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On Fall-Colorable Graphs

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Abstract: A fall k-coloring of a graph G is a proper k-coloring of G such that each vertex has at least one neighbor in each of the other color classes. A graph G which has a fall k-coloring is equivalent to having a partition of the vertex set $V(G)$ in k independent dominating sets. In this paper, we first prove that for any fall k-colorable graph G with order n, the number of edges of G is at least $(n(k-1)+r(k-r))/2$, where $r \equiv n \pmod{k}$ and $0 \leq r \leq k-1$, and the bound is tight. Then, we obtain that if G is k-colorable ($k \geq 2$) and the minimum degree of G is at least $\frac{k-2}{k-1}n$, then G is fall k-colorable and this condition of minimum degree is the best possible. Moreover, we give a simple proof for an NP-hard result of determining whether a graph is fall k-colorable, where $k \geq 3$. Finally, we show that there exist an infinite family of fall k-colorable planar graphs for $k \in \{5, 6\}$.

Keywords: fall k-coloring; fall k-colorable graph; computational complexity; domination problem

MSC: 05C15; 05C69

1. Introduction

In this paper, we only consider simple and undirected graphs. For a graph $G = (V(G), E(G))$, we use $V(G)$ and $E(G)$ to represent the sets of vertices and edges of G, respectively. We use $d_G(v)$ to represent the degree of a vertex $v \in V(G)$, that is, the number of neighbors of v in G. If $d_G(v) = r$ for any $v \in V(G)$, then the graph G is called an r-regular graph. For a vertex $v \in V(G)$, let $N_G(v) = \{u : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and the *closed neighborhood* of v, respectively. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. When no confusion can arise, $N_G(v)$, $N_G[v]$, $\Delta(G)$, and $\delta(G)$ are simplified by $N(v)$, $N[v]$, Δ, and δ, respectively. A *plane graph* is a graph drawn in the plane such that its edges intersect only at their ends; a *planar graph* is a graph that can be drawn as a plane graph.

Let G be a graph. A (proper) *k-coloring* f of G is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ for any $uv \in E(G)$. Hence, a k-coloring can be regarded as a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$, where V_i denotes the set of vertices assigned color i, and is called a *color class* of f, where $i = 1, 2, \ldots, k$. If a graph G admits a k-coloring, the G is called *k-colorable*. The minimum number k such that G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$.

Let f be a k-coloring of a graph G. If a vertex $v \in V(G)$ has all colors in its closed neighborhood under f, namely $|f(N[v])| = k$, then the vertex v is called *colorful* with respect to f. Furthermore, the coloring f is called *colorful* whenever each of its color classes contains at least one colorful vertex. The maximum order of a colorful coloring of a graph G is called the b-chromatic number of G, and is denoted by $\varphi(G)$. A *fall k-coloring* of a graph G is a k-coloring of G such that every vertex is colorful.

The problem of b-chromatic numbers was introduced by Irving and Manlove in 1999 [1] and studied extensively in the literature (see the survey in [2]), whereas fall

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coloring was introduced in [3] and studied in [4–6]. It follows from [6] that fall coloring strongly chordal graphs is doable in polynomial time, even with an unbounded number of colors.

A *dominating set* in a graph G is a subset $S \subseteq V(G)$ such that each vertex in $V(G)$ is either in S or has at least one neighbor in S. If S is a dominating set and independent, then S is an independent dominating set (IDS) of G. The independent domination number $\gamma_i(G)$ is the minimum cardinality of an IDS of G. A graph G has a fall k-coloring if and only if $V(G)$ can be partitioned into k independent dominating sets [7].

Note that a graph may have no fall coloring. For instance, the cycle C_n has a fall coloring only when n is a multiple of three or is even [3]. Hence, determining which graphs are fall-colorable is an interesting problem. In fact, in 1976 Cockayne and Hedetniemi [7] first studied fall-colorable graphs but used another term, *indominable graphs*. They found several families of graphs which have fall colorings.

In this paper, we further discuss fall-colorable graphs. First, the size of a k-colorable graph is determined, including the boundaries. Then, a sufficient condition of a graph to be k-colorable ($k \geq 2$) is proposed and the tightness of this condition is discussed. Moreover, we give a simple proof for an NP-hard result of determining whether a graph is fall k-colorable, where $k \geq 3$. Finally, we show that there exist an infinite family of fall k-colorable planar graphs for $k \in \{5, 6\}$ and find some sufficient conditions for a maximal planar graph to be fall-colorable.

For other notations and terminologies in graph theory, we refer to [8].

2. Some Properties of Fall-Colorable Graphs

In this section, we discuss some properties of fall k-colorable graphs. The following, Lemmas 1 and 2, can be obtained straight from previous studies, such as [3,7].

Lemma 1 ([3])**.** *Let G be a fall k-colorable graph and f a fall k-coloring. We have the following:*

(i) $\delta(G) \geq k-1$;

(ii) The subgraph induced by the union of any r color classes under f is fall r-colorable, where $r \leq k$.

Lemma 2 ([7])**.** *A graph G is fall k-colorable if and only if G has a k-coloring such that the subgraph induced by the union of any two color classes has no isolated vertices.*

Theorem 1. *Let G be a fall k-colorable graph of order n. Then,*

$$|E(G)| \geq \frac{n(k-1) + r(k-r)}{2},$$

where $r \equiv n \pmod{k}$ and $0 \leq r \leq k-1$.

Proof. Let $f = (V_1, V_2, \ldots, V_k)$ be a fall k-coloring of G and $|V_i| = n_i$, where $i = 1, 2, \ldots, k$. Then, $n = n_1 + n_2 + \cdots + n_k$. Without a loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. For any two color classes V_i and V_j with $i < j$, by Theorem 2, we know that the subgraph $G_{i,j}$ induced by $V_i \cup V_j$ has no isolated vertices. Since $G_{i,j}$ is a bipartite graph, we have $|E(G_{i,j})| \geq |V_j| = n_j$. Hence,

$$\begin{aligned} |E(G)| &= \sum_{1\leq i<j\leq k} |E(G_{i,j})| \geq \sum_{1\leq i<j\leq k} n_j \\ &= n_2 + 2n_3 + \cdots + (k-1)n_k \\ &= (\sum_{i=1}^{k} i \cdot n_i) - n. \end{aligned} \tag{1}$$

Now, we prove that if $\sum_{i=1}^{k} i \cdot n_i$ is the minimum then $(V_1, V_2, \ldots, V_k)$ is an equitable partition of $V(G)$, namely $|n_p - n_q| \leq 1$, for any $p, q \in \{1, 2, \ldots, k\}$.

Suppose, to the contrary, that $(V_1, V_2, \ldots, V_k)$ is not an equitable partition of $V(G)$. Then, there exists $a \in \{1, 2, \ldots, k\}$ such that $n_{a+1} - n_a \geq 2$ or $b, c \in \{1, 2, \ldots, k\}$ with $b < c$ such that $n_{b+1} - n_b = n_{c+1} - n_c = 1$. If the former occurs, let $n'_a = n_a + 1$, $n'_{a+1} = n_{a+1} - 1$, and $n'_i = n_i$ for any $i \in \{1, 2, \ldots, k\} \setminus \{a, a+1\}$. Then,

$$\begin{aligned}
\sum_{i=1}^{k} i \cdot n_i - \sum_{i=1}^{k} i \cdot n'_i &= \sum_{i=1}^{k} i \cdot (n_i - n'_i) \\
&= a(n_a - n'_a) + (a+1)(n_{a+1} - n'_{a+1}) \\
&= a(n_a - n_a - 1) + (a+1)(n_{a+1} - n_{a+1} + 1) \\
&= 1 > 0.
\end{aligned}$$

However, this contradicts the minimality of $\sum_{i=1}^{k} i \cdot n_i$.

If the latter occurs, let $n'_b = n_b + 1$, $n'_{c+1} = n_{c+1} - 1$, and $n'_i = n_i$ for any $i \in \{1, 2, \ldots, k\} \setminus \{b, c+1\}$. Similar to the former case, we can obtain $\sum_{i=1}^{k} i \cdot n_i - \sum_{i=1}^{k} i \cdot n'_i = c - b + 1 > 0$, which is a contradiction. Therefore, if $\sum_{i=1}^{k} i \cdot n_i$ is a minimum then $(V_1, V_2, \ldots, V_k)$ is an equitable partition of $V(G)$.

Let $n = kt + r$, where $r \equiv n \pmod{k}$ and $0 \leq r \leq k-1$. Now, we consider the case of $\sum_{i=1}^{k} i \cdot n_i$ as the minimum. Note that $n_1 \leq n_2 \leq \cdots \leq n_k$. It therefore follows that $n_1 = \cdots = n_{k-r} = t$ and $n_{k-r+1} = \cdots = n_k = t+1$. Hence,

$$\begin{aligned}
\sum_{i=1}^{k} i \cdot n_i &= \sum_{i=1}^{k-r} i \cdot t + \sum_{i=k-r+1}^{k} i \cdot (t+1) \\
&= \sum_{i=1}^{k} i \cdot t + \sum_{i=k-r+1}^{k} i \\
&= t \cdot \frac{k(k+1)}{2} + \frac{r(2k-r+1)}{2} \\
&= \frac{n-r}{k} \cdot \frac{k(k+1)}{2} + \frac{r(2k-r+1)}{2} \\
&= \frac{n(k+1) + r(k-r)}{2}.
\end{aligned} \tag{2}$$

Together with Formulae (1) and (2), we have

$$\begin{aligned}
|E(G)| &\geq (\sum_{i=1}^{k} i \cdot n_i) - n \\
&\geq \frac{n(k+1) + r(k-r)}{2} - n \\
&= \frac{n(k-1) + r(k-r)}{2}.
\end{aligned}$$

□

Theorem 2. *For any fall k-colorable graph G with order n, if G is $(k-1)$-regular, then $n \equiv 0 \pmod{k}$. Moveover, for any fall k-coloring f of G, each color class of f has exactly $\frac{n}{k}$ vertices.*

Proof. Let V_i be any color class of the fall k-coloring f of G. Then, for any two vertices u and v in V_i; we can obtain $N_G(u) \cap N_G(v) = \emptyset$. Otherwise, if there exists a vertex $x \in N_G(u) \cap N_G(v)$, since G is $(k-1)$-regular, we can deduce that x is adjacent to at most $k-2$ color classes, which implies that x is not a colorful vertex of f; this is a contradiction. Let $V_i = \{v_1, v_2, \ldots, v_t\}$. Then, $N_G[v_1], N_G[v_2], \ldots, N_G[v_t]$ is a t-partition of $V(G)$. Since

$|N_G[v_j]| = k$ for each $j = 1, 2, \ldots, t$, we have $n = kt$ and so $n \equiv 0 \pmod{k}$. Note that $|V_i| = t = \frac{n}{k}$, we can discover that each color class of f has exactly $\frac{n}{k}$ vertices. □

3. A Sufficient Condition

In 2010, Balakrishnan and Kavaskar [9] showed that any graph G with $\delta(G) \geq |V(G)| - 2$ admits a fall coloring. In this section, we improve this result by relaxing the condition $\delta(G) \geq |V(G)| - 2$ to $\delta(G) > \frac{k-2}{k-1}|V(G)|$ for any $k \geq 2$ and prove that the condition of $\delta(G)$ is the best possible. First, we give a useful lemma obtained by Zarankiewicz [10]:

Lemma 3 ([10])**.** *Let G be a k-colorable graph with n vertices and $\delta(G) > \frac{k-2}{k-1}n$, where $k \geq 2$. We have $\chi(G) = k$.*

Theorem 3. *Let G be a k-colorable graph with n vertices and $\delta(G) > \frac{k-2}{k-1}n$, where $k \geq 2$. Then, G is fall k-colorable.*

Proof. If $k = 2$, then $\delta(G) > \frac{k-2}{k-1}n = 0$ and G has no isolated vertices. Hence, G is fall 2-colorable.

Now, assume that $k \geq 3$. Let v be an arbitrary vertex of G and G_v be the subgraph of G induced by $N_G(v)$. Then, $|V(G_v)| = |N_G(v)| = d_G(v) > \frac{k-2}{k-1}n$. Hence, for any vertex x in G_v, we have

$$\begin{aligned} d_{G_v}(x) &> \frac{k-2}{k-1}n - (n - |V(G_v)|) \\ &= |V(G_v)| - \frac{1}{k-1}n \\ &> |V(G_v)| - \frac{1}{k-1} \cdot \frac{k-1}{k-2}|V(G_v)| \\ &= \frac{(k-1)-2}{(k-1)-1}|V(G_v)|. \end{aligned}$$

Note that G is k-colorable, so G_v is $(k-1)$-colorable. Hence, by Lemma 3, we can see that $\chi(G_v) = k-1$, which yields that $|f(N_G(v))| = k-1$ for any k-coloring f of G. That is to say, v is a colorful vertex with respect to f. Since v is an arbitrary vertex of G, we can deduce that f is a fall k-coloring of G. Hence, the graph G is fall k-colorable. □

Now, we show that the condition $\delta(G) > \frac{k-2}{k-1}n$ in Theorem 3 is the best possible. We will construct a family of graphs that are not fall k-colorable, G_ℓ, with $\delta(G_\ell) = \frac{k-2}{k-1}|V(G_\ell)|$.

We use K_n to denote the complete graph of order n and use $T_{r,s}$ to denote the complete r-partite graph with s vertices in each class, where $r \geq 2$. The *join* of two graphs G and H, denoted as $G \vee H$, is the graph obtained from the disjointed union of G and H, and we add edges joining every vertex of G to every vertex of H.

For any $k \geq 3$ and $\ell \geq 1$, let $G_\ell^1 = \overline{K_\ell}$, $G_\ell^2 = T_{2,\ell}$, $G_\ell^3 = T_{k-2,3\ell}$, and $G_\ell = (G_\ell^1 \cup G_\ell^2) \vee G_\ell^3$. For example, when $k = 4$ and $\ell = 1$, the graph G_ℓ is shown in Figure 1.

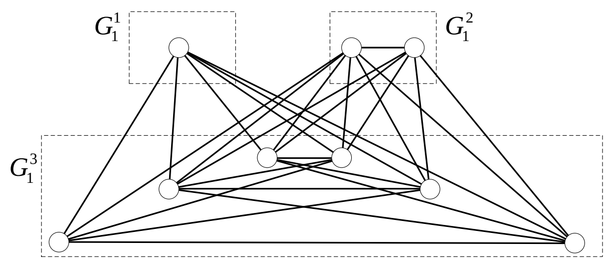

Figure 1. The graph G_ℓ when $k = 4$ and $\ell = 1$.

Then, $|V(G_\ell)| = |V(G_\ell^1)| + |V(G_\ell^2)| + |V(G_\ell^3)| = \ell + 2\ell + 3\ell(k-2) = 3\ell(k-1)$.

For any $v \in V(G_\ell^1) \cap V(G_\ell)$, $d_{G_\ell}(v) = 3\ell(k-2)$; for any $v \in V(G_\ell^2) \cap V(G_\ell)$, $d_{G_\ell}(v) = 3\ell(k-2) + \ell = (3k-5)\ell$; for any $v \in V(G_\ell^3) \cap V(G_\ell)$, $d_{G_\ell}(v) = 3\ell(k-3) + \ell + 2\ell = 3\ell(k-2)$. Hence, $\delta(G_\ell) = 3\ell(k-2) = \frac{k-2}{k-1}|V(G_\ell)|$.

Note that for any k-coloring of G_ℓ, $|f(V(G_\ell^3))| = k-2$. Hence, each vertex in $V(G_\ell^1)$ is not a colorful vertex with respect to f. So, G_ℓ is not fall k-colorable.

4. Complexity

The problem of determining whether a graph is fall k-colorable ($k \geq 3$) has been shown to be NP-complete [3,11–13]. In this section, we give a simple proof for the NP-complete result of the FALL k-COLORABLE problem, which is defined as follows:

FALL k-COLORABLE:
Instance: Given a graph $G = (V, E)$ and a positive integer k.
Question: Is G fall k-colorable?

k-COLORABLE:
Instance: Given a graph $G = (V, E)$ and a positive integer k.
Question: Is G k-colorable?

It is well known that the k-COLORABLE problem is NP-hard for any $k \geq 3$ [14]. We will prove that the fall k-colorable problem is NP-hard by using a reduction from the k-COLORABLE problem.

Theorem 4. *FALL k-COLORABLE is NP-complete for any $k \geq 3$.*

Proof. We show that the FALL k-COLORABLE problem is NP-complete by a reduction from k-COLORABLE. For any graph G of order n with the vertex set $\{v_1, v_2, \cdots, v_n\}$, we construct a graph G' as follows:

First, take n copies $K_k^1, K_k^2, \cdots, K_k^n$ of the complete graph K_k. Then, add these n copies of K_k to G and identify v_i and a vertex of K_k^i into a single vertex, where $i = 1, 2, \cdots, n$.

We claim that G is k-colorable if and only if G' is fall k-colorable.

Let G' be fall k-colorable. Let f' be a fall k-coloring of G'. By this definition, f' is a proper k-coloring of G'. Then, the restriction of f' to $V(G)$ is a k-coloring of G. So, G is k-colorable. Conversely, assume that G has a k-coloring f. By the construction of G', f can be extended to a k-coloring f' of G'. Since every vertex of G' belongs to a subgraph of G' which is isomorphic to K_k, we can see that f' is a fall k-coloring of G'. So, G' is fall k-colorable. □

Furthermore, the k-COLORABLE problem remains NP-hard under several restrictions. Garey and Johnson [15] proved the following:

Lemma 4 ([15])**.** *Three-COLORABLE is NP-complete even when restricted to planar graphs with a maximum degree of four.*

By Lemma 4 and using a similar approach to that in the proof of Theorem 4, we can obtain the following result.

Corollary 1. *FALL 3-COLORABLE is NP-complete even when restricted to planar graphs with a maximum degree of six.*

5. Fall Colorings of Planar Graphs

In this section, we discuss the fall colorings of planar graphs. Since $\delta(G) \leq 5$ for any planar graph G, it follows from Lemma 1 (i) that $\psi_f(G) \leq 6$. In [7], Cockayne and Hedetniemi found that each uniquely k-colorable graph is fall k-colorable. Note that for any integer $k \leq 4$, there exist an infinite family of planar graphs that are uniquely k-colorable [16–21], but uniquely five-colorable planar graphs do not exist [18]. Hence, there exist an infinite family of planar fall k-colorable graphs for any $k \leq 4$. Now, we show that there also exist an infinite family of planar fall k-colorable graphs for $k \in \{5, 6\}$.

We can see that the icosahedron G_{12} in Figure 2, which is a planar graph, has a fall five-coloring (Figure 2a) and a fall six-coloring (Figure 2b).

Theorem 5. *There exist an infinite family of planar fall k-colorable graphs for $k \in \{5, 6\}$.*

Proof. From the icosahedron G_{12}, we can construct a family of graphs $\{H_i\}$ as follows:

(1) $H_1 = G_{12}$;

(2) For integer $i \geq 2$, H_i can be obtained by embedding a copy of G_{12} in some interior face of H_{i-1} and identifying the boundaries of this face and the exterior face of G_{12}.

It can be checked that H_i is a planar graph of order $9i + 3$. Note that every three-coloring of the exterior triangle of G_{12} can be extended to a fall five-coloring by Figure 2a or a fall six-coloring by Figure 2b of G_{12}. We can recursively obtain a fall five-coloring and a fall six-coloring of H_i. Hence, for any integer i, H_i is a planar fall k-colorable graph for $k \in \{5, 6\}$. $\square$

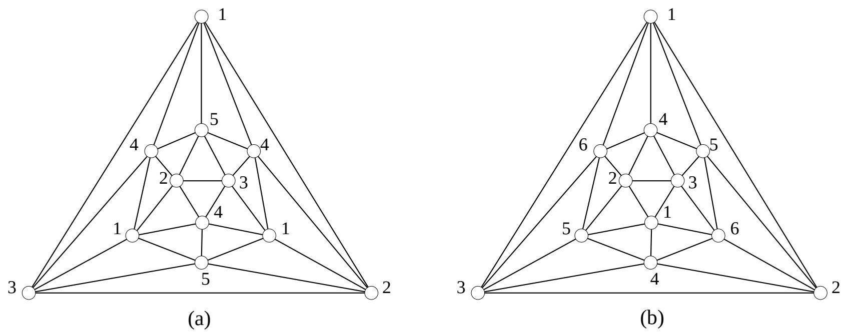

Figure 2. Two fall colorings of the icosahedron G_{12}; (**a**) a fall 5-coloring and (**b**) a fall 6-coloring.

Now, we discuss the fall colorings of maximal planar graphs. A planar graph G is *maximal* if $G + uv$ is not planar for any two nonadjacent vertices u and v of G. For example, the icosahedron G_{12} in Figure 2 is a maximal planar graph.

Theorem 6 ([12]). *If a maximal planar graph G is three-colorable, then G is fall three-colorable.*

Since a maximal planar graph G is three-colorable if and only if every vertex in G has an even degree [22,23], we can obtain the following result:

Corollary 2. *Let G be a maximal planar graph. If each vertex in G has an even degree, then G is fall three-colorable.*

Theorem 7. *Let G be a maximal planar graph. If each vertex in G has an odd degree, then G is fall four-colorable.*

Proof. It follows from the Four Color Theorem [24,25] that G is four-colorable. Let f be a four-coloring of G. Since G is a maximal planar graph, we know that the neighbors of each vertex v form a cycle C_v of order $d_G(v)$. Note that v has an odd degree in G. Hence, C_v contains three colors under the coloring f, that is, v is colorful with respect to f. So, f is a fall four-coloring of G. $\square$

6. Conclusions and Open Problems

In this paper, we first show that $|E(G)| \geq (n(k-1) + r(k-r))/2$ for any fall k-colorable graph G with order n, where $r \equiv n \pmod{k}$ and $0 \leq r \leq k-1$, and this bound is tight. Then, we obtain that if G is k-colorable ($k \geq 2$) and the minimum degree $\delta(G) \geq \frac{k-2}{k-1}n$, then G is fall k-colorable and this condition of the minimum degree is the best possible.

Moreover, we give a simple proof for an NP-hard result of determining whether a graph is fall k-colorable, where $k \geq 3$.

For a maximal planar graph G, if G has a fall k-coloring, then by Theorems 1, 5, 6, and 7 we can obtain that $k \in \{3, 4, 5, 6\}$. This prompts us to propose the following problem:

Problem 1. *For each $k \in \{3, 4, 5, 6\}$, which maximal planar graphs G have fall k-colorings?*

For any outerplane graph G, note that the minimum degree $\delta(G) \leq 2$. If G has a fall k-coloring, then by Lemma 1 we have $k \leq \delta(G) + 1 \leq 3$.

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