mathring{\mathsf{W}}_{2,2}(z)]. \tag{38}$$

Therefore, it follows from (35), (36) and (38) that

$$\mathsf{B}(z) = \mathring{\mathsf{W}}(z)\mathrm{Diag}(\alpha|\mathsf{C}|^2, -\alpha)\mathring{\mathsf{W}}^{\mathsf{H}}(z). \tag{39}$$

Finally, define

$$\mathsf{U}_3 := \begin{cases}\mathring{\mathsf{W}}, & \text{if } \alpha > 0, \\ \mathring{\mathsf{W}}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}, & \text{if } \alpha < 0,\end{cases} \qquad \mathsf{D} := \begin{cases}\mathrm{Diag}(\sqrt{\alpha}|\mathsf{C}|, \sqrt{\alpha}) & \text{if } \alpha > 0, \\ \mathrm{Diag}(\sqrt{-\alpha}, \sqrt{-\alpha}|\mathsf{C}|) & \text{if } \alpha < 0.\end{cases}$$

It is trivial that $\mathsf{B} = \mathsf{U}_3\mathsf{D}\mathrm{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}_3^{\mathsf{H}}$. Furthermore,

- if $\alpha > 0$, then

$$\mathsf{SU}_3(z) = \begin{bmatrix}1 \\ -z^{-2k}\end{bmatrix}\begin{bmatrix}-\epsilon_w z^m & \epsilon_w z^m\end{bmatrix},$$

$$\mathsf{SD}(z) = \mathsf{S}[\mathrm{Diag}(1,-1)](z) = \begin{bmatrix}-\epsilon_w z^{-m} \\ \epsilon_w z^{-m}\end{bmatrix}\begin{bmatrix}-\epsilon_w z^m & \epsilon_w z^m\end{bmatrix},$$

- if $\alpha < 0$, then

$$\mathsf{SU}_3(z) = \begin{bmatrix}1 \\ -z^{-2k}\end{bmatrix}\begin{bmatrix}\epsilon_w z^m & -\epsilon_w z^m\end{bmatrix},$$

$$\mathsf{SD}(z) = \mathsf{S}[\mathrm{Diag}(1,-1)](z) = \begin{bmatrix}\epsilon_w z^{-m} \\ -\epsilon_w z^{-m}\end{bmatrix}\begin{bmatrix}\epsilon_w z^m & -\epsilon_w z^m\end{bmatrix}.$$

In either case, all multiplications are compatible. This completes the justification. □

3. Quasi-Tight Framelets with Symmetry

In this section, we apply the main result Theorem 1 to construct quasi-tight framelets with symmetry.

3.1. Basics on Framelets

We first recall some basic concepts and facts about framelets.

For $p \in [1, \infty)$, denote $L_p(\mathbb{R})$, the linear space of functions f that satisfies

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

When $p = 2$, the space $L_2(\mathbb{R})$ is a Hilbert space with the following inner product:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} dx, \quad \forall f, g \in L_2(\mathbb{R}).$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a function. For any real number c and any positive number λ, define

$$f_{\lambda;c}(x) := \sqrt{\lambda} f(\lambda x - c), \quad \forall x \in \mathbb{R}.$$

For $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, we say that $\{\phi; \psi^1, \ldots, \psi^s\}$ is a *framelet* (i.e., a wavelet frame) in $L_2(\mathbb{R})$ if there exist positive constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|^2_{L_2(\mathbb{R})} \leqslant \sum_{k \in \mathbb{Z}} |\langle f, \phi_{1;k} \rangle|^2 + \sum_{\ell=1}^{s} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} |\langle f, \psi^{\ell}_{2^j;k} \rangle|^2 \leqslant C_2 \|f\|^2_{L_2(\mathbb{R})}, \qquad \forall\, f \in L_2(\mathbb{R}).$$

For $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$ and $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$, we say that $\{\phi; \psi^1, \ldots, \psi^s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is *a quasi-tight framelet* in $L_2(\mathbb{R})$ if $\{\phi; \psi^1, \ldots, \psi^s\}$ is a framelet in $L_2(\mathbb{R})$ and

$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{1;k} \rangle \phi_{1;k} + \sum_{\ell=1}^{s} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \epsilon_\ell \langle f, \psi^{\ell}_{2^j;k} \rangle \psi^{\ell}_{2^j;k}, \qquad \forall\, f \in L_2(\mathbb{R}), \tag{40}$$

with the above series converging unconditionally in $L_2(\mathbb{R})$. It is well-known that (40) implies that $\{\psi^1, \ldots, \psi^s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is *a homogeneous quasi-tight framelet* in $L_2(\mathbb{R})$, that is,

$$f = \sum_{\ell=1}^{s} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \epsilon_\ell \langle f, \psi^{\ell}_{2^j;k} \rangle \psi^{\ell}_{2^j;k}, \qquad \forall\, f \in L_2(\mathbb{R}), \tag{41}$$

with the above series converging unconditionally in $L_2(\mathbb{R})$. If in addition, $\epsilon_1 = \cdots = \epsilon_s = 1$, then $\{\phi; \psi^1, \ldots, \psi^s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is called *a tight framelet* in $L_2(\mathbb{R})$. In this case, $\{\psi^1, \ldots, \psi^s\}$ is *a homogeneous tight framelet* in $L_2(\mathbb{R})$.

To reduce computational complexity in applications, we hope the framelet functions are compactly supported, and such framelets are often derived from compactly supported *refinable functions* from *extension principles*. To better discuss this in more detail, we introduce some notations here. First, for any integrable function $f \in L_1(\mathbb{R})$, recall that its *Fourier transform* is defined via $\int_{\mathbb{R}} f(x) e^{-ix\xi} dx$. The definition of the Fourier transform can be extended to $L_2(\mathbb{R})$ functions and tempered distributions. Next, denote $l_0(\mathbb{Z})$, the set (space) of all finitely supported sequences $u : \mathbb{Z} \to \mathbb{C}$; that is, every $u = \{u[k]\}_{k \in \mathbb{Z}} \in l_0(\mathbb{Z})$ only has finitely many non-zero terms. For every $u \in l_0(\mathbb{Z})$, its *symbol* is the Laurent polynomial u that is defined by

$$\mathsf{u}(z) := \sum_{k \in \mathbb{Z}} u[k] z^k, \quad \forall z \in \mathbb{C} \backslash \{0\}.$$

Suppose we have a finitely supported filter $a \in l_0(\mathbb{Z})$ such that $\mathsf{a}(1) = 1$. It is well-known in wavelet and framelet theory that one can define a compactly supported distribution ϕ through

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(e^{-i2^{-j}\xi}), \qquad \xi \in \mathbb{R}. \tag{42}$$

The distribution ϕ defined as (42) is called the *standard refinable function* associated with the finitely supported filter a, and we call a the *refinement filter* associated with ϕ. It is easy to verify that the following *refinement equation* holds:

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} a[k]\phi(2x-k), \quad \forall x \in \mathbb{R}, \quad \text{or equivalently}, \quad \widehat{\phi}(2\xi) = \mathsf{a}(e^{-i\xi})\widehat{\phi}(\xi), \quad \forall \xi \in \mathbb{R}. \tag{43}$$

To construct a compactly supported quasi-tight framelet, one can apply the unitary extension principle (UEP) (see e.g., [8,23,29,30]). Here, we state the quasi-tight framelet version of the UEP.

Theorem 2. *Let $a \in l_0(\mathbb{Z})$ be such that $\mathsf{a}(1) = 1$ and define the standard refinable function ϕ associated with a as in (42). If*

(1) $\phi \in L_2(\mathbb{R})$;
(2) there exist $\epsilon_1, \dots, \epsilon_s \in \{-1, 1\}$ and finitely supported filters $b_1, \dots, b_s \in l_0(\mathbb{Z})$ such that
- *$\mathsf{b}_1(1) = \cdots = \mathsf{b}_s(1) = 0$, where b_ℓ is the symbol of b_ℓ for all $\ell = 1, \dots, s$;*
- *$\{a; b_1, \dots, b_s\}_{(\epsilon_1,\dots,\epsilon_s)}$ forms a* quasi-tight framelet filter bank*, i.e., the symbols $\mathsf{a}, \mathsf{b}_1, \dots, \mathsf{b}_s$ satisfy*

$$\mathsf{a}^*(z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \epsilon_\ell \mathsf{b}_\ell^*(z)\mathsf{b}_\ell(z) = 1, \qquad z \in \mathbb{C}\backslash\{0\}, \tag{44}$$

$$\mathsf{a}^*(z)\mathsf{a}(-z) + \sum_{\ell=1}^{s} \epsilon_\ell \mathsf{b}_\ell^*(z)\mathsf{b}_\ell(-z) = 0, \qquad z \in \mathbb{C}\backslash\{0\}, \tag{45}$$

then, by letting

$$\psi^\ell := 2\sum_{k\in\mathbb{Z}} b_\ell(k)\phi(2\cdot -k), \tag{46}$$

$\{\phi; \psi^1, \dots, \psi^s\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight framelet in $L_2(\mathbb{R})$.

In a quasi-tight framelet filter bank $\{a; b_1, \dots, b_s\}_{(\epsilon_1,\dots,\epsilon_s)}$, a is called the *low-pass filter* and $b_1, \dots, b_s$ are called the *high-pass filters*.

Let us make some comments on Theorem 2. First, to verify if ϕ defined in (42) is an element of $L_2(\mathbb{R})$, one can check the *L_2-smoothness exponent* of the refinement filter a, denoted by $\operatorname{sm}(a)$. The quantity $\operatorname{sm}(a)$ is technical; we refer the readers to [(5.6.44) [8]] for its definition and [Corollary 5.8.5 [8]] for the method to compute it. According to [Theorem 6.3.3 [8]], if $\operatorname{sm}(a) > 0$, then

$$\|\phi\|^2_{H^\tau(\mathbb{R})} := \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 (1 + |\xi|^2)^\tau d\xi < \infty$$

holds for every $\tau \in [0, \operatorname{sm}(a))$. In other words, ϕ belongs to the *Sobolev space $H^\tau(\mathbb{R})$* and thus $\phi \in L_2(\mathbb{R})$.

Next, we see from Theorem 2 that the construction of a quasi-tight framelet essentially reduces to the design of an underlying quasi-tight framelet filter bank $\{a; b_1, \dots, b_s\}_{(\epsilon_1,\dots,\epsilon_s)}$ that satisfies (44) and (45). Define

$$\mathcal{M}_a(z) := \begin{bmatrix} 1 - \mathsf{a}^*(z)\mathsf{a}(z) & -\mathsf{a}^*(z)\mathsf{a}(-z) \\ -\mathsf{a}^*(-z)\mathsf{a}(z) & 1 - \mathsf{a}^*(-z)\mathsf{a}(-z) \end{bmatrix}. \tag{47}$$

To construct a quasi-tight framelet filter bank, it suffices to find $b = (b_1, \dots, b_s)^\mathsf{T} \in (l_0(\mathbb{Z}))^{s\times 1}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that $\mathsf{b}_1(1) = \cdots = \mathsf{b}_s(1) = 0$ and

$$\mathcal{M}_a(z) = \begin{bmatrix} \mathsf{b}(z) & \mathsf{b}(-z) \end{bmatrix}^* \operatorname{Diag}(\epsilon_1, \dots, \epsilon_s) \begin{bmatrix} \mathsf{b}(z), \mathsf{b}(-z) \end{bmatrix}. \tag{48}$$

When $\epsilon_1 = \cdots = \epsilon_s = 1$, then (48) becomes a spectral factorization of the matrix $\mathcal{M}_a$. In this case, the refinement filter a necessarily satisfies the *sub-QMF condition*:

$$\det(\mathcal{M}_a(z)) = 1 - \mathsf{a}^*(z)\mathsf{a}(z) - \mathsf{a}^*(-z)\mathsf{a}(-z) \geqslant 0, \qquad \forall z \in \mathbb{T}. \tag{49}$$

Unfortunately, many refinement filters fail to satisfy (49). For example, consider the refinement filter $a \in l_0(\mathbb{Z})$ with

$$\mathsf{a}(z) = \frac{1}{16}(-z^2 + 4z + 10 + 4z^{-1} - z^{-2}), \tag{50}$$

we have

$$1 - \mathsf{a}^*(z)\mathsf{a}(z) - \mathsf{a}^*(-z)\mathsf{a}(-z) = -\frac{(z - z^{-1})^4}{128} \leqslant 0, \quad \forall z \in \mathbb{T}.$$

Therefore, constructing a tight framelet is impossible in such cases, and we must work with the quasi-tight framelets. In fact, with the refinement filter a given in (50), we have the first observed example of a quasi-tight framelet filter bank in [Example 3.2.2 [8]].

3.2. Vanishing Moments and Symmetry

Two highly desired features for quasi-tight framelets are high-order vanishing moments and symmetry on the generators. Let us briefly discuss them here.

The orders of *vanishing moments* of the framelet generators $\psi^\ell, \ell = 1, \ldots, s$ reflect the sparsity of the framelet expansion (40). To better understand this, let us introduce some notations. For a finitely supported filter $u \in l_0(\mathbb{Z})$ and $m \in \mathbb{N}_0$, we say that

- u has *order m sum rules* if

$$\mathsf{u}(z) = (1+z)^m \tilde{\mathsf{u}}(z), \quad \forall z \in \mathbb{C} \backslash \{0\}, \tag{51}$$

  for some Laurent polynomial $\tilde{\mathsf{u}}$. Denote $\mathrm{sr}(u) := m$ with m being the largest positive integer such that (51) holds;
- u has *order m vanishing moments* if

$$\mathsf{u}(z) = (1-z)^m \mathsf{v}(z), \quad \forall z \in \mathbb{C} \backslash \{0\}, \tag{52}$$

  for some Laurent polynomial v. Denote $\mathrm{vm}(u) := m$ with m being the largest positive integer such that (52) holds.

For a function $\psi \in L_2(\mathbb{R})$, we say that ψ has *order m vanishing moments* if

$$\widehat{\psi}^{(j)}(0) = 0, \quad \forall j = 0, 1, \ldots, m-1. \tag{53}$$

Denote $\mathrm{vm}(\psi) := m$ with m being the largest positive integer such that (53) holds. Suppose $\{\phi; \psi^1, \ldots, \psi^s\}_{(\epsilon_1, \ldots, \epsilon_s)}$ is a quasi-tight framelet that is derived through the UEP from an underlying quasi-tight framelet filter bank $\{a; b_1, \ldots, b_s\}_{(\epsilon_1, \ldots, \epsilon_s)}$. It is easy to show from (44), (45) and the definition of ψ^ℓ in (46) that

$$\min\{\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)\} = \min\{\mathrm{vm}(\psi^1), \ldots, \mathrm{vm}(\psi^s)\} \leqslant \min\left\{\mathrm{sr}(a), \tfrac{1}{2}\,\mathrm{vm}(u_a)\right\},$$

where $u_a \in l_0(\mathbb{Z})$ is the filter whose symbol is given by

$$\mathsf{u}_a(z) := 1 - \mathsf{a}^*(z)\mathsf{a}(z). \tag{54}$$

We always try to make $n := \min\{\mathrm{vm}(\psi^1), \ldots, \mathrm{vm}(\psi^s)\}$ as large as possible.

On the other hand, the symmetry of the framelet generators is critically important in handling boundary artifacts in many applications. To derive a symmetric quasi-tight framelet, we start with a refinement filter $a \in l_0(\mathbb{Z})$ such that $\mathsf{a}(1) = 1$ and $\mathsf{a} \in \mathrm{Sym}_{\mathbb{F}}$. Then, the matrix $\mathcal{M}_a$ defined in (47) is an element of $\mathrm{HS}_{\mathbb{F},2}$. To construct the high-pass filters $b_1, \ldots, b_s$ with symmetry, we must find a generalized matrix spectral factorization of $\mathcal{M}_a$ as in (48) and ensure that all multiplications preserve the symmetry structure. It is well-known that a framelet filter bank with symmetry necessarily has $s \geqslant 2$ high-pass filters. Quite often, we hope to have fewer high-pass filters in a framelet filter bank to make algorithms using framelet transforms efficient. For the construction of symmetric tight framelet filter banks, we refer the readers to [1,7,10] for the construction of symmetric tight framelet filter banks with $s = 3$ and [6,9] for symmetric tight framelet filter banks with $s = 2$. The construction of symmetric quasi-tight framelets is much less investigated. To the best of the author's knowledge, [2] is the only existing paper that studies symmetric quasi-tight framelets but with coefficients in the field $\mathbb{F} = \mathbb{C}$.

3.3. The Main Theorem

From the previous discussion, we see that the smaller the number of generators is, the better. Therefore, we focus on symmetric quasi-tight framelet filter banks with two generators. Specifically, we work on characterizing and designing a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$ such that $\mathsf{a} \in \mathrm{Sym}_{\mathbb{F}}$, $\mathsf{b}_j = c_j \breve{\mathsf{b}}_j$ for some $\breve{\mathsf{b}}_1, \breve{\mathsf{b}}_2 \in \mathrm{Sym}_{\mathbb{F}}$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2, |c_2|^2 \in \mathbb{F}$. Moreover, we want our high-pass filters to have the highest possible order of vanishing moments. The construction of such a quasi-tight framelet filter bank will directly apply Theorem 1.

Before we present our main theorem, let us provide a brief guideline on constructing a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$. Let $a \in l_0(\mathbb{Z})$ be such that $\mathsf{a}(1) = 1$ and $\mathsf{a} \in \mathbb{F}[z, z^{-1}]$. Define $\mathcal{M}_a$ as in (47). To derive a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$ with high-order vanishing moments, it suffices to find $\mathsf{b} := (b_1, b_2)^{\mathsf{T}} \in (l_0(\mathbb{Z}))^{2\times 1}$ such that

$$\mathcal{M}_a(z) = \begin{bmatrix} \mathsf{b}(z) & \mathsf{b}(-z) \end{bmatrix}^{\mathsf{H}} \mathrm{Diag}(1, -1) \begin{bmatrix} \mathsf{b}(z) & \mathsf{b}(-z) \end{bmatrix}, \tag{55}$$

and $\min\{\mathrm{vm}(b_1), \mathrm{vm}(b_2)\}$ is as large as possible. Choose $n \in \mathbb{N}$ with $n \leqslant \min\{\mathrm{sr}(a), \frac{1}{2}\mathrm{vm}(u_a)\}$, where $u_a \in l_0(\mathbb{Z})$ is defined as in (54). Define

$$\mathcal{M}_{a,n}(z) := \begin{bmatrix} \dfrac{1 - \mathsf{a}^*(z)\mathsf{a}(z)}{(1 - z^{-1})^n (1 - z)^n} & \dfrac{-\mathsf{a}^*(z)\mathsf{a}(-z)}{(1 - z^{-1})^n (1 + z)^n} \\ \dfrac{-\mathsf{a}^*(-z)\mathsf{a}(z)}{(1 + z^{-1})^n (1 - z)^n} & \dfrac{1 - \mathsf{a}^*(-z)\mathsf{a}(-z)}{(1 + z^{-1})^n (1 + z)^n} \end{bmatrix}. \tag{56}$$

By the choice of n, it is easy to see that all entries of $\mathcal{M}_{a,n}$ are Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ and $\mathcal{M}_{a,n}^{\mathsf{H}} = \mathcal{M}_{a,n}$. Define

$$\tilde{\mathsf{a}}(z) := 1 - \mathsf{a}^*(z)\mathsf{a}(z), \quad \tilde{\mathsf{b}}(z) := -\mathsf{a}^*(z)\mathsf{a}(-z), \tag{57}$$

$$\mathring{\mathsf{a}}(z) := \frac{\tilde{\mathsf{a}}(z)}{(1 - z^{-1})^n (1 - z)^n}, \quad \mathring{\mathsf{b}}(z) := \frac{\tilde{\mathsf{b}}(z)}{(1 - z^{-1})^n (1 + z)^n}. \tag{58}$$

Then, we have $\mathcal{M}_a(z) := \begin{bmatrix} \tilde{\mathsf{a}}(z) & \tilde{\mathsf{b}}(z) \\ \tilde{\mathsf{b}}(-z) & \tilde{\mathsf{a}}(-z) \end{bmatrix}$ and $\mathcal{M}_{a,n}(z) := \begin{bmatrix} \mathring{\mathsf{a}}(z) & \mathring{\mathsf{b}}(z) \\ \mathring{\mathsf{b}}(-z) & \mathring{\mathsf{a}}(-z) \end{bmatrix}$, where $\mathcal{M}_a$ is defined as in (47). Note that

$$\det(\mathcal{M}_a(z)) = \tilde{\mathsf{a}}(z) + \tilde{\mathsf{a}}(-z) - 1,$$

from which it is not hard to see that $\gcd(\tilde{\mathsf{a}}, \tilde{\mathsf{b}}, \tilde{\mathsf{b}}(-\cdot), \tilde{\mathsf{a}}(-\cdot)) = 1$. Therefore, we conclude that $\gcd(\mathring{\mathsf{a}}, \mathring{\mathsf{b}}, \mathring{\mathsf{b}}(-\cdot), \mathring{\mathsf{a}}(-\cdot)) = 1$.

For any Laurent polynomial $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k) z^k \in \mathbb{F}[z, z^{-1}]$, define $\mathsf{u}^{[0]}, \mathsf{u}^{[1]} \in \mathbb{F}[z, z^{-1}]$ via

$$\mathsf{u}^{[0]}(z) := \sum_{k\in\mathbb{Z}} u(2k) z^k, \quad \mathsf{u}^{[1]}(z) := \sum_{k\in\mathbb{Z}} u(2k+1) z^k.$$

It is trivial to see that

$$\mathsf{u}^{[0]}(z^2) := \frac{\mathsf{u}(z) + \mathsf{u}(-z)}{2}, \quad \mathsf{u}^{[1]}(z^2) := \frac{\mathsf{u}(z) - \mathsf{u}(-z)}{2z}. \tag{59}$$

Define

$$\mathcal{N}_{a,n}(z) = \frac{1}{2}\begin{bmatrix} \mathring{\mathsf{a}}^{[0]}(z) + \mathring{\mathsf{b}}^{[0]}(z) & \mathring{\mathsf{a}}^{[1]}(z) - \mathring{\mathsf{b}}^{[1]}(z) \\ z\left[\mathring{\mathsf{a}}^{[1]}(z) + \mathring{\mathsf{b}}^{[1]}(z)\right] & \mathring{\mathsf{a}}^{[0]}(z) - \mathring{\mathsf{b}}^{[0]}(z) \end{bmatrix} =: \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^*(z) & \mathsf{q}(z) \end{bmatrix}. \tag{60}$$

It is easy to verify that

$$\mathcal{N}_{a,n}(z^2) := \frac{1}{4}\mathsf{F}(z)\mathcal{M}_{a,n}(z)\mathsf{F}(z)^{\mathsf{H}}, \quad \text{where } \mathsf{F}(z) := \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}. \tag{61}$$

Now, to construct a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$, it suffices to find a matrix $\mathsf{U}(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$ of Laurent polynomials in $\mathbb{F}[z, z^{-1}]$ and a diagonal matrix $\mathsf{D} = \mathrm{Diag}(c_1, c_2)$ for some $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2, |c_2|^2 \in \mathbb{F}$ such that $\mathcal{N}_{a,n} = \mathsf{U}\mathsf{D}\mathrm{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}$. Once this is done, define $b := (b_1, b_2)^{\mathsf{T}} \in (l_0(\mathbb{Z}))^{2\times 1}$ via

$$\mathsf{b}_1(z) := c_1(1-z)^n[\mathsf{U}_{1,1}^*(z^2) + z\mathsf{U}_{2,1}^*(z^2)], \quad \mathsf{b}_2(z) := c_2(1-z)^n[\mathsf{U}_{1,2}^*(z^2) + z\mathsf{U}_{2,2}^*(z^2)], \tag{62}$$

then it is easy to verify that $\{a; b_1, b_2\}_{(1,-1)}$ is a quasi-tight framelet filter bank such that $\min\{\mathrm{vm}(b_1), \mathrm{vm}(b_2)\} \geqslant n$. Moreover, by letting

$$\breve{\mathsf{b}}_1(z) := (1-z)^n[\mathsf{U}_{1,1}^*(z^2) + z\mathsf{U}_{2,1}^*(z^2)], \quad \breve{\mathsf{b}}_2(z) := (1-z)^n[\mathsf{U}_{1,2}^*(z^2) + z\mathsf{U}_{2,2}^*(z^2)], \tag{63}$$

we have $\breve{\mathsf{b}}_j \in \mathbb{F}[z, z^{-1}]$ and $\mathsf{b}_j = c_j\breve{\mathsf{b}}_j$ for $j = 1, 2$.

Theorem 3. *Let $a \in l_0(\mathbb{Z})$ be such that $\mathsf{a}(1) = 1$ and $\mathsf{a} \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{Sa}(z) = z^c$ for some $c \in \mathbb{Z}$. Let $n \in \mathbb{N}$ be such that $n \leqslant \min\{\mathrm{sr}(a), \frac{1}{2}\mathrm{vm}(u_a)\}$ where u_a is defined as in (54). Define $\mathcal{M}_{a,n}$ as in (56) and $\mathcal{N}_{a,n}$ as in (60). The following statements are equivalent:*

(1) $\det(\mathcal{N}_{a,n}(z)) = -|\mathsf{C}|^2\mathsf{d}_n(z)\mathsf{d}_n^*(z)$ for some $\mathsf{d}_n \in \mathrm{Sym}_{\mathbb{F}}$ and $\mathsf{C} \in \mathbb{C}$ with $|\mathsf{C}|^2 \in \mathbb{F}$;

(2) there exist $b_1, b_2 \in l_0(\mathbb{Z})$ such that

- *(i) $\{a; b_1, b_2\}_{(1,-1)}$ is a quasi-tight framelet filter bank;*
- *(ii) $\mathsf{b}_1(z) = c_1\breve{\mathsf{b}}_1(z)$ and $\mathsf{b}_2(z) = c_2\breve{\mathsf{b}}_2(z)$ for some $\breve{\mathsf{b}}_1, \breve{\mathsf{b}}_2 \in \mathrm{Sym}_{\mathbb{F}}$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2, |c_2|^2 \in \mathbb{F}$;*
- *(iii) $\min\{\mathrm{vm}(b_1), \mathrm{vm}(b_2)\} = n$.*

Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(-e^{i2^{-j}\xi})$ for all $\xi \in \mathbb{R}$. If in addition $\phi \in L_2(\mathbb{R})$, define

$$\psi^{\ell}(x) := 2\sum_{k\in\mathbb{Z}} b_{\ell}(k)\phi(2x-k), \quad \forall x \in \mathbb{R}, \quad \ell = 1, 2, \tag{64}$$

then $\{\phi; \psi^1, \psi^2\}_{(1,-1)}$ is a quasi-tight framelet in $L_2(\mathbb{R})$ such that all ϕ, ψ^1, ψ^2 have symmetry and $\min(\mathrm{vm}(\psi^1), \mathrm{vm}(\psi^2)) \geqslant n$.

Proof. (2) $\Rightarrow$ (1): Denote $\mathsf{S}\breve{\mathsf{b}}_j(z) = \breve{\epsilon}_j z^{k_j}$ where $\breve{\epsilon}_j \in \{-1, 1\}$ and $k_j \in \mathbb{Z}$ for $j = 1, 2$. We first show that $\mathrm{prt}(k_1) = \mathrm{prt}(k_2) = \mathrm{prt}(c)$. Assume otherwise, say $\mathrm{prt}(k_1) = \mathrm{prt}(k_2) \neq \mathrm{prt}(c)$. Since $\{a; b_1, b_2\}_{(1,-1)}$ is a quasi-tight framelet filter bank, we have

$$-\mathsf{a}^*(z)\mathsf{a}(-z) = -|c_1|^2\breve{\mathsf{b}}_1^*(z)\breve{\mathsf{b}}_1(-z) + |c_2|^2\breve{\mathsf{b}}_2^*(z)\breve{\mathsf{b}}_2(-z) = 0.$$

As $\mathsf{S}[\mathsf{a}^*\mathsf{a}(-\cdot)](z) = (-1)^c$, $\mathsf{S}[-|c_1|^2\breve{\mathsf{b}}_1^*\breve{\mathsf{b}}_1(-\cdot)](z) = (-1)^{k_1}$ and $\mathsf{S}[|c_2|^2\breve{\mathsf{b}}_2^*\breve{\mathsf{b}}_2(-\cdot)](z) = (-1)^{k_2}$, we conclude that $(-1)^c = (-1)^{k_1} = (-1)^{k_2}$, which is a contradiction. Therefore, $\mathrm{prt}(k_1) = \mathrm{prt}(k_2) \neq \mathrm{prt}(c)$ cannot happen. Similarly $\mathrm{prt}(k_1) \neq \mathrm{prt}(k_2) = \mathrm{prt}(c)$ and $\mathrm{prt}(k_2) \neq \mathrm{prt}(k_1) = \mathrm{prt}(c)$ cannot happen either. Consequently, we must have $\mathrm{prt}(k_1) = \mathrm{prt}(k_2) = \mathrm{prt}(c)$.

By items (ii) and (iii), there exist $\mathsf{v}_1, \mathsf{v}_2 \in \mathrm{Sym}_{\mathbb{F}}$ such that

$$\breve{\mathsf{b}}_j(z) = (1-z)^n \mathsf{v}_j(z), \quad j = 1, 2, \tag{65}$$

and $\mathsf{S}\mathsf{v}_j(z) = (-1)^n \breve{\epsilon}_j z^{k_j+n}$ for $j = 1, 2$. Define $\mathsf{V}(z) := \begin{bmatrix} \mathsf{v}_1(z) & \mathsf{v}_1(-z) \\ \mathsf{v}_2(z) & \mathsf{v}_2(-z) \end{bmatrix}$. As $\{a; b_1, b_2\}_{(1,-1)}$ is a quasi-tight framelet filter bank, it is easy to derive from (55) that

$$\mathcal{M}_{a,n}(z) = \mathsf{V}^{\mathsf{H}}(z)\mathrm{Diag}(c_1, c_2)\mathrm{Diag}(1, -1)\mathrm{Diag}(c_1, c_2)^{\mathsf{H}}\mathsf{V}(z),$$

where $\mathcal{M}_{a,n}$ is defined as (56). Now, let F be the same as in (61), we have

$$\mathcal{N}_{a,n}(z^2) = \frac{1}{4}\,\mathsf{F}(z)\mathcal{M}_{a,n}(z)\,\mathsf{F}(z)^{\mathsf{H}} = \tilde{\mathsf{V}}^{\mathsf{H}}(z^2)\mathrm{Diag}(c_1, c_2)\mathrm{Diag}(1, -1)\mathrm{Diag}(c_1, c_2)^{\mathsf{H}}\tilde{\mathsf{V}}(z^2),$$

where

$$\tilde{\mathsf{V}}(z) := \begin{bmatrix} \mathsf{v}_1^{[0]}(z) & \mathsf{v}_1^{[1]}(z) \\ \mathsf{v}_2^{[0]}(z) & \mathsf{v}_2^{[1]}(z) \end{bmatrix}.$$

We show that $\det(\tilde{\mathsf{V}}) \in \mathrm{Sym}_{\mathbb{F}}$. Consider the following two cases:

- $c + n \in 2\mathbb{Z}$: In this case, we have $\mathrm{prt}(n) = \mathrm{prt}(c) = \mathrm{prt}(k_1) = \mathrm{prt}(k_2)$. Thus, $k_1 + k_2 + 2n \in 2\mathbb{Z}$ and direct calculation yields

$$\mathsf{S}\mathsf{v}_1(z)\mathsf{S}\mathsf{v}_2(z) = \mathsf{S}\mathsf{v}_1(-z)\mathsf{S}\mathsf{v}_2(z) = \mathsf{S}\mathsf{v}_1(z)\mathsf{S}\mathsf{v}_2(-z) = \mathsf{S}\mathsf{v}_1(-z)\mathsf{S}\mathsf{v}_2(-z) = \breve{\epsilon}_1\breve{\epsilon}_2 z^{k_1+k_2+2n}.$$

  Therefore, $\mathsf{S}[\det(\tilde{\mathsf{V}})](z) = \breve{\epsilon}_1\breve{\epsilon}_2 z^{(k_1+k_2)/2+n-1}$ and thus $\det(\tilde{\mathsf{V}}) \in \mathrm{Sym}_{\mathbb{F}}$.
- $c + n \in 2\mathbb{Z} + 1$: In this case, we have $\mathrm{prt}(n) \neq \mathrm{prt}(c) = \mathrm{prt}(k_1) = \mathrm{prt}(k_2)$. Let F be the same as in (61). Direct calculation yields

$$\tilde{\mathsf{V}}(z^2)\,\mathsf{F}(z^2) = \begin{bmatrix} (1+z)\mathsf{v}_1(z) + (1-z)\mathsf{v}_1(-z) & (1-z)\mathsf{v}_1(z) + (1+z)\mathsf{v}_1(-z) \\ (1+z)\mathsf{v}_2(z) + (1-z)\mathsf{v}_2(-z) & (1-z)\mathsf{v}_2(z) + (1+z)\mathsf{v}_2(-z) \end{bmatrix},$$

  and

$$\mathsf{S}[\tilde{\mathsf{V}}\,\mathsf{F}](z^2) = \begin{bmatrix} (-1)^n\breve{\epsilon}_1 z^{k_1+n+1} & (-1)^{n+1}\breve{\epsilon}_1 z^{k_1+n+1} \\ (-1)^n\breve{\epsilon}_2 z^{k_2+n+1} & (-1)^{n+1}\breve{\epsilon}_2 z^{k_2+n+1} \end{bmatrix}.$$

  It follows that $\mathsf{S}[\det(\tilde{\mathsf{V}}\,\mathsf{F})](z) = -\breve{\epsilon}_1\breve{\epsilon}_2 z^{(k_1+k_1)/2+n+1}$ and thus $\det(\tilde{\mathsf{V}}) \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{S}[\det(\tilde{\mathsf{V}})](z) = -\breve{\epsilon}_1\breve{\epsilon}_2 z^{(k_1+k_1)/2+n-1}$.

Now, by letting $\mathsf{d}_n(z) := \det(\tilde{\mathsf{V}}(z)) \in \mathrm{Sym}_{\mathbb{F}}$ and $C := c_1c_2$, we see that item (1) holds.

(1) $\Rightarrow$ (2): Define $\tilde{\mathsf{a}}, \tilde{\mathsf{b}}$ as in (57) and $\mathring{\mathsf{a}}, \mathring{\mathsf{b}}$ as in (58). Write $\mathcal{N}_{a,n} = \begin{bmatrix} \mathsf{p} & \mathsf{r} \\ \mathsf{r}^* & \mathsf{q} \end{bmatrix}$ as in (60). By (61) and the fact that $\gcd(\mathring{\mathsf{a}}, \mathring{\mathsf{b}}, \mathring{\mathsf{b}}(-\cdot), \mathring{\mathsf{a}}(-\cdot)) = 1$, it is not difficult to see that $\gcd(\mathsf{p}, \mathsf{r}, \mathsf{r}^*, \mathsf{q}) = 1$.

By the symmetry type of a, we have $\mathsf{S}\mathring{\mathsf{a}}(z) = 1$ and $\mathsf{S}\mathring{\mathsf{b}}(z) = (-1)^{c+n}$. Consider the following two cases:

- $c + n \in 2\mathbb{Z}$: In this case, we have $\mathsf{S}\mathring{\mathsf{b}}(z) = 1$. Using (59) and the definition of $\mathcal{N}_{a,n}$ in (60), it is easy to see that $\mathsf{S}[\mathcal{N}_{a,n}](z) = \begin{bmatrix} 1 & z^{-1} \\ z & 1 \end{bmatrix}$. Hence, $\mathcal{N}_{a,n} \in \mathrm{HS}_{\mathbb{F},2}$ satisfies all assumptions of Theorem 1. Therefore, there exist $\mathsf{U} = \begin{bmatrix} \mathsf{U}_{1,1} & \mathsf{U}_{1,2} \\ \mathsf{U}_{2,1} & \mathsf{U}_{2,2} \end{bmatrix} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2, |c_2|^2 \in \mathbb{F}$ such that

$$\mathcal{N}_{a,n}(z) = \mathsf{U}(z)\mathrm{Diag}(c_1, c_2)\mathrm{Diag}(1, -1)\mathrm{Diag}(c_1, c_2)^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z),$$

  and

$$\frac{\mathsf{S}\mathsf{U}_{1,1}(z)}{\mathsf{S}\mathsf{U}_{2,1}(z)} = \frac{\mathsf{S}\mathsf{U}_{1,2}(z)}{\mathsf{S}\mathsf{U}_{2,2}(z)} = \mathsf{S}\mathsf{r}(z) = z^{-1}. \tag{66}$$

Define $b_1, b_2 \in l_0(\mathbb{Z})$ via (62). It is straightforward to verify that $\{a; b_1, b_2\}_{(1,-1)}$ is a quasi-tight framelet filter bank with $\min\{\mathrm{vm}(b_1), \mathrm{vm}(b_2)\} \geqslant n$. Moreover, we have $\mathsf{b}_j = c_j \breve{\mathsf{b}}_j$, where $\breve{\mathsf{b}}_j$ is defined in (63), and it follows from (66) that $\breve{\mathsf{b}}_1, \breve{\mathsf{b}}_2 \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{S}\breve{\mathsf{b}}_1(z) = (-1)^n z^n \mathsf{SU}^*_{1,1}(z)$ and $\mathsf{S}\breve{\mathsf{b}}_2(z) = (-1)^n z^n \mathsf{SU}^*_{1,2}(z)$.

- $c + n \in 2\mathbb{Z} + 1$: In this case, we have $\mathsf{S}\mathring{\mathsf{b}}(z) = -1$. Define $\mathsf{P} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathcal{N}(z) := \mathsf{P}\mathcal{N}_{a,n}(z)\mathsf{P}^{\mathsf{H}}$. Direct calculation yields $\mathcal{N}(z) = \begin{bmatrix} \mathcal{N}_{1,1}(z) & \mathcal{N}_{1,2}(z) \\ \mathcal{N}_{2,1}(z) & \mathcal{N}_{2,2}(z) \end{bmatrix}$, where

$$\mathcal{N}_{1,1}(z) := \mathring{\mathsf{a}}^{[0]}(z) + \frac{1}{2}(z+1)\mathring{\mathsf{a}}^{[1]}(z) + \frac{1}{2}(z-1)\mathring{\mathsf{b}}^{[1]}(z),$$

$$\mathcal{N}_{1,2}(z) := \mathring{\mathsf{b}}^{[0]}(z) + \frac{1}{2}(z+1)\mathring{\mathsf{b}}^{[1]}(z) + \frac{1}{2}(z-1)\mathring{\mathsf{a}}^{[1]}(z),$$

$$\mathcal{N}_{2,1}(z) := \mathring{\mathsf{b}}^{[0]}(z) - \frac{1}{2}(z+1)\mathring{\mathsf{b}}^{[1]}(z) - \frac{1}{2}(z-1)\mathring{\mathsf{a}}^{[1]}(z),$$

$$\mathcal{N}_{2,2}(z) := \mathring{\mathsf{a}}^{[0]}(z) - \frac{1}{2}(z+1)\mathring{\mathsf{a}}^{[1]}(z) - \frac{1}{2}(z-1)\mathring{\mathsf{b}}^{[1]}(z).$$

Using (59) and the definition of $\mathcal{N}_{a,n}$ in (60), we see that $\mathsf{S}\mathcal{N}(z) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence, $\mathcal{N} \in \mathrm{HS}_{\mathbb{F},2}$. Moreover, since $\det(\mathcal{N}) = 4\det(\mathcal{N}_{a,n})$, we see that $\gcd(\mathcal{N}_{1,1}, \mathcal{N}_{1,2}, \mathcal{N}_{2,1}, \mathcal{N}_{2,2}) = 1$. Therefore, $\mathcal{N}$ satisfies all assumptions of Theorem 1, and thus there exist $\mathsf{V} = \begin{bmatrix} \mathsf{V}_{1,1} & \mathsf{V}_{1,2} \\ \mathsf{V}_{2,1} & \mathsf{V}_{2,2} \end{bmatrix} \in \mathrm{Sym}_{\mathbb{F}^{2\times 2}}$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2, |c_2|^2 \in \mathbb{F}$ such that

$$\mathcal{N}(z) = \mathsf{V}(z)\mathrm{Diag}(c_1, c_2)\mathrm{Diag}(1, -1)\mathrm{Diag}(c_1, c_2)^{\mathsf{H}}\mathsf{V}^{\mathsf{H}}(z),$$

and

$$\frac{\mathsf{SV}_{1,1}(z)}{\mathsf{SV}_{2,1}(z)} = \frac{\mathsf{SV}_{1,2}(z)}{\mathsf{SV}_{2,2}(z)} = \mathsf{S}\mathcal{N}_{1,2}(z) = -1. \tag{67}$$

It follows that

$$\mathcal{N}_{a,n}(z) = \mathsf{U}(z)\mathrm{Diag}(c_1, c_2)\mathrm{Diag}(1, -1)\mathrm{Diag}(c_1, c_2)^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z),$$

where $\mathsf{U} := \mathsf{P}^{-1}\mathsf{V} = \begin{bmatrix} \mathsf{U}_{1,1} & \mathsf{U}_{1,2} \\ \mathsf{U}_{2,1} & \mathsf{U}_{2,2} \end{bmatrix}$. Now, define $b_1, b_2 \in l_0(\mathbb{Z})$ via (62) and $\breve{\mathsf{b}}_1, \breve{\mathsf{b}}_2$ via (63). It is clear that items (i) and (iii) hold. Moreover, direct calculation yields

$$\breve{\mathsf{b}}_1(z) = \frac{(1-z)^n}{2}[(1+z)\mathsf{V}^*_{1,1}(z^2) + (1-z)\mathsf{V}^*_{2,1}(z^2)],$$

$$\breve{\mathsf{b}}_2(z) = \frac{(1-z)^n}{2}[(1+z)\mathsf{V}^*_{1,2}(z^2) + (1-z)\mathsf{V}^*_{2,2}(z^2)].$$

Therefore, using (67), we conclude that $\breve{\mathsf{b}}_1, \breve{\mathsf{b}}_2 \in \mathrm{Sym}_{\mathbb{F}}$ with $\mathsf{S}\breve{\mathsf{b}}_1(z) = (-1)^n z^{n+1}\mathsf{SV}^*_{1,1}(z)$ and $\mathsf{S}\breve{\mathsf{b}}_2(z) = (-1)^n z^{n+1}\mathsf{SV}^*_{1,2}(z)$. This proves item (ii).

To this end, we have finished the proofs of (i)–(iii) in item (2).

If in addition $\phi \in L_2(\mathbb{R})$, where $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(e^{-i2^{-j}\xi})$, define ψ^1 and ψ^2 as in (64). It follows from Theorem 2 that $\{\phi; \psi^1, \psi^2\}_{(1,-1)}$ is a quasi-tight framelet in $L_2(\mathbb{R})$. Moreover, we have $\min\{\mathrm{vm}(\psi^1), \mathrm{vm}(\psi^2)\} = \min\{\mathrm{vm}(b_1), \mathrm{vm}(b_2)\} \geqslant n$. Finally, the symmetry of ϕ, ψ^1 and ψ^2 follows immediately from the symmetry of a, b_1 and b_2. $\square$

4. Illustrative Examples

This section presents several examples to illustrate our main theorems on the generalized spectral factorization and quasi-tight framelets with symmetry.

Example 1. *Let $\mathbb{F} = \mathbb{Q}$ and consider the refinement filter $a \in l_0(\mathbb{Z})$ with*

$$\mathsf{a}(z) := -\frac{(1+z)^3}{16z^2}(z^2 - 4z + 1).$$

Define u_a as in (54). We have $\mathsf{Sa}(z) = z^3$, $\operatorname{sr}(a) = 3$ and $\operatorname{vm}(u_a) = 2$. Let $n = 1$, define $\mathcal{M}_{a,1}$ via (56) and $\mathcal{N}_{a,1}$ via (60). Specifically, we have

$$\mathcal{N}_{a,1}(z) := \frac{1}{256}\begin{bmatrix} z^2 - 16z + 30 - 16z^{-1} + z^{-2} & -16(1+z^{-1}) \\ -16(1+z) & 0 \end{bmatrix} =: \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^*(z) & \mathsf{q}(z) \end{bmatrix},$$

and

$$\det(\mathcal{N}_{a,1})(z) = -\frac{1}{256}\mathsf{d}_1^*(z)\mathsf{d}_1^*(z) \text{ where } \mathsf{d}_1(z) := 1 + z.$$

Hence, by Theorem 3, we obtain a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$ such that b_1, b_2 have symmetry and at least order 1 vanishing moments.

We follow the steps in the proof of Theorem 1 to factorize $\mathcal{N}_{a,1}$:

Step 1. Note that $\gcd(\mathsf{p}, \mathsf{r}) = \gcd(\mathsf{r}, \mathsf{q}) = 1$, so the entries of $\mathcal{N}_{a,1}$ are already mutually coprime and we move on to step 2.

Step 2. Define $\mathsf{U}_1(z) := Diag(1, 1+z)$. Clearly $\mathsf{U}_1 \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$. Moreover, by letting

$$\mathsf{B}(z) := \mathsf{U}_1(z)^{-1}\mathcal{N}_{a,1}(z)\mathsf{U}_1(z)^{-\mathsf{H}} = \frac{1}{256}\begin{bmatrix} z^2 - 16z + 30 - 16z^{-1} + z^{-2} & -16 \\ -16 & 0 \end{bmatrix},$$

We have $\mathsf{B} \in \mathrm{HS}_{\mathbb{Q},2}$ and $\det(\mathsf{B}) = -\frac{1}{256}$.

Step 3. Note that $\mathsf{SB}_{1,2}(z) = 1$, so we follow the justification of step 3 for case 1 and define

$$\begin{aligned} \mathsf{U}_2(z) &:= \begin{bmatrix} \dfrac{\mathsf{B}_{1,1}(z)+1}{2} & \dfrac{\mathsf{B}_{1,1}(z)-1}{2} \\ \mathsf{B}_{1,2}^*(z) & \mathsf{B}_{1,2}^*(z) \end{bmatrix} \\ &= \frac{1}{512}\begin{bmatrix} z^2 - 16z + 286 - 16z^{-1} + z^{-2} & z^2 - 16z - 226 - 16z^{-1} + z^{-2} \\ -32 & -32 \end{bmatrix}. \end{aligned}$$

Then, $\mathsf{U}_2(z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and thus $\mathsf{U}_2 \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$. Moreover, we have $\mathsf{U}_2 Diag(1, -1)\mathsf{U}_2^{\mathsf{H}} = \mathsf{B}$.

Define

$$\mathsf{U}(z) := \mathsf{U}_1(z)\mathsf{U}_2(z) = \frac{1}{512}\begin{bmatrix} z^2 - 16z + 286 - 16z^{-1} + z^{-2} & z^2 - 16z - 226 - 16z^{-1} + z^{-2} \\ -16(1+z) & -16(1+z) \end{bmatrix},$$

We see that $\mathsf{SU}(z) = \begin{bmatrix} 1 & 1 \\ z & z \end{bmatrix}$ and thus $\mathsf{U} \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$. Moreover, we have $\mathcal{N}_{a,1} = \mathsf{U}Diag(1, -1)\mathsf{U}^{\mathsf{H}}$. Finally, derive the high-pass filters b_1, b_2 via (62) with $c_1 = c_2 = 1$ and $n = 1$, we have

$$\mathsf{b}_1(z) = \frac{1}{512}(1-z)(z^4 - 16z^2 - 32z + 286 - 32z^{-1} - 16z^{-2} + z^{-4}),$$

$$\mathsf{b}_2(z) = \frac{1}{512}(1-z)(z^4 - 16z^2 - 32z - 226 - 32z^{-1} - 16z^{-2} + z^{-4}).$$

Note that $\operatorname{vm}(b_1) = \operatorname{vm}(b_2) = 1$ and $\mathsf{Sb}_1(z) = \mathsf{Sb}_2(z) = -z$.

Define ϕ via (42). As $\operatorname{sm}(a) \approx 1.4408$, we have $\phi \in L_2(\mathbb{R})$. Hence, by letting ψ^ℓ as in (46) for $\ell = 1, 2$, we see that $\{\phi; \psi^1, \psi^2\}_{(1,-1)}$ is a quasi-tight framelet in $L_2(\mathbb{R})$. Moreover, ϕ, ψ^1, ψ^2 all have symmetry and $\operatorname{vm}(\psi^1) = \operatorname{vm}(\psi^2) = 1$.

Example 2. *Let $\mathbb{F} = \mathbb{Q}$ and consider the refinement filter $a \in l_0(\mathbb{Z})$ with*

$$\mathsf{a}(z) := \frac{1}{1024}(-z^6 + 18z^4 - 32z^3 - 63z^2 + 288z + 604 + 288z^{-1} - 63z^{-2} - 32z^{-3} + 18z^{-4} - z^{-6}).$$

Define u_a as in (54). We have $\mathsf{Sa}(z) = 1$, $\mathrm{sr}(a) = 4$ and $\mathrm{vm}(u_a) = 8$. Let $n = 4$, define $\mathcal{M}_{a,4}$ via (56) and $\mathcal{N}_{a,4}$ via (60). Specifically, we have

$$\mathcal{N}_{a,4}(z) := \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^*(z) & \mathsf{q}(z) \end{bmatrix},$$

where

$$\mathsf{p}(z) := \frac{1}{1\,048\,576}(-z^4 + 16z^3 - 156z^2 + 2800z + 1850 + 2800z^{-1} - 156z^{-2} + 16z^{-3} - z^{-4}),$$

$$\mathsf{r}(z) := \frac{1}{262\,144}(-z^3 + 13z^2 - 81z + 2245 + 2245z^{-1} - 81z^{-2} + 13z^{-3} - z^{-4}),$$

$$\mathsf{q}(z) := \frac{1}{65\,536}(-z^3 + 10z^2 - 15z + 1996 - 15z^{-1} + 10z^{-2} - z^{-3}).$$

Note that

$$\det(\mathcal{N}_{a,4})(z) = -\frac{1}{2\,097\,152}\mathsf{d}_4^*(z)\mathsf{d}_4^*(z), \text{ where } \mathsf{d}_4(z) := z - 14 + z^{-1}.$$

Hence, by Theorem 3, we obtain a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$ such that b_1, b_2 have symmetry and at least order 4 vanishing moments.

We follow the steps in the proof of Theorem 1 to factorize $\mathcal{N}_{a,4}$:

Step 1. Note that $\gcd(\mathsf{p}, \mathsf{r}) = \gcd(\mathsf{r}, \mathsf{q}) = 1$, so the entries of $\mathcal{N}_{a,4}$ are already mutually coprime and we move on to step 2.

Step 2. Define $\mathsf{U}_1(z) := \begin{bmatrix} \mathsf{p}_1(z) & \mathsf{r}(z) \\ \mathsf{s}_1(z) & \mathsf{q}(z) \end{bmatrix}$ where

$$\begin{aligned}\mathsf{p}_1(z) :=& \frac{1}{32}(z^2 - 14z + 1)\Big[1\,375\,343(z^9 + z^{-9}) - 17\,801\,154(z^8 + z^{-8}) \\ &+110\,390\,161(z^7 + z^{-7}) - 3\,081\,371\,452(z^6 + z^{-6}) - 3\,263\,011\,343(z^5 + z^{-5}) \\ &-76\,359\,874(z^4 + z^{-4}) - 24\,287\,153(z^3 + z^{-3})\Big],\end{aligned}$$

$$\begin{aligned}\mathsf{s}_1(z) :=& \frac{1}{8}(z^2 - 14z + 1)\Big[1\,375\,343(z^9 + z^{-8}) - 13\,675\,125(z^8 + z^{-7}) + 19\,852\,438(z^7 + z^{-6}) \\ &-2\,744\,063\,146(z^6 + z^{-5}) - 135\,589\,845(z^5 + z^{-4}) - 23\,238\,577(z^4 + z^{-3})\Big].\end{aligned}$$

We have $\mathsf{SU}_1(z) = \begin{bmatrix} z^2 & z^{-1} \\ z^3 & 1 \end{bmatrix}$ and thus $\mathsf{U}_1 \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$. Moreover, we have $\mathsf{B} := \mathsf{U}_1^{-1}\mathcal{N}_{a,4}\mathsf{U}_1^{-\mathsf{H}} \in \mathrm{HS}_{\mathbb{Q},2}$ and $\det(\mathsf{B}) = -\frac{1}{2\,097\,152}$.

Step 3. Note that $\mathsf{SB}_{1,2}(z) = z^{-3}$, so we follow the justification of step 3 for case 1 and define

$$\mathsf{U}_2(z) := \begin{bmatrix} \mathsf{p}_2(z) & 1 \\ \mathsf{q}_2(z) & \mathsf{s}_2(z) \end{bmatrix},$$

where

$$\mathsf{p}_2(z) := \frac{1}{2\,097\,152}(-1 + 5z^{-1} + 5z^{-2} - z^{-3}),$$

$$\mathsf{q}_2(z) := \frac{1}{256}\Big[-1\,375\,343(z^8+z^{-8}) + 26\,053\,212(z^7+z^{-7}) - 89\,290\,186(z^6+z^{-6}) \\ -95\,839\,588(z^5+z^{-5}) + 20\,617\,137(z^4+z^{-4}) - 262\,144(z^3+z^{-3})\Big],$$

$$\mathsf{s}_2(z) := 8192\Big[1\,375\,343(z^8+z^{-5}) - 19\,176\,497(z^7+z^{-4}) + 284\,416(z^6+z^{-3}) \\ +38\,408(z^5+z^{-2}) + 640(z^4+z^{-1}) + 128(z^3+1)\Big].$$

Then, $\mathsf{SU}_2(z) = \begin{bmatrix} z^{-3} & 1 \\ 1 & z^3 \end{bmatrix}$ *and thus* $\mathsf{U}_2 \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$*. Moreover, we have* $\mathsf{B} = \mathsf{U}_2 \mathit{Diag}(32, -\frac{1}{67\,108\,864})\mathsf{U}_2^{\mathsf{H}}$.

Define

$$\mathsf{U}(z) := \mathsf{U}_1(z)\mathsf{U}_2(z) = \begin{bmatrix} \frac{1}{128}(1+z^{-1}) & -8z^3+64z^2-368z+64-8z^{-1} \\ \frac{1}{32} & -32z^3+160z^2+160z-32 \end{bmatrix},$$

we see that $\mathsf{SU}(z) = \begin{bmatrix} z^{-1} & z^2 \\ 1 & z^3 \end{bmatrix}$ *and thus* $\mathsf{U} \in \mathrm{Sym}_{\mathbb{Q}^{2\times 2}}$*. By letting* $\mathsf{D} := \mathit{Diag}(\sqrt{32}, \sqrt{\frac{1}{67\,108\,864}})$ $=\mathit{Diag}(4\sqrt{2}, \frac{1}{8192})$*, we have* $\mathcal{N}_{a,4}(z) = \mathsf{U}(z)\mathsf{D}\mathit{Diag}(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z)$*. Finally, derive the high-pass filters* b_1, b_2 *via (62) with* $c_1 := 4\sqrt{2}$, $c_2 = \frac{1}{8192}$ *and* $n = 4$*, we have*

$$\mathsf{b}_1(z) = \frac{\sqrt{2}}{32}(1-z)^4(z^2+4z+1),$$

$$\mathsf{b}_2(z) = \frac{z^{-2}}{1024}(1-z)^4(-z^4-4z^3+8^2+20z-46+20z^{-1}+8z^{-2}-4z^{-3}-z^{-4}).$$

Note that $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 4$, $\mathsf{Sb}_1(z) = z^6$ *and* $\mathsf{Sb}_2(z) = 1$.

Define ϕ *via (42). As* $\mathrm{sm}(a) \approx 1.6821$*, we have* $\phi \in L_2(\mathbb{R})$*. Hence, by letting* ψ^ℓ *as in (46) for* $\ell = 1, 2$*, we see that* $\{\phi; \psi^1, \psi^2\}_{(1,-1)}$ *is a quasi-tight framelet in* $L_2(\mathbb{R})$*. Moreover,* ϕ, ψ^1, ψ^2 *all have symmetry and* $\mathrm{vm}(\psi^1) = \mathrm{vm}(\psi^2) = 4$.

Example 3. *Let* $\mathbb{F} = \mathbb{Q}(\sqrt{226})$ *and consider the refinement filter* $a \in l_0(\mathbb{Z})$ *with*

$$\mathsf{a}(z) := -\frac{1}{4096}z^{-4}(1+z)^3\Big[2\sqrt{226}(z-1)^6 - 86z^6 + 198z^5 - 426z^4 - 396z^3 - 426z^2 + 198z - 86\Big].$$

Define u_a *as in (54). We have* $\mathsf{Sa}(z) = z$, $\mathrm{sr}(a) = 3$ *and* $\mathrm{vm}(u_a) = 6$*. Let* $n = 3$*, define* $\mathcal{M}_{a,3}$ *via (56) and* $\mathcal{N}_{a,3}$ *via (60). Specifically, we have*

$$\mathcal{N}_{a,3}(z) := \begin{bmatrix} \mathsf{p}(z) & \mathsf{r}(z) \\ \mathsf{r}^*(z) & \mathsf{q}(z) \end{bmatrix},$$

where

$$\mathsf{p}(z) := \frac{1}{16\,777\,216}\Big[(-172\sqrt{226}+2753)(z^3+z^{-3}) + (1320\sqrt{226}-22\,710)(z^2+z^{-2}) \\ (-9876\sqrt{226}+203\,343)(z+z^{-1}) - 48\,080\sqrt{226}+821\,068\Big],$$

$$\mathsf{r}(z) := \frac{1}{4\,194\,304}\Big[(645-30\sqrt{226})(z^2+z^{-3}) + (218\sqrt{226}-5767)(z+z^{-2}) \\ +(120\,834-4284\sqrt{226})(1+z^{-1})\Big],$$

$$\mathsf{q}(z) := \frac{1}{1\,048\,576}\Big[225(z^2+z^{-2}) - 1920(z+z^{-1}) + 61\,246\Big].$$

Note that

$$\det(\mathcal{N}_{a,3})(z) = -\frac{-5679+396\sqrt{226}}{16\,777\,216}\mathsf{d}_3^*(z)\mathsf{d}_3^*(z) \text{ where } \mathsf{d}_3(z) := z-1.$$

Hence, by Theorem 3, we obtain a quasi-tight framelet filter bank $\{a; b_1, b_2\}_{(1,-1)}$ *such that* b_1, b_2 *have symmetry and at least order 3 vanishing moments.*

We follow the steps in the proof of Theorem 1 to factorize $\mathcal{N}_{a,3}$*:*

Step 1. Note that $\gcd(\mathsf{p},\mathsf{r}) = \gcd(\mathsf{r},\mathsf{q}) = 1$, *so the entries of* $\mathcal{N}_{a,3}$ *are already mutually coprime and we move on to step 2.*

Step 2. Define $\mathsf{U}_1(z) := \begin{bmatrix} \mathsf{p}_1(z) & \mathsf{r}(z) \\ \mathsf{s}_1(z) & \mathsf{q}(z) \end{bmatrix}$, *where*

$$\begin{aligned}\mathsf{p}_1(z) := &\frac{1}{3\,603\,993\,750}\Big[(1\,091\,704\,285\,875 + 9\,791\,074\,100\sqrt{226}))(z^7 - z^{-6}) \\ &-(13\,257\,411\,426\,695 + 398\,623\,540\,340\sqrt{226})(z^6 - z^{-5}) \\ &+(281\,188\,303\,563\,542 + 9\,390\,986\,903\,768\sqrt{226}))(z^5 - z^{-4}) \\ &+(2\,552\,371\,608\,810 - 80\,926\,623\,996\sqrt{226}))(z^4 - z^{-3}) \\ &-(282\,308\,839\,799\,417 + 9\,410\,777\,977\,868\sqrt{226}))(z^3 - z^{-2}) \\ &+(10\,733\,871\,767\,885 + 1\,207\,889\,780\,300\sqrt{226}))(z^2 - z^{-1})\Big],\end{aligned}$$

$$\begin{aligned}\mathsf{s}_1(z) := &\frac{z}{1\,801\,996\,875}\Big[(1\,774\,249\,199\,550 + 96\,330\,944\,700\sqrt{226})(z^6 - z^{-6}) \\ &-(16\,934\,499\,950\,460 + 921\,952\,067\,640\sqrt{226})(z^5 - z^{-5}) \\ &+(498\,282\,206\,778\,368 + 27\,077\,560\,570\,112\sqrt{226})(z^4 - z^{-4}) \\ &-(503\,642\,983\,325\,828 + 28\,049\,119\,633\,352\sqrt{226})(z^3 - z^{-3}) \\ &+(20\,521\,027\,298\,370 + 1\,797\,180\,186\,180\sqrt{226})(z^2 + z^{-2})\Big].\end{aligned}$$

We have $\mathsf{SU}_1(z) = \begin{bmatrix} -z & z^{-1} \\ -z^2 & 1 \end{bmatrix}$ *and thus* $\mathsf{U}_1 \in \mathrm{Sym}_{\mathbb{Q}(\sqrt{226})^{2\times 2}}$. *Moreover, we have* $\mathsf{B} := \mathsf{U}_1^{-1}\mathcal{N}_{a,3}\mathsf{U}_1^{-\mathsf{H}} \in \mathrm{HS}_{\mathbb{Q}(\sqrt{226}),2}$ *and* $\det(\mathsf{B}) = \frac{5679}{16\,777\,216} - \frac{99\sqrt{226}}{4{,}194\,304}$.

Step 3. Note that $\mathsf{SB}_{1,2}(z) = -z^{-2}$, *so we follow the justification of step 3 for case 2 and define*

$$\mathsf{U}_2(z) := \begin{bmatrix} \mathsf{p}_2(z) & \mathsf{r}_2(z) \\ \mathsf{q}_2(z) & \mathsf{s}_2(z) \end{bmatrix},$$

where

$$\mathsf{p}_2(z) := \frac{45\sqrt{226}-180}{32\,768}(1-z^2),$$

$$\mathsf{r}_2(z) := z^2 - \frac{482+32\sqrt{226}}{15}z + 1,$$

$$
\begin{aligned}
\mathsf{q}_2(z) :=& \frac{z^2}{381\,375(44\sqrt{226}-631)}\Big[(6\,166\,090\,041\,856 - 312\,570\,426\,944\sqrt{226})(z^5+z^{-5}) \\
&+ (273\,675\,365\,504\sqrt{226} - 5\,615\,090\,615\,296)(z^4+z^{-4}) \\
&+ (351\,399\,233\,984\sqrt{226} - 6\,722\,543\,842\,816)(z^3+z^{-3}) \\
&+ (5\,618\,397\,085\,696 - 273\,472\,815\,104\sqrt{226})(z^2+z^{-2}) \\
&+ (560\,749\,608\,960 - 39\,101\,399\,040\sqrt{226})(z+z^{-1})\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{s}_2(z) :=& \frac{z^2}{5\,405\,990\,625}\Big[-(24\,805\,801\,715\,564\,544 + 1\,346\,804\,222\,263\,296\sqrt{226})(z^5-z^{-5}) \\
&+ (9\,759\,344\,745\,565\,388\sqrt{226} + 1\,471\,517\,599\,115\,247\,616)(z^4-z^{-4}) \\
&- (99\,799\,177\,382\,330\,368\sqrt{226} + 1\,504\,645\,431\,213\,162\,496)(z^3-z^{-3}) \\
&+ (51\,856\,679\,217\,659\,904 + 3\,179\,890\,706\,743\,296\sqrt{226})(z^2-z^{-2}) \\
&+ (6\,076\,954\,595\,819\,520 + 372\,643\,442\,196\,480\sqrt{226})(z-z^{-1})\Big].
\end{aligned}
$$

Then, $\mathsf{SU}_2(z) = \begin{bmatrix} -z^2 & z^2 \\ z^4 & -z^4 \end{bmatrix}$ *and thus* $\mathsf{U}_2 \in \mathrm{Sym}_{\mathbb{Q}(\sqrt{226})^{2\times 2}}$. *Moreover, we have* $\mathsf{U}_2 Diag(|c_1|^2, -|c_2|^2)\mathsf{U}_2^{\mathsf{H}} = \mathsf{B}$ *where*

$$
c_1 := \frac{16-\sqrt{226}}{256}, \quad c_2 := \frac{(631\sqrt{2}-88\sqrt{113})(7033-508\sqrt{226})\sqrt{7033+508\sqrt{226}}}{206\,438\,400\,000}.
$$

Note that $|c_1|^2, |c_2|^2 \in \mathbb{Q}(\sqrt{226})$.

Define $\mathsf{U}(z) := \mathsf{U}_1(z)\mathsf{U}_2(z) = \begin{bmatrix} \mathsf{U}_{1,1}(z) & \mathsf{U}_{1,2}(z) \\ \mathsf{U}_{2,1}(z) & \mathsf{U}_{2,2}(z) \end{bmatrix}$, *where*

$$
\mathsf{U}_{1,1}(z) := \frac{1}{2640\sqrt{226}-37\,860}\Big[(47\,021-3154\sqrt{226})(z^3+1) + (23\,835+210\sqrt{226})(z^2+z)\Big],
$$

$$
\mathsf{U}_{1,2}(z) := \frac{1}{75}\Big[-(9216+384\sqrt{226})(z^3-1) + (31\,744+2176\sqrt{226})(z^2-z)\Big],
$$

$$
\mathsf{U}_{2,1}(z) := -z^3 + \frac{1016\sqrt{226}+14\,066}{660\sqrt{226}-9465}z^2 - z,
$$

$$
\mathsf{U}_{2,2}(z) := -\frac{86\,528\sqrt{226}+1\,411\,072}{2925}(z^3-z).
$$

We see that $\mathsf{SU}(z) = \begin{bmatrix} z^3 & -z^{-3} \\ z^4 & -z^{-4} \end{bmatrix}$ *and thus* $\mathsf{U} \in \mathrm{Sym}_{\mathbb{Q}(\sqrt{226})^{2\times 2}}$. *By letting* $\mathsf{D} := Diag(c_1, c_2)$, *we have* $\mathcal{N}_{a,3}(z) = \mathsf{U}(z)\mathsf{D}Diag(1,-1)\mathsf{D}^{\mathsf{H}}\mathsf{U}^{\mathsf{H}}(z)$. *Finally, derive the high-pass filters* b_1, b_2 *via* (62) *with* $n = 3$, *we obtain a quasi-tight framelet filter bank* $\{a; b_1, b_2\}_{(1,-1)}$ *such that* $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 3$, $\mathsf{S}b_1(z) = -z^{-3}$ *and* $\mathsf{S}b_2(z) = z^{-3}$.

Define ϕ *via* (42). *As* $\mathrm{sm}(a) \approx 1.4635$, *we have* $\phi \in L_2(\mathbb{R})$. *Hence, by letting* ψ^ℓ *as in* (46) *for* $\ell = 1, 2$, *we see that* $\{\phi; \psi^1, \psi^2\}_{(1,-1)}$ *is a quasi-tight framelet in* $L_2(\mathbb{R})$. *Moreover,* ϕ, ψ^1, ψ^2 *all have symmetry and* $\mathrm{vm}(\psi^1) = \mathrm{vm}(\psi^2) = 3$.

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