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# On the Oscillatory Behavior of Solutions of Second-Order Non-Linear Differential Equations with Damping Term

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**Abstract:** In this paper, we discuss the oscillatory behavior of solutions of two general classes of nonlinear second-order differential equations. New criteria are obtained using Riccati transformations and the integral averaging techniques. The obtained results improve and generalize some recent criteria in the literature. Moreover, a traditional condition is relaxed. Three examples are given to justify the results.

**Keywords:** differential equations; oscillation; damping term; Riccati transformation; integral average technique

**MSC:** 34C10; 34C29



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## 1. Introduction

The purpose of this paper is to discuss the oscillatory behavior of solutions of two general classes of nonlinear second-order differential equations of the forms

$$\left[r(\iota)U(\chi(\iota))(\chi'(\iota))\right]' + P(\iota)\chi'(\iota) + Q(\iota)z(\chi(\iota))g(\chi'(\iota)) = 0, \tag{1}$$

and

$$[r(\iota)U(\chi(\iota))\chi'(\iota)]' + P(\iota)\chi'(\iota) + Q(\iota)z(\chi(\iota)) = 0, \tag{2}$$

where $r(\iota), P(\iota), Q(\iota) \in C([\iota_0, \infty), \mathbb{R})$ and $U(\chi), z(\chi), g(\chi') \in C(\mathbb{R}, \mathbb{R})$.

In the last few decades, there has been a great interest in studying the oscillatory behavior of differential equations due to its great importance in describing many real applications in Physics, Biology, and Engineering; see [1–29] and references therein. The most important tools in the study of oscillatory behavior of solutions of these equations are the averaging techniques and Riccati transformations. For the author of the papers concerned with particular cases of Equations (1) and (2), we mention here, for example, Yan [27], who studied the special case of $U(\chi(\iota)) \equiv 1, z(\chi(\iota)) = \chi, g(\chi') \equiv 1$ and deduced the oscillation criterion without requiring that $Q(\iota)$ be integrable or bounded on $[\iota_0, \infty)$. Rogovchenko [21] and Elabbasy, Hassan and Saker [4] discussed the special case of $U(\chi(\iota)) \equiv 1$ of Equation (2) using a Riccati transformation technique where the authors established some oscillation criteria of Kamanev and Philos type with no sign conditions on $P(\iota)$ and $Q(\iota)$. Their results can be considered as affirmative answer to the question posed by Rogovechenko et al. [22].

Meanwhile, the undamped special case of Equation (2) was discussed by El-Sheikh [5], where the author considered function $U(\chi(\iota))$ to be bounded from above by function $\xi(\iota)$.

In fact, several authors were concerned with the oscillation of Equation (2) itself. We mention here, as examples, Grace and Lalli [13], who studied the oscillatory behavior of Equation (2) using integral averaging techniques in the case $r(\iota)$ and $Q(\iota)$ are not assumed to be non-negative for all values of $\iota$. In [14], Grace established new oscillation results in the spirit of those obtained by Kamenev, Philos, and Yan for a broad class of second-order nonlinear equations of Type (2). In [25], Tiryaki and Zafer made use of Philos's technique [20] to establish new oscillation criteria for Equation (2) in the case where $P(\iota)$ and $Q(\iota)$ are allowed to change signs on $[\iota_0, \infty)$. Their theorems improve and generalize some previous results of Grace [14] and Yan [27]. In [16], Kirane and Rogovchenko answered the questions addressed in [14] for the case of Equation (2) with non-monotonic nonlinearity, without any condition on the sign of the damping coefficient at all and $z(\chi(\iota))$ not needing to be non-decreasing. Their criteria cover new classes of equations discussed by the authors in [11,13,14,27]. In [22], S. Rogovchenko and Yuri Rogovchenko discussed the oscillatory behavior for different second-order equations like (2) and some equations more general than (2). Moreover, in [23], Rogovchenko and Tuncay used the averaging technique and strengthened the results of Kirane and Rogovchenko [16], Grace [14], Philos [20], and Yan [27]. They relaxed some of the restrictions introduced by Grace [11,12]. Also, Manojlovic [18] discussed the oscillation of (2) using a general class of parameter functions $M(\iota, \Theta)$ in the averaging techniques. An essential feature of the results in [18] is that the assumption of positivity of function $U(\chi(\iota))$ is not required. Jiang et al. [15] discussed a class of equations more general than (1) and (2), where they employed the generalized Riccati transformation and a class of functions to establish several oscillation criteria for monotonic and non-monotonic functions to improve some results obtained in the literature. In [26], Wang and Song studied a class of second-order nonlinear differential equations with a damping term which is more general than (1) and (2). They established some new sufficient conditions by using the refined integral averaging technique. Zhang et al. [29] established two new oscillation criteria using a generalized Riccati technique and the integral averaging technique of the Philos type, where they considered the case when the sign of $P(\iota)$ and $Q(\iota)$ may change. It is notable (see [23]) that the importance of function $U(\chi(\iota))$ is closely related to the presence of a damping term in Equation (2), which makes reduction to simpler differential equations either very complicated or impossible. We note that in all of those papers, the authors assume that function $U(\chi(\iota))$ is bounded by constants as $0 < c \leq U(\chi(\iota)) \leq c_1$, even in papers [15,26], where the authors consider equations more general than (1) and (2), however, they also assumed constant bounds for the function $U(\chi(\iota))$ as well. Moreover, the authors in [22] claimed that most of oscillation criteria for Equation (2) require that function $U(\chi(\iota))$ must be bounded away from zero by positive constant $c$. In fact, our results in the present article show that this claim is not always necessary. In this article, we impose the more general condition on function $0 < \xi_1(\iota) \leq U(\chi(\iota)) \leq \xi_2(\iota) < \infty$. Unlike the trend of the preceding works, we think that our criteria may cover new classes of equations to which known results do not apply.

Nontrivial solution $\chi(\iota)$ of each of these differential equations is called oscillatory if it has an infinite number of zeros; otherwise, it is called non-oscillatory. The differential equation is oscillatory if all its solutions are oscillatory.

The aim of this article is to establish new oscillation criteria of a Philos type for Equations (1) and (2) using the integral averaging technique. Our results improve and generalize some known results in the literature. To illustrate the obtained results, we offer three examples.

Now, we define the Philos-type function as follows. We let $D = \{(\iota, \Theta) : \iota_0 \leq \Theta \leq \iota < \infty\}$ and $D_0 = \{(\iota, \Theta) : \iota_0 \leq \Theta < \iota < \infty\}$.

Function $M = M(\iota, \Theta) \in C(D, [0, \infty))$ is said to be belong to function class $W$ if

(i) $M(\iota, \iota) = 0$ for $\iota \geq \iota_0$; $M(\iota, \Theta) \geq 0$ for $\iota > \Theta$ and
(ii) $M$ has partial derivative $\frac{\partial M}{\partial \Theta} = -m(\iota, \Theta)\sqrt{M(\iota, \Theta)}$, where $m$ is locally integrable with respect to $\Theta$ in $D_0$.

Also, we use notation $F_+(\iota) = \max\{F(\iota), 0\}$. Throughout the paper, we assume the following conditions:

(A1) $0 < \xi_1(\iota) \leq U(\chi(\iota)) \leq \xi_2(\iota) < \infty$;

(A2) $z'(\chi)$ exists, with $z'(\chi) \geq \gamma_1 > 0$;
(A3) $g \in C^1([\iota_0, \infty))$, $g(\chi') \geq \gamma_2 > 0$;
(A4) The function $M \in (W, \mathbb{R})$ satisfies $0 < \inf_{\Theta \geq \iota_0}\left[\lim_{\iota\to\infty} inf \frac{M(\iota,\Theta)}{M(\iota,\iota_0)}\right] \leq \infty$;
(A5) $\frac{z(\chi)}{\chi} \geq \gamma_3 > 0$.

## 2. Main Results

We start with two new oscillation criteria for Equation (1). In the first result, we do not need any restriction on the sign of $P(\iota)$ or $Q(\iota)$. The second result is dealing with Equation (2) in the case of monotonic and nonmonotonic function $z(\chi(\iota))$.

**Theorem 1.** *Supposethat conditions (A1)–(A3) hold. If for a continuously differentiable function $\varrho(\iota) : [\iota_0, +\infty) \to (0, +\infty)$, we have*

$$\lim_{\iota\to\infty} \sup \frac{1}{M(\iota,\iota_0)} \int_{\iota_0}^{\iota} \Bigg( M(\iota,\Theta)\varrho(\Theta)\Phi_1(\Theta) - \frac{r(\Theta)\xi_2(\Theta)M(\iota,\Theta)}{4\gamma_1\varrho(\Theta)} \times \left[\frac{\varrho(\Theta)P(\Theta)}{r(\Theta)\xi_2(\Theta)} + \frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{M(\iota,s)}} - \varrho'(\Theta)\right]^2 \Bigg) d\Theta = \infty, \tag{3}$$

*where*

$$\Phi_1(\iota) = \gamma_2 Q(\iota) + \left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\frac{P^2(\iota)}{4\gamma_1 r(\iota)}, \tag{4}$$

*then Equation (1) is oscillatory.*

**Proof.** We suppose the contrary, that $\chi(\iota) \neq 0$, $\iota \geq \iota_0$ is a nonoscillatory solution of Equation (1). We define the following general Riccati transformation:

$$\varpi_1(\iota) = \frac{r(\iota)U(\chi(\iota))\chi'(\iota)}{z(\chi(\iota))}. \tag{5}$$

Differentiating (5) and using (1), we obtain

$$\begin{aligned}
\varpi_1'(\iota) &= \frac{[r(\iota)U(\chi(\iota))\chi'(\iota)]'z(\chi(\iota)) - z'(\chi(\iota))\chi'(\iota)[r(\iota)U(\chi(\iota))\chi'(\iota)]}{z^2(\chi(\iota))} \\
&= -P(\iota)\frac{\chi'(\iota)}{z(\chi(\iota))} - Q(\iota)g(\chi'(\iota)) - z'(\chi(\iota))\frac{\varpi_1^2(\iota)}{r(\iota)U(\chi(\iota))} \\
&\leq -P(\iota)\frac{\varpi_1(\iota)}{r(\iota)U(\chi(\iota))} - Q(\iota)g(\chi'(\iota)) - z'(\chi(\iota))\frac{\varpi_1^2(\iota)}{r(\iota)U(\chi(\iota))}.
\end{aligned}$$

In view of (A3), we obtain

$$\varpi_1'(\iota) \leq -P(\iota)\frac{\varpi_1(\iota)}{r(\iota)U(\chi(\iota))} - \gamma_2 Q(\iota) - z'(\chi(\iota))\frac{\varpi_1^2(\iota)}{r(\iota)U(\chi(\iota))}. \tag{6}$$

$$\varpi_1'(\iota) \leq -\gamma_2 Q(\iota) - \frac{\gamma_1}{r(\iota)U(\chi(\iota))}\left[\varpi_1(\iota) + \frac{P(\iota)}{2\gamma_1}\right]^2 + \frac{P^2(\iota)}{4\gamma_1 r(\iota)U(\chi(\iota))}. \tag{7}$$

Applying Condition (A1), we obtain

$$\varpi_1'(\iota) \leq -\gamma_2 Q(\iota) - \frac{\gamma_1}{r(\iota)\xi_2(\iota)}\left[\varpi_1(\iota) + \frac{P(\iota)}{2\gamma_1}\right]^2 + \frac{P^2(\iota)}{4\gamma_1 r(\iota)\xi_1(\iota)}$$

$$\leq -\gamma_2 Q(\iota) - \frac{p^2(\iota)}{4\gamma_1 r(\iota)}\left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right) - \frac{\gamma_1}{r(\iota)\xi_2(\iota)}\left[\varpi_1^2(\iota) + \frac{\varpi_1(\iota)P(\iota)}{\gamma_1}\right]$$
$$\leq -\Phi_1(\iota) - \frac{\gamma_1}{r(\iota)\xi_2(\iota)}\left[\varpi_1^2(\iota) + \frac{\varpi_1(\iota)P(\iota)}{\gamma_1}\right],$$

where $\Phi_1(\iota)$ is defined as in (4).

Now, multiplying by $\varrho(\iota)M(\iota,\Theta)$ and integrating from $\iota_0$ to $\iota$, we obtain

$$\int_{\iota_0}^{\iota} M(\iota,\Theta)\varpi_1'(\Theta)\varrho(\Theta)M(\iota,\Theta)d\Theta \leq -\int_{\iota_0}^{\iota}\Phi_1(\Theta)\varrho(\Theta)M(\iota,\Theta)d\Theta$$
$$-\int_{\iota_0}^{\iota}\frac{\gamma_1}{r(\Theta)\xi_2(\Theta)}\left[\varpi_1^2(\Theta) + \frac{\varpi_1(\Theta)P(\Theta)}{\gamma_1}\right]\varrho(\Theta)M(\iota,\Theta)d\Theta.$$

Applying (i) and (ii), we find

$$-\varrho(\iota_0)M(\iota,\iota_0)\varpi_1(\iota_0) + \int_{\iota_0}^{\iota}\left[-\varrho'(\Theta) + \frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{m(\iota,\Theta)}}\right]M(\iota,\Theta)\varpi_1(\Theta)d\Theta \leq$$
$$-\int_{\iota_0}^{\iota}\Phi_1(\Theta)\varrho(\Theta)M(\iota,\Theta)d\Theta - \int_{\iota_0}^{\iota}\varrho(\Theta)M(\iota,\Theta)\frac{\gamma_1}{r(\Theta)\xi_2(\Theta)}\left[\varpi_1^2(\Theta) + \frac{\varpi_1(\Theta)P(\Theta)}{\gamma_1}\right]d\Theta.$$

Then,

$$\int_{\iota_0}^{\iota}\Phi_1(\Theta)\varrho(\Theta)M(\iota,\Theta)d\Theta \leq \varrho(\iota_0)M(\iota,\iota_0)\varpi_1(\iota_0) - \int_{\iota_0}^{\iota}\varrho(\Theta)M(\iota,\Theta)\varpi_1^2(\Theta)\frac{\gamma_1}{r(\Theta)\xi_2(\Theta)}d\Theta. \tag{8}$$

By completing the squares, we obtain

$$\int_{\iota_0}^{\iota}\Phi_1(\Theta)\varrho(\Theta)M(\iota,\Theta)d\Theta \leq \varrho(\iota_0)M(\iota,\iota_0)\varpi_1(\iota_0)$$
$$-\int_{\iota_0}^{\iota}\left[\varpi_1(\Theta)\sqrt{\frac{\gamma_1 M(\iota,\Theta)}{\varrho(\Theta)}r(\Theta)\xi_2(\Theta)}\right.$$
$$\left.+\sqrt{\frac{r(\Theta)\xi_2(\Theta)M(\iota,\Theta)}{4\gamma_1\varrho(\Theta)}}\left(\varrho(\Theta)m(\iota,\Theta) - \frac{P(\Theta)\varrho(\Theta)}{r(\Theta)\xi_2(\Theta)} - \varrho'(\Theta)\right)^2\right]d\Theta$$
$$+\int_{\iota_0}^{\iota}\left[\frac{r(\Theta)\xi_2(\Theta)M(\iota,\Theta)}{4\gamma_1\varrho(\Theta)}\left(\frac{\varrho(\Theta)P(\Theta)}{r(\Theta)\xi_2(\Theta)} + \frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{M(\iota,\Theta)}} - \varrho'(\Theta)\right)^2\right]d\Theta.$$

Then,

$$\int_{\iota_0}^{\iota}\left[\Phi_1(\Theta)\varrho(\Theta)M(\iota,\Theta) - \frac{r(\Theta)\xi_2(\Theta)M(\iota,\Theta)}{4\gamma_1\varrho(\Theta)} \times \left(\frac{\varrho(\Theta)P(\Theta)}{r(\Theta)\xi_2(\Theta)} + \frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{M(\iota,\Theta)}} - \varrho'(\Theta)\right)^2\right]d\Theta$$
$$\leq \varrho(\iota_0)M(\iota,\iota_0)\varpi_1(\iota_0)$$

$$-\int_{\iota_0}^{\iota}\left\{\varpi_1(\Theta)\sqrt{\frac{\gamma_1\varrho(\Theta)M(\iota,\Theta)}{r(\Theta)\xi_2(\Theta)}} + \frac{r(\Theta)\xi_2(\iota)(\Theta)M(\iota,\Theta)}{4\gamma_1\varrho(\Theta)} \times \left[\frac{\varrho(\Theta)P(\Theta)}{r(\Theta)\xi_2(\Theta)} + \frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{M(\iota,\Theta)}} - \varrho'(\Theta)\right]\right\}^2 d\Theta \tag{9}$$
$$\leq \varrho(\iota_0)M(\iota,\iota_0)\varpi_1(\iota_0).$$

Dividing (9) by $M(\iota,\iota_0)$ and taking the limit as $\iota \to \infty$, we obtain

$$\limsup_{\iota\to\infty}\frac{1}{M(\iota,\iota_0)}\int_{\iota_0}^{\iota}\left(M(\iota,\Theta)\varrho(\Theta)\Phi_1(\Theta)-\frac{r(\Theta)\xi_2(\Theta)H(\iota,\Theta)}{4\gamma_1\varrho(\Theta)}\times\left[\frac{\varrho(\Theta)P(\Theta)}{r(\Theta)\xi_2(\Theta)}+\frac{\varrho(\Theta)m(\iota,\Theta)}{\sqrt{M(\iota,\Theta)}}-\varrho'(\Theta)\right]^2\right)d\Theta$$
$$\leq \varpi_1(\iota_0)\varrho(\iota_0),$$

which contradicts (3). Then, (1) oscillates. □

**Remark 1.** *In the special case of $g(\chi')\equiv 1$ in Equation (1), Criterion (3) generalizes Criterion (5) of [29] in the special form of our Condition (A1) as $0<c_1\leq U(\chi(\iota))\leq c_2$.*

**Theorem 2.** *Supposethat (A1)–(A4) hold. If, for some $\beta>1$, we have*

$$\limsup_{\iota\to\infty}\left[\frac{1}{M(\iota,l)}\int_{l}^{\iota}\left(M(\iota,\Theta)\Phi(\Theta)-\frac{\beta\xi_2(\Theta)\aleph(\Theta)r(\Theta)}{4\gamma_1}m^2(\iota,\Theta)\right)d\Theta\right]\geq F(l), \tag{10}$$

*and*

$$\limsup_{\iota\to\infty}\int_{\iota_0}^{\iota}\frac{F_+^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta=\infty, \tag{11}$$

*where*

$$\aleph(\iota)=exp\left[-2\gamma_1\int^{\iota}\frac{1}{\xi_2(\Theta)}\left(\frac{\eta(\Theta)}{r(\Theta)}-\frac{P(\Theta)}{2\gamma_1 r(\Theta)}\right)d\Theta\right], \tag{12}$$

*and*

$$\Phi(\iota)=\aleph(\iota)\left[\gamma_2 q(\iota)+\frac{\gamma\eta^2(\iota)}{\xi_2(\iota)r(\iota)}-\frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)}-\eta'(\iota)+\left(\frac{1}{\xi_2(\iota)}-\frac{1}{\xi_1(\iota)}\right)\frac{P^2(\iota)}{4\gamma_1 r(\iota)}\right] \tag{13}$$

*for all $\iota>l\geq t_0$, then Equation (1) is oscillatory.*

**Proof.** We let that $\chi(\iota)$ be a nonoscillatory solution of Equation (1). Then, there exists $l_0\geq\iota_0$ such that $\chi(\iota)\neq 0$ for all $\iota\geq l_0$. Without loss of generality, we may assume that $\chi(\iota)>0$ for all $\iota\geq l_0$. We define a generalized Riccati transformation as

$$\varpi_2(\iota)=\aleph(\iota)\left[\frac{r(\iota)U(\chi(\iota))\chi'(\iota)}{z(\chi(\iota))}+\eta(\iota)\right], \tag{14}$$

where $\aleph(\iota)$ is given by (12). Differentiating (14) and using (1), we obtain

$$\begin{aligned}\varpi_2'(\iota)=&\aleph'(\iota)\left[\frac{r(\iota)U(\chi(\iota))\chi'(\iota)}{z(\chi(\iota))}+\eta(\iota)\right]+\\&\aleph(\iota)\left[\eta'(\iota)+\frac{(r(\iota)U(\chi(\iota))\chi'(\iota))'}{f(\chi(\iota))}-r(\iota)U(\chi(\iota))z'(\chi(\iota))\left(\frac{\chi'(\iota)}{z(\chi(\iota))}\right)^2\right]\\=&\aleph(\iota)\left\{-\frac{P(\iota)}{r(\iota)U(\chi(\iota))}\left(\frac{\varpi_2(\iota)}{\aleph(\iota)}-\eta(\iota)\right)-Q(\iota)g(\chi'(\iota))+\eta'(\iota)-\right.\\&\left.\frac{z'(\chi(\iota))}{r(\iota)U(\chi(\iota))}\left(\frac{\varpi_2(\iota)}{\aleph(\iota)}-\eta(\iota)\right)^2\right\}+\frac{\aleph'(\iota)}{\aleph(\iota)}\varpi_2(\iota),\end{aligned}$$

for all $\iota\geq l_0$.

$$\varpi_2'(\iota)\leq\aleph(\iota)\left\{-\frac{P(\iota)}{r(\iota)U(\chi(\iota))}\left(\frac{\varpi_2(\iota)}{\aleph(\iota)}-\eta(\iota)\right)-\gamma_2 Q(\iota)+\eta'(\iota)-\frac{\gamma_1}{r(\iota)U(\chi(\iota))}\left(\frac{\varpi_2(\iota)}{\aleph(\iota)}-\eta(\iota)\right)^2\right\}+\frac{\aleph'(\iota)}{\aleph(\iota)}\varpi_2(\iota). \tag{15}$$

This with (14) and (A3) leads to

$$\varpi_2'(\iota) \leq -\Phi(\iota) - \frac{\gamma_1}{r(\iota)\xi_2(\iota)\aleph(\iota)}\varpi_2^2(\iota), \tag{16}$$

for all $\iota \geq l_0$. Applying $(i)-(ii)$, multiplying (16) by $M(\iota,\Theta)$, and integrating from $l$ to $\iota$, we obtain, for every $\iota \geq l \geq l_0$,

$$\int_l^\iota M(\iota,\Theta)\Phi(\Theta)d\Theta \leq -\int_l^\iota M(\iota,\Theta)\varpi_2'(\Theta)d\Theta - \int_l^\iota M(\iota,\Theta)\frac{\gamma_1}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)d\Theta$$

$$= -M(\iota,\Theta)\varpi_2(\Theta)|_l^\iota - \int_l^\iota \left[-\frac{\partial M(\iota,\Theta)}{\partial\Theta}\varpi_2(\Theta) + M(\iota,\Theta)\frac{\gamma_1}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)\right]d\Theta$$

$$= M(\iota,l)\varpi_2(l) - \int_l^\iota \left[m(\iota,\Theta)\sqrt{M(\iota,\Theta)}\varpi_2(\Theta) + Y(\iota,\Theta)\frac{\gamma_1}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)\right]d\Theta.$$

For any $\beta > 1$, we obtain

$$\begin{aligned}\int_l^\iota M(\iota,\Theta)\Phi(\Theta)d\Theta \leq M(\iota,l)\varpi_2(l) - \int_l^\iota \left(\sqrt{\frac{\gamma_1 M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}}u_2(\Theta) + \sqrt{\frac{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{4\gamma_1}}m(\iota,\Theta)\right)^2 d\Theta \\ + \frac{\beta}{4\gamma_1}\int_l^\iota \xi_2(\Theta)r(\Theta)\aleph(\Theta)m^2(\iota,\Theta)d\Theta - \int_l^\iota \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)d\Theta,\end{aligned} \tag{17}$$

and for all $\iota \geq l \geq l_0$,

$$\begin{aligned}&\int_l^\iota \left[M(\iota,\Theta)\Phi(\Theta) - \frac{\beta}{4\gamma_1}\xi_2(\Theta)r(\Theta)\aleph(\Theta)m^2(\iota,\Theta)\right]d\Theta \leq \\ &M(\iota,l)\varpi_2(l) - \int_l^\iota \left(\sqrt{\frac{\gamma_1 M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}}\varpi_2(\Theta) + \sqrt{\frac{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{4\gamma_1}}m(\iota,\Theta)\right)^2 d\Theta \\ &- \int_l^\iota \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)d\Theta,\end{aligned} \tag{18}$$

From (18), it follows that for any $\iota > l \geq l_0$,

$$\begin{aligned}&\frac{1}{M(\iota,l)}\int_l^\iota \left[M(\iota,\Theta)\Phi(\Theta) - \frac{\beta}{4\gamma_1}\xi_2(\Theta)r(\Theta)\aleph(\Theta)m^2(\iota,\Theta)\right]d\Theta \leq \\ &\varpi_2(l) - \frac{1}{M(\iota,l)}\int_l^\iota \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)d\Theta \leq \\ &\varpi_2(l) - \frac{1}{M(\iota,l)}\int_l^\iota \left(\sqrt{\frac{\gamma_1 M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}}\varpi_2(\Theta) + \sqrt{\frac{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{4\gamma_1}}m(\iota,\Theta)\right)^2 d\Theta \\ &- \frac{1}{M(\iota,l)}\int_l^\iota \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{\varpi_2}^2(\Theta)d\Theta,\end{aligned}$$

and

$$\lim_{\iota\to\infty} \sup \frac{1}{M(\iota,l)} \int\limits_{l}^{\iota} \left[ M(\iota,\Theta)\Phi(\Theta) - \frac{\beta}{4\gamma_1}\xi_2(\Theta)r(\Theta)\aleph(\Theta)m^2(\iota,\Theta) \right] d\Theta \leq \varpi_2(l) - \lim_{l\to\infty} \inf \frac{1}{M(\iota,l)} \int\limits_{l}^{\iota} \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)} {\varpi_2}^2(\Theta)d\Theta \tag{19}$$

In view of (19), for all $l \geq l_0$,

$$\varpi_2(l) \geq F(l) + \lim_{l\to\infty} \inf \frac{1}{M(\iota,l)} \int\limits_{l}^{\iota} \frac{\gamma_1(\beta-1)M(\iota,\Theta)}{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)} {\varpi_2}^2(\Theta)d\Theta.$$

Consequently,

$$\varpi_2(l) \geq F(l), \quad for\, all\, l \geq l_0, \tag{20}$$

and

$$\lim_{\iota\to\infty} \inf \frac{1}{M(\iota,l_0)} \int\limits_{l}^{\iota} \frac{M(\iota,\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} {\varpi_2}^2(\Theta)d\Theta \leq \frac{\beta}{\gamma_1(\beta-1)}(\varpi_2(l_0) - F(l_0)) < \infty \tag{21}$$

Now, to show that

$$\int\limits_{l_0}^{\infty} \frac{{\varpi_2}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} d\Theta < \infty, \tag{22}$$

we assume the contrary, that

$$\int\limits_{l_0}^{\infty} \frac{{\varpi_2}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} d\Theta = \infty. \tag{23}$$

From (A4), it may be deduced that constant $\delta > 0$ such that

$$\inf_{\Theta\geq\iota_0} \left[ \lim_{\iota\to\infty} inf \frac{M(\iota,\Theta)}{M(\iota,\iota_0)} \right] > \delta. \tag{24}$$

Conversely, by considering (23), there is a $l_1 > l_0$ such that, for any positive number $\sigma$,

$$\int\limits_{l_0}^{\iota} \frac{{\varpi_2}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} d\Theta \geq \frac{\sigma}{\delta}, \quad for\, all\, \iota \geq l_1.$$

Thus, by the integration by parts, we obtain, for all $\iota \geq l_1$,

$$\frac{1}{M(\iota,\iota_0)} \int\limits_{l_0}^{\iota} M(\iota,\Theta) \frac{{\varpi_2}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} d\Theta =$$

$$\frac{1}{M(\iota,\iota_0)} \int\limits_{l_0}^{\iota} \left[ -\frac{\partial M(\iota,\Theta)}{\partial\Theta} \right] \left[ \int\limits_{l_0}^{\Theta} \frac{{\varpi_2}^2(\varepsilon)}{\xi_2(\varepsilon)r(\varepsilon)\aleph(\varepsilon)} d\varepsilon \right] d\Theta$$

$$\geq \frac{1}{M(\iota,\iota_0)} \int\limits_{l_1}^{\iota} \left[ -\frac{\partial M(\iota,\Theta)}{\partial\Theta} \right] \left[ \int\limits_{l_0}^{\Theta} \frac{{\varpi_2}^2(\varepsilon)}{\xi_2(\varepsilon)r(\varepsilon)\aleph(\varepsilon)} d\varepsilon \right] d\Theta$$

Thus,

$$\frac{1}{M(\iota,\iota_0)}\int_{l_0}^{\iota} M(\iota,\Theta)\frac{\varpi_2{}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta \geq \frac{\sigma}{\delta}\frac{1}{M(\iota,\iota_0)}\int_{l_1}^{\iota}\left[-\frac{\partial M(\iota,\Theta)}{\partial\Theta}\right]d\Theta = \frac{\sigma}{\delta}\frac{M(\iota,l_1)}{M(\iota,\iota_0)}. \tag{25}$$

From (24), it is obvious that

$$\lim_{\iota\to\infty} inf\frac{M(\iota,\Theta)}{M(\iota,\iota_0)} > \delta > 0,$$

and there exists $l_2 \geq l_1$ such that

$$\frac{M(\iota,l_1)}{M(\iota,\iota_0)} \geq \delta, \quad for\, all\, \iota \geq l_2.$$

Thus, using (25),

$$\frac{1}{M(\iota,l_0)}\int_{l_0}^{\iota} M(\iota,\Theta)\frac{\varpi_2{}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta \geq \sigma, \quad for\, all\, \iota \geq l_2.$$

For an arbitrary constant $\sigma$, we obtain

$$\lim_{\iota\to\infty} inf\frac{1}{M(\iota,l_0)}\int_{l_0}^{\iota} M(\iota,\Theta)\frac{\varpi_2{}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta = \infty,$$

which contradicts (21). Thus, (22) should hold, and in view of (20), we obtain

$$\int_{l_0}^{\infty}\frac{F_+{}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta \leq \int_{l_0}^{\infty}\frac{\varpi_2{}^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta < \infty,$$

which contradicts (11) and completes the proof. □

**Corollary 1.** *Assume that (A1)–(A2) and (A4) hold. If for any $\iota \geq l \geq \iota_0$ and for some $\beta > 1$, we have*

$$\lim_{\iota\to\infty}\sup\left[\frac{1}{M(\iota,l)}\int_{l}^{\iota}\left(M(\iota,\Theta)\Phi(\Theta) - \frac{\beta\xi_2(\Theta)\aleph(\Theta)r(\Theta)}{4\gamma_1}m^2(\iota,\Theta)\right)d\Theta\right] \geq F(l), \tag{26}$$

*and*

$$\lim_{\iota\to\infty}\sup\int_{\iota_0}^{\iota}\frac{F_+^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta = \infty, \tag{27}$$

*where*

$$\aleph(\iota) = exp\left[-2\gamma_1\int^{\iota}\frac{1}{\xi_2(\Theta)}\left(\frac{\eta(\Theta)}{r(\Theta)} - \frac{P(\Theta)}{2\gamma_1 r(\Theta)}\right)d\Theta\right], \tag{28}$$

*and*

$$\Phi(\iota) = \aleph(\iota)\left[Q(\iota) + \frac{\gamma_1\eta^2(\iota)}{\xi_2(\iota)r(\iota)} - \frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)} - \eta'(\iota) + \left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\frac{P^2(\iota)}{4\gamma_1 r(\iota)}\right] \tag{29}$$

*for some $\gamma_1 > 0$, then (2) oscillates.*

**Remark 2.** *In the special case of $\xi_1(\iota) = C_1$ and $\xi_2(\iota) = C_2$, the criteria of Corollary 1 include those of Theorem 2 in [5] and Theorem 3 in [23].*

Now, we consider the case of non-monotonic function $z(x(\iota))$.

**Theorem 3.** *Suppose that (A1), (A4), (A5) and (11) hold. Assume further that there exist functions $M \in W$, $\eta(\iota) \in C^1([\iota_0,\infty),\mathbb{R})$, $F \in C([\iota_0,\infty)$, and $Q(\iota) > 0$. If for some $\beta > 1$, and all $\iota > l \geq \iota_0$, we have*

$$\lim_{\iota\to\infty} \sup \frac{1}{M(\iota,l)} \int_l^\iota \left( M(\iota,\Theta)\Phi(\Theta) - \frac{\beta\xi_2(\Theta)r(\Theta)\aleph(\Theta)}{4} m^2(\iota,\Theta) \right) d\Theta \geq F(l), \tag{30}$$

*where*

$$\aleph(\iota) = \left( -2\int^\iota \frac{1}{r(\Theta)\xi_2(\Theta)} \left[ \eta(\Theta) - \frac{1}{2}P(\Theta) \right] d\Theta \right) \tag{31}$$

*and*

$$\Phi(\iota) = \aleph(\iota)\left( \gamma_3 Q(\iota) + \frac{\eta^2(\iota)}{\xi_2(\iota)r(\iota)} - \frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)} - \eta'(\iota) + \left[ \frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)} \right] \frac{P^2(\iota)}{4r(\iota)} \right), \tag{32}$$

*then Equation (2) is oscillatory.*

**Proof.** The proof is similar to the proof of Theorem 2 and so it is omitted. □

## 3. Examples

In this section, we introduce three examples which numerically justify our analytic results.

**Example 1.** *Consider the non-linear second-order differential equation*

$$\left( \iota^2 \frac{2+\chi^4(\iota)}{1+\chi^4(\iota)} \chi'(\iota) \right)' + 2(e^\iota + sin\iota)\chi'(\iota) + Q(\iota)\left( \chi(\iota) - \frac{1}{\chi(\iota)} \right)\left( 2 + \frac{1}{(\chi'(\iota))^2} \right) = 0, \tag{33}$$
$$for \quad \iota > 0.$$

*Here, $r(\iota) = \iota^2$, $P(\iota) = 2(e^\iota + sin\iota)$, $g(\chi'(\iota)) = 2 + \frac{1}{(\chi'(\iota))^2}$ and $z(\chi(\iota)) = \chi(\iota) - \frac{1}{\chi(\iota)}$. So, $z'(\chi) = 1 + \frac{1}{\chi^2(\iota)} \geq 1 = \gamma_1$ and $g(\chi'(\iota)) = 2 + \frac{1}{(\chi'(\iota))^2} \geq 2 = \gamma_2$.*

*Moreover, $\xi_1(\iota) = \frac{1}{1+\iota^2} \leq U(\chi(\iota)) = \frac{2+\chi^4(\iota)}{1+\chi^4(\iota)} = 1 + \frac{1}{1+\chi^4(\iota)} \leq 2 + \frac{1}{1+\iota^2} = \xi_2(\iota)$.*

*Now, we let $\eta(\iota) = e^\iota + sin\iota$; then,*

$$\begin{aligned} \aleph(\iota) &= exp\left[ -2\gamma_1 \int^\iota \frac{1}{r(\Theta)\xi_2(\Theta)} \left( \eta(\Theta) - \frac{P(\Theta)}{2\gamma_1} \right) d\Theta \right] \\ &= exp(0) = 1 \end{aligned}$$

*and*

$$\begin{aligned} \Phi(\iota) &= \aleph(\iota)\left[ \gamma_2 Q(\iota) + \frac{\gamma_1\eta^2(\iota)}{\xi_2(\iota)r(\iota)} - \frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)} - \eta'(\iota) + \left( \frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)} \right) \frac{P^2(\iota)}{4\gamma_1 r(\iota)} \right] \\ &= 2Q(\iota) - (e^\iota + cos\iota) + \frac{(e^\iota + sin\iota)^2}{e^\iota\xi_2(\iota)} - \frac{2(e^\iota + sin\iota)^2}{e^\iota\xi_2(\iota)} + \left( \frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)} \right) \frac{4(e^\iota + sin\iota)^2}{4e^\iota} \\ &= 2Q(\iota) - e^\iota - cos\iota - \frac{(1+\iota^2)}{e^\iota}(e^\iota + sin\iota)^2. \end{aligned}$$

*Choosing $Q(\iota) = 1 + \frac{1}{2}\left[ e^\iota + cos\iota + e^{-\iota}(1+\iota^2)(e^\iota + sin\iota)^2 \right]$; then, $\Phi(\iota) = 2$.*

*Now, for $\beta = 1$, we have*

$$\lim_{\iota\to\infty} sup\, \iota^{1-n} \int_l^\iota \left[ (\iota-\Theta)^{n-1}\Phi(s) - \frac{\beta\xi_2(\iota)(n-1)^2}{4\gamma_1}(\iota-\Theta)^{n-3}\aleph(\Theta)r(\Theta) \right] d\Theta$$

$$= \lim_{\iota\to\infty} sup\, \frac{1}{\iota^2} \int_l^\iota \left[ 2(\iota-\Theta)^2 - \Theta^2\left(2 + \frac{1}{1+\Theta^2}\right) \right] d\Theta$$

$$= -2l.$$

*Since*

$$\frac{F_+^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)} d\Theta = O(\iota^2) \qquad as\ \iota \to \infty,$$

*then (11) is satisfied, and by Theorem 2, Equation (33) oscillates.*

**Example 2.** *Consider the non-linear second-order differential equation*

$$\left[ (1+\iota^2)\left(\frac{2+\chi^2(\iota)}{1+\chi^2(\iota)}\right)\chi'(\iota) \right]' + 2\iota\sqrt{\iota^2+1}\chi'(\iota) + Q(\iota)[\chi(\iota)+\chi^3(\iota)] = 0, \quad for\ \iota > 0 \tag{34}$$

*Here, $r(\iota) = 1+\iota^2$, $P(\iota) = 2\iota\sqrt{\iota^2+1}$ and $z(\chi(\iota)) = \chi(\iota)+\chi^3(\iota)$. So, $z'(\chi) = 1 + 3\chi^2(\iota) \geq 1 = \gamma_1$.*

*Also, $\xi_1(\iota) = \frac{1}{1+\iota^2} \leq U(\chi(\iota)) = \frac{2+\chi^2(\iota)}{1+\chi^2(\iota)} = 1 + \frac{1}{1+\chi^2(\iota)} \leq 2 + \frac{1}{1+\iota^2} = \xi_2(\iota)$.*

*Now, we let $\eta(\iota) = \iota\sqrt{\iota^2+1}$. Then,*

$$\begin{aligned} \aleph(\iota) &= exp\left[ -2\gamma_1 \int^\iota \frac{1}{r(\Theta)\xi_2(\Theta)}\left(\eta(\Theta) - \frac{P(\Theta)}{2\gamma_1}\right) d\Theta \right] \\ &= exp(0) = 1 \end{aligned}$$

*and*

$$\begin{aligned} \Phi(\iota) &= \aleph(\iota)\left[ Q(\iota) + \frac{\gamma_1\eta^2(\iota)}{\xi_2(\iota)r(\iota)} - \frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)} - \eta'(\iota) + \left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\frac{P^2(\iota)}{4\gamma_1 r(\iota)} \right] \\ &= Q(\iota) + \frac{\iota^2(\iota^2+1)}{\xi_2(\iota)(\iota^2+1)} - \frac{2\iota^2(\iota^2+1)}{\xi_2(\iota)(\iota^2+1)} - \sqrt{\iota^2+1} - \frac{\iota^2}{\sqrt{\iota^2+1}} \\ &\quad + \left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\frac{4\iota^2(\iota^2+1)}{4(\iota^2+1)} \\ &= Q(\iota) - \sqrt{\iota^2+1} - \frac{\iota^2}{\sqrt{\iota^2+1}} - \iota^2(\iota^2+1). \end{aligned}$$

*Choosing $Q(\iota) = 2 + \iota^4 + \iota^2 + \sqrt{\iota^2+1} + \frac{\iota^2}{\sqrt{\iota^2+1}}$; then, $\Phi(\iota) = 2$.*

*Now, for $\beta = 1$, we have*

$$\lim_{\iota\to\infty} sup\, \iota^{1-n} \int_l^\iota \left[ (\iota-\Theta)^{n-1}\Phi(\Theta) - \frac{\beta\xi_2(\iota)(n-1)^2}{4\gamma_1}(\iota-\Theta)^{n-3}\aleph(\Theta)r(\Theta) \right] d\Theta$$

$$\begin{aligned} &= \lim_{\iota\to\infty} sup\, \frac{1}{\iota^2} \int_l^\iota \left[ 2(\iota-\Theta)^2 - (1+\Theta^2)\left(2 + \frac{1}{1+\Theta^2}\right) \right] d\Theta \\ &= \lim_{\iota\to\infty} sup\, \frac{1}{\iota^2}\left[ -3\iota + 3l + \frac{2}{3}(\iota-l)^3 - \frac{2}{3}(\iota^3 - l^3) \right] \\ &= -2l = F(l). \end{aligned}$$

*Since*

$$\frac{F_+^2(\Theta)}{\xi_2(\Theta)r(\Theta)\aleph(\Theta)}d\Theta = O(\iota^2) \qquad as\, \iota \to \infty, \tag{35}$$

*then Condition (27) is satisfied, and so by Corollary 1, Equation (34) oscillates.*

**Example 3.** *Consider the non-linear second-order differential equation*

$$\left[\iota^2\left(\frac{1}{2}+\frac{e^{-|\chi|}}{2}\right)\chi'(\iota)\right]' + 2\iota^2\chi'(\iota) + Q(\iota)\chi(\iota)\left(\frac{1}{9}+\frac{1}{\chi^2+1}\right) = 0, \ for\, \iota > 0 \tag{36}$$

*From the given example, we have* $r(\iota) = \iota^2$, $P(\iota) = 2\iota^2$ *and* $z(\chi(\iota)) = \chi(\iota)\left(\frac{1}{9}+\frac{1}{\chi^2+1}\right)$. *So,* $\frac{z(\chi)}{\chi} = \frac{1}{9} + \frac{1}{\chi^2+1} \geq \frac{1}{9} = \gamma_3$.

*Also,* $\xi_1(\iota) = \frac{2}{4+\iota^2} \leq U(\chi(\iota)) = \frac{1}{2} + \frac{e^{-|\chi|}}{2} \leq 1 + \frac{1}{4+\iota^2} = \xi_2(\iota)$.

*Now, we let* $\eta(\iota) = \iota^2$. *Then,*

$$\begin{aligned}\aleph(\iota) &= exp\left[-2\int^{\iota}\frac{1}{r(\Theta)\xi_2(\Theta)}\left(\eta(\Theta) - \frac{P(\Theta)}{2}\right)d\Theta\right]\\ &= exp(0) = 1\end{aligned}$$

*and*

$$\begin{aligned}\Phi(\iota) &= \aleph(\iota)\left[\gamma_3 Q(\iota) + \frac{\gamma_3\eta^2(\iota)}{\xi_2(\iota)r(\iota)} - \frac{P(\iota)\eta(\iota)}{\xi_2(\iota)r(\iota)} - \eta'(\iota) + \left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\frac{P^2(\iota)}{4r(\iota)}\right]\\ &= \gamma_3 Q(\iota) + \frac{\iota^4}{\iota^2\xi_2(\iota)} - \frac{2\iota^4}{\iota^2\xi_2(\iota)} - 2\iota + \iota^2\left(\frac{1}{\xi_2(\iota)} - \frac{1}{\xi_1(\iota)}\right)\\ &= \gamma_3 Q(\iota) - 2\iota - \frac{\iota^2}{2}(4+\iota^2).\end{aligned}$$

*Choosing* $Q(\iota) = \frac{9}{2}\iota^4 + 18\iota^2 + 18\iota + 9$, *we obtain* $\Phi(\iota) = 1$.

*Now, for* $\beta = 1$

$$\begin{aligned}&\lim_{\iota\to\infty} sup\, \iota^{1-n}\int_l^{\iota}\left[(\iota-\Theta)^{n-1}\Phi(\Theta) - \frac{\beta\xi_2(\iota)(n-1)^2}{4}(\iota-\Theta)^{n-3}\aleph(\Theta)r(\Theta)\right]d\Theta\\ &= \lim_{\iota\to\infty} sup\,\frac{1}{\iota^2}\int_l^{\iota}\left[(\iota-\Theta)^2 - \Theta^2\left(1+\frac{1}{4+\Theta^2}\right)\right]d\Theta\\ &= \lim_{\iota\to\infty} sup\frac{1}{\iota^2}\left[-\iota^2 l + \iota l^2 - \iota + 2tan^{-1}\left(\frac{\iota}{2}\right) + l - 2tan^{-1}\left(\frac{l}{2}\right)\right]\\ &= -l = F(l).\end{aligned}$$

*It is clear that Conditions (27) and (35) are satisfied, and so by Theorem 3, Equation (36) is oscillatory.*

## 4. Conclusions

In this article, we discussed the oscillation of two general classes of second-order nonlinear differential equations o types (1) and (2). New oscillation criteria are given using a new condition on function $U(\chi(\iota))$ in the case of monotonic and non-monotonic function $z(\chi)$, which is more general than the traditional restriction used by other authors. Three examples were presented to justify our results of the article. In future work, we plan to consider the case of delayed differential equations by which we can produce richer results; see [3,6,19,24].

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## References


1. Abdullah, H.K. A note on the oscillation of the second order differential equations. *Czech. Math. J.* **2004**, *54*, 949–954. [CrossRef]
2. Attia, E.R.; Benekas, V.; El-Morshedy, H.A.; Stavroulakis, I.P. Oscillation of first order linear differential equations with several non-monotone delays. *Open Math.* **2018**, *16*, 83–94. [CrossRef]
3. Braverman, E.; Domoshnitsky, A.; Stavroulakis, J.I. Semicycles and correlated asymptotics of oscillatory solutions to second-order delay differential equations. *J. Math. Anal. Appl.* **2024**, *531*, 127875. [CrossRef]
4. Elabbasy, E.M.; Hassan, T.S.; Saker, S.H. Oscillation of second-order nonlinear differential equations with a damping term. *Electron. J. Differ. Equ.* **2005**, *76*,1–13. [CrossRef]
5. El-Sheikh, M.M.A. Oscillation and nonoscillation criteria for second order nonlinear differential equations. *J. Math. Anal. Appl.* **1993**, *179*, 14–27. [CrossRef]
6. El-Sheikh, M.M.A.; Sallam, R.A.; El-Saedy, E.I. On the oscillatory behavior of solutions of second order nonlinear neutral delay differential equations. *WSEAS Trans. Math.* **2018**, *17*, 51–57.
7. Euat Tallah, S.; El-sheikh, M.M.A.; Ismail, G.A.F. On the oscillation and non-oscillation of solutions of forced second order differential equations. *J. Math. Comput. Sci.* **2023**, *30*, 101–115. [CrossRef]
8. Grace, S.R.; Lalli, B.S. Oscillation and convergence to zero of solutions of damped second-order nonlinear differential equations. *J. Math. Anal. Appl.* **1984**, *102*, 539–548. [CrossRef]
9. Grace, S.R.; Lalli, B.S. Oscillation theorems for second order nonlinear differential equations. *J. Math. Anal. Appl.* **1987**, *124*, 213–224. [CrossRef]
10. Grace, S.R.; Lalli, B.S. Oscillations in second order differential equations with alternating coefficients. *Period. Math. Hungar.* **1988**, *19*, 69–78. [CrossRef]
11. Grace, S.R. Oscillation theorems for second order nonlinear differential equations with damping. *Math. Nachr.* **1989**, *141*, 117–127. [CrossRef]
12. Grace, S.R.; Lalli, B.S. Oscillation theorems for perturbed nonlinear second order differential equations with a damping term. *Comment. Math. Univ. Carolin.* **1989**, *30*, 691–697.
13. Grace, S.R.; Lalli, B.S. Integral averaging technique for the oscillation of second order nonlinear differential equations. *J. Math. Anal. Appl.* **1990**, *149*, 277–311. [CrossRef]
14. Grace, S.R. Oscillation theorems for nonlinear differential equations of second order. *J. Math. Anal. Appl.* **1992**, *171*, 220–241. [CrossRef]
15. Jiang, C.; Tian, Y.; Jiang, Y.; Li, T. Some oscillation results for nonlinear second-order differential equations with damping. *Adv. Differ. Equ.* **2015**, *354*, 1–14. [CrossRef]
16. Kirane, M.; Rogovchenko, Y.V. Oscillation results for a second order damped differential equation with nonmonotonous nonlinearity. *J. Math. Anal. Appl.* **2000**, *250*, 118–138. [CrossRef]
17. Lalli, B.S.; Grace, S.R. *Oscillations of Second Order General Nonlinear Differential Equations in Recent Trends in Differential Equations*; Agarwal, R.P., Ed.; World Scientific: Singapore, 1992; pp. 413–429.
18. Manojlovic, J.V. Oscillation criteria for sublinear differential equations with damping. *Acta Math. Hungar.* **2004**, *104*, 153–169. [CrossRef]
19. Meng, Q.; Yan, J. Bounded oscillation for second order non-linear neutral delay differential equations in critical and non-critical cases. *Nonlinear Anal. Theory Methods Appl.* **2006**, *64*, 1543–1561. [CrossRef]
20. Philos, C.G. Oscillation Theorems for Linear Differential Equations of Second Order. *Arch. Math.* **1989**, *53*, 482–492. [CrossRef]
21. Rogovchenko, Y.V. Oscillation theorems for second-order equations with damping. *Nonlinear Anal.* **2000**, *41*, 1005–1028. [CrossRef]
22. Rogovchenko, S.P.; Rogovchenko, Y.V. Oscillation of differential equations with damping. *Dyn. Contin. Discret. Impuls. Syst. Ser. Math. Anal.* **2003**, *10*, 447–461.
23. Rogovchenko, Y.V.; Tuncay, F. Oscillation theorems for a class of second order nonlinear differential equations with damping. *Taiwan. J. Math.* **2009**, *13*, 1909–1928. [CrossRef]
24. Temimi, H.; Ben-Romdhane, M.; El-Borgi, S.; Cha, Y.J. Time-delay effects on controlled seismically excited linear and nonlinear structures. *Int. J. Struct. Stab. Dyn.* **2016**, *16*, 1550031. [CrossRef]
25. Tiryaki, A.; Zafer, A. Oscillation criteria for second order nonlinear differential equations with damping. *Turk. J. Math.* **2000**, *24*, 185–196.
26. Wang, X.; Song, G. Oscillation theorems for a class of nonlinear second order differential equations with damping. *Adv. Pure Math.* **2013**, *3*, 226–233. [CrossRef]

27. Yan, J. Oscillatory Theorems for Second Order Linear Differential Equations with Damping. *Proc. Am. Math. Soc.* **1986**, *98*, 276–282. [CrossRef]
28. Zhang, Q.; Yan, J. Oscillatory Behavior of the Second Order Nonlinear Delay Differential Equation. *Pure Appl. Math.* **2008**, *24*, 646–653.
29. Zhang, Q.; Song, X.; Liu, S. New Oscillation Criteria for the Second Order Nonlinear Differential Equations with Damping. *J. Appl. Math. Phys.* **2016**, *4*, 1179–1185. [CrossRef]