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Multidimensional Model of Information Struggle with Impulse Perturbation in Terms of Levy Approximation

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Abstract: The focus of this research was on building a decision support system for a model that characterizes the conflict interaction of n-dimensional complex systems with non-trivial internal structures. The interpretation of the new model was focused on information warfare as the impact of rare events that quickly change certain perceptions of a large number of people. Consequently, the support for various ideas experiences stochastic jumps, a phenomenon observable through a non-classical Levy approximation scheme. The essence of our decision support system lies in its ability to navigate the complex dynamics of conflict interaction among multifaceted systems. Through the utilization of advanced modeling techniques, our aim is to illuminate the complicated interplay of factors influencing information warfare and its cascading effects on societal perceptions and behaviors. Key components of our decision support system encompass model development, simulation capabilities, data integration, and visualization tools. The significance of our work lies in its potential to inform policy formulation, conflict resolution strategies, and societal resilience in the face of information warfare. By providing decision-makers with actionable intelligence and foresight into emerging threats and opportunities, our decision support system serves as a valuable tool for navigating the complexities of modern conflict dynamics. In conclusion, developing a decision support system for modeling conflict interaction in complex systems represents an essential step toward enhancing our understanding of information warfare and its consequences. Through interdisciplinary collaboration and innovative modeling techniques, we aim to provide stakeholders with the insights and capabilities needed to navigate the developing landscape of conflict and ensure the stability and resilience of society.

Keywords: simulation and optimization; control of dynamic systems; integrator systems; dynamic system forecast; random evolution; Levy approximation; multidimensional model

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1. Introduction

The objective of an information threat is to compromise the integrity, confidentiality, or availability of information, thereby causing harm to individuals, organizations, or states [1]. It is evident that these threats require effective mitigation strategies. In the context of information warfare, adversaries may exploit information threats to discredit opponents, undermine public trust in specific entities, or even manipulate election processes. Information threats manifest in various forms within the realm of information warfare. In order to counter information threats effectively, it is imperative to develop robust security measures [2]. Primarily, this involves the construction of mathematical models that consider a multitude of factors capable of influencing the evolving situation.

The Lotka-Volterra model, which describes predator-prey interactions, is one of the main types of models of many processes in applied mathematics, social sciences, and economics [1,3–6]. The classical Lotka-Volterra model, well-known for its applications in predator-prey interactions, encounters significant limitations when applied to the dynamics of information dissemination. One of the primary drawbacks of this model lies in its inability to accommodate the dynamic nature of information influence. Specifically, the model assumes constancy in the characteristics (intensities) of information impact, failing to account for sudden and unpredictable events that profoundly affect the perceptions of information consumers Other approaches to modeling the spread of information are models of population dynamics, in particular, Gompertz dynamics, which are exponential at small moments of time and pass to some asymptotic level at large moments of time [7–9], game models [10] or models described by linear partial equations derivatives [11]. However, in real-world scenarios, the dissemination of information is characterized by volatility and unpredictability, influenced by a myriad of factors such as breaking news, viral content, and social media trends. The model's static nature precludes it from capturing the nuanced dynamics of information dissemination in contemporary society [12–14].

Considering this gap, the construction and analysis of a new model of information warfare take into account both frequent events that occur with high probabilities and rare ones that quickly change some beliefs of a large number of people. As a result, the quantities of adherents of different ideas make stochastic jumps that we may see applying the Levy approximation scheme. We suppose that such a model could be essential as soon as now breaking news produces quick and astonishing influence on the audience through social media and the Internet or TV channels. The behavior of the model could not be analyzed obviously for any fixed moment of time as it was done in a classical case. But, as it is usual for stochastic models, we may obtain functional limit theorems that present the behavior at large time intervals [14–19]. In order to streamline the decision-making process and facilitate result visualization, a software implementation was developed for the described process. JetBrains PyCharm Community Edition 2023.3 software, along with the Matplotlib 3.6.2 library calculation visualization package, was selected as the programming language for this purpose. Additionally, it was decided not to utilize off-the-shelf solutions for the primary calculations, thus ensuring complete control over the construction of the numerical solution.

Hence, this study introduces several innovative contributions. Primarily, it presents a multi-dimensional model of information warfare that incorporates the influence of random factors, offering a more comprehensive understanding of the dynamics of information conflicts. Secondly, a software decision support system was developed based on the analyzed stochastic system. This system enables real-time construction, analysis, and management of various information scenarios, providing decision-makers with valuable insights for effective response strategies.

2. Model of Information Warfare

The application of this approach to the model of information warfare was initially proposed in [1]. The authors conceptualize a specific social community characterized by a constant population size denoted as N_0, which faces potential exposure to various types of information threats. The values $N_1(t), N_2(t), \ldots, N_n(t)$ are the number of "supporters" depending on time t who have perceived new information, ideas, norms, etc. of type $1, 2, \ldots, n$ respectively.

In the "classical" version, the model is described by the Lotka-Volterra equations:

$$\begin{cases} \frac{dN_1}{dt} = (\alpha_1 + \beta_1 N_1(t))(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)), \; N_1(t_0 = 0) = N_1(0) \geq 0 \\ \frac{dN_2}{dt} = (\alpha_2 + \beta_2 N_2(t))(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)), \; N_2(t_0 = 0) = N_2(0) \geq 0 \\ \ldots \\ \frac{dN_n}{dt} = (\alpha_n + \beta_n N_n(t))(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)), \; N_n(t_0 = 0) = N_n(0) \geq 0 \end{cases} \quad (1)$$

Key assumptions about the model (1):

1. There are multiple concepts and n ideas that are disseminated among the community through 2 information channels:
   - the first is "external" to the community, for example, an advertising media campaign. Its intensity is characterized by the parameters $\alpha_1 > 0, \ldots \alpha_n > 0$, respectively, and the parameters $\alpha_i, i = 1, \ldots, n$ are considered independent of time and environment;
   - the second "internal" channel is the interpersonal communication between members of the social community (its intensity, i.e., the number of equivalent information contacts characterized by the parameters $\beta_1 > 0, \ldots, \beta_n > 0$, respectively, which also do not depend on time and environment). As a result, supporters of the first idea who have already been "recruited" (their number is equal to $N_1(\mathrm{t})$) make their contribution to the process of spreading the idea among the community by influencing its "unrecruited" members (their number is equal to $N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)$). The same applies to supporters of other ideas.
2. The rate of change of the number of supporters $N_1(t), \ldots N_n(t)$ (i.e., the number of supporters of the respective idea "recruited" per unit of time) consists of:
   - the external recruitment rate (proportional to the product of intensities $\alpha_1, \ldots, \alpha_\mathrm{n}$ by the number of individuals not yet recruited $N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)$), i.e.,

$$
\begin{gathered}
\alpha_1(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)) \\
\alpha_2(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)) \\
\cdots \\
\alpha_n(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t))
\end{gathered}
$$

   accordingly; internal rate of recruitment (it is proportional to the product of intensities $\beta_1, \beta_2, \ldots, \beta_n$ by the corresponding number of active supporters $N_1(t)$, $N_2(t), \ldots, N_n(t)$ and the number of unrecruited $N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)$, i.e.,

$$
\begin{gathered}
\beta_1 N_1(t)(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)) \\
\beta_2 N_2(t)(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)), \\
\cdots \\
\beta_n N_n(t)(N_0 - N_1(t) - N_2(t) - \cdots - N_n(t)),
\end{gathered}
$$

   accordingly; thus, the model can be described by equations (1) of the Lotka-Volterra type (for more details on possible solutions and characteristics of the dynamic system, see [5]).

3. Model of Information Warfare with Impulse Influence and Markov Switches

We extend the approach to building an information warfare model proposed in [1] to the multidimensional case:

$$
dN^\varepsilon(t) = C\Big(N^\varepsilon(t), x\Big(t/\varepsilon^2\Big)\Big)dt + d\eta^\varepsilon(t) \tag{2}
$$

where,

$$
C\Big(N^\varepsilon(t), x\Big(\tfrac{t}{\varepsilon^2}\Big)\Big) =
$$

$$
= \begin{pmatrix}
-\alpha_1(x) + \beta_1(x)N_0 - \beta_1(x)N_1^\varepsilon(t) & -\alpha_1(x) - \beta_1(x)N_1^\varepsilon(t) & -\alpha_1(x) - \beta_1(x)N_1^\varepsilon(t) & -\alpha_1(x) - \beta_1(x)N_1^\varepsilon(t) \\
-\alpha_2(x) - \beta_2(x)N_2^\varepsilon(t) & -\alpha_2(x) + \beta_2(x)N_0 - \beta_2(x)N_2^\varepsilon(t) & -\alpha_2(x) - \beta_2(x)N_2^\varepsilon(t) & -\alpha_2(x) - \beta_2(x)N_2^\varepsilon(t) \\
\cdots & \cdots & \cdots & \cdots \\
-\alpha_n(x) - \beta_n(x)N_n^\varepsilon(t) & -\alpha_n(x) - \beta_n(x)N_n^\varepsilon(t) & -\alpha_n(x) - \beta_n(x)N_n^\varepsilon(t) & -\alpha_n(x) + \beta_n(x)N_0 - \beta_n(x)N_n^\varepsilon(t)
\end{pmatrix}
$$

$$
\times \begin{pmatrix} N_1^\varepsilon(t) \\ N_2^\varepsilon(t) \\ \cdots \\ N_n^\varepsilon(t) \end{pmatrix} + \begin{pmatrix} \alpha_1 N_0 \\ \alpha_2 N_0 \\ \cdots \\ \alpha_n N_0 \end{pmatrix},
$$

where

(A) $N^{\varepsilon}(t)$ is an n-dimensional vector of solutions, the components of which are the numbers of supporters of different ideas;
(B) $x(t/\varepsilon^2)$ is a uniformly ergodic Markov process that models the influence of the environment on the intensity of information dissemination. This means that in the period between the moments of resumption of such a Markov process, the values of $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x), \beta_1(x), \beta_2(x), \ldots, \beta_n(x)$ are constant, as in the classical model, while at the moments of resumption the values change instantly. This models random events that occur independently of society and have a significant impact on people's views.

The Markov random process $x(t/\varepsilon^2)$ is defined in the standard phase space (X, X), which is given by the generator [14–17].

$$Q\varphi(x) = q(x) \int_X P(x, dy)[\varphi(y) - \varphi(x)]$$

on the Banach space $B(X)$ of bounded functions with real values and a supremum norm

$$||\varphi(x)|| = \sup_{x \in X} |\varphi(x)|$$

The stochastic kernel $P(x, B), x \in X, B \in X$ defines a uniformly ergodic nested Markov chain $x_n = x(\tau_n), n \geq 0$, where τ_n are the jump moments of the nested chain, which has a stationary distribution $\rho(B), x \in X, B \in X$. The stationary distribution $\Pi(B), B \in X$, of the Markov process $x(t), t \geq 0$ can be determined from the relation [18]

$$\Pi(dx)q(x) = q\rho(dx)$$

where

$$q = \int_X \Pi(dx)q(x)$$

Let us define the potential operator $\boldsymbol{R}_0$ for the generator $\boldsymbol{Q}$ using the relation

$$\boldsymbol{R}_0 = \boldsymbol{\Pi} - \left(\boldsymbol{\Pi} + \boldsymbol{Q}\right)^{-1}$$

in which $\boldsymbol{\Pi}\varphi(x) = \int_X \Pi(dy)\varphi(y)$ is a projector onto the subspace $N_Q = \varphi : \boldsymbol{Q}\varphi = 0$ of the zeros of the operator $\boldsymbol{Q}$.

The impulsive perturbation process (IPP) [14,16,18,20,21] $\eta^{\varepsilon}(t), t \geq 0$, which components under the Levy approximation scheme is given by

$$\eta_i^{\varepsilon}(t) = \int_0^t \eta_i^{\varepsilon}\left(ds, x^{\varepsilon}\left(\frac{s}{\varepsilon^2}\right)\right),\ i = 1, \ldots, n.$$

where the set of processes with independent increments $\eta_i^{\varepsilon}(t, x), t \geq 0, x \in X$, is determined by the generators

$$\boldsymbol{\Gamma}^{\varepsilon}(x)\varphi(w) = \varepsilon^{-2} \int_R (\varphi(w + v) - \varphi(w))\Gamma^{\varepsilon}(dv, x), x \in X$$

and satisfy the conditions of the Levy approximation.

L1: Approximation of averages:

$$\int_R v\Gamma^{\varepsilon}(dv, x) = \varepsilon a_1(x) + \varepsilon^2(a_2(x) + \theta_a(x)), \theta_a(x) \to 0, \varepsilon \to 0$$

and

$$\int_R v^2\Gamma^{\varepsilon}(dv, x) = \varepsilon^2(b(x) + \theta_b(x)), \theta_b(x) \to 0, \varepsilon \to 0$$

L2: Condition on the distribution function:

$$\int_R g(v)\Gamma^\varepsilon(dv,x) = \varepsilon^2\left(\Gamma_g(x) + \theta_g(x)\right), \theta_g(x) \to 0, \varepsilon \to 0$$

for all $g(v) \in C_3(R)$. Here, the measure $\Gamma_g(x)$ is bounded for all $g(v) \in C_3(R)$ and is defined by the relation

$$\Gamma_g(x) = \int_R g(v)\Gamma_0(dv,x), g(v) \in C_3(R)$$

where $C_3(R)$ is a class of functions that define a measure and contains bounded functions with real values such that $g(v)/|v|^2 \to 0$ when $v \to 0$.

L3: Uniform quadratic integrability:

$$\lim_{c\to\infty}\sup_{x\in X}\int_{|v|>c} v^2\Gamma_0(dv,x)$$

4. Asymptotic Model Analysis

Let's assume that the balance condition is met

$$\hat{a}_1 := \int_X \Pi(dx)a_1(x) = 0 \tag{3}$$

We present the asymptotic properties of the impulsive perturbation process obtained in [18].

Theorem 1. *If the balance condition (3) and the Levy approximation conditions* ***L1–L3*** *are satisfied, then there is a weak convergence*

$$\eta^\varepsilon(t) \to \eta^0(t), \varepsilon \to 0$$

The limit process $\eta^0(t)$ is determined by the generator

$$\Gamma\varphi(w) = \hat{a}_2\varphi\prime(w) + \frac{1}{2}\sigma^2\varphi''(w) + \int_R[\varphi(w+v) - \varphi(v)]\hat{\Gamma}_0(dv)$$

where

$$\hat{a}_2 = \int_X \Pi(dx)(a_2(x) - a_0(x))$$

$$\sigma^2 = \int_X \Pi(dx)(b(x) - b_0(x)) + 2\int_X \Pi(dx)a_1(x)R_0a_1(x)$$

$$a_0(x) = \int_R v\Gamma_0(dv,x), b_0(x) = \int_R v^2\Gamma_0(dv,x), \hat{\Gamma}_0(v) = \int_X \Pi(dx)\Gamma_0(v,x)$$

Thus, in the limit, we have obtained a Levy process with three components: a deterministic shift, a diffusion component, and a Poisson jump part.

Proof. Since more general statements were already proved by us earlier [18], for our model (2), we will present the main stages of the proof with more detailed explanations in the proof of Theorem 2 for the entire dynamic system.

As we know [14], generators of processes with independent increments $\eta^\varepsilon(t,x), t \geq 0, x \in X$, on test functions $\varphi(\omega) \in C^3(R)$ under conditions L1–L3 admit the asymptotic representation

$$\boldsymbol{\Gamma}^\varepsilon(x)\varphi(\omega) = \varepsilon^{-1}\Gamma_1(x)\varphi(\omega) + \Gamma_2(x)\varphi(\omega),$$

where

$$\begin{gathered}\Gamma_1(x)\varphi(\omega) = a_1(x)\varphi\prime(\omega),\\ \Gamma_2(x)\varphi(\omega) = (a_2(x) - a_0(x))\varphi\prime(\omega) + \tfrac{1}{2}(b(x) - b_0(x))\varphi''(\omega) + \\ + \int_R[\varphi(\omega+v) - \varphi(v)]\Gamma_0(dv,x)\,.\end{gathered}$$

It is also a well-known fact [14] that the generator of the two-component Markov process $\left(\eta^{\varepsilon}, x\left(t/\varepsilon^2\right)\right), t \geq 0$ has the form

$$\hat{\Gamma}^{\varepsilon}(x)\varphi(\omega,x) = \varepsilon^{-2}\boldsymbol{Q}\varphi(\omega,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(\omega,x) + \\ + \Gamma_2(x)\varphi(\omega,x) + \Gamma^{\varepsilon}(x)\varphi(\omega,x),$$

where the remainder term $\|\Gamma^{\varepsilon}(x)\varphi(\omega,x)\| \to 0$ at $\varepsilon \to 0, \varphi(\omega,\cdot) \in \mathrm{C}^3(\mathrm{R})$.

Let us consider the so-called truncated generator:

$$\Gamma_0^{\varepsilon}(x)\varphi(\omega) = \varepsilon^{-2}\boldsymbol{Q}\varphi(\omega,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(\omega,x) + \Gamma_2(x)\varphi(\omega,x).$$

When the balance condition (3) is fulfilled, the solution of the singular perturbation problem for the truncated operator $\Gamma_0^{\varepsilon}(\mathrm{x})$ on the test functions

$$\varphi^{\varepsilon}(\omega,x) = \varphi(\omega) + \varepsilon\varphi_1(\omega,x) + \varepsilon^2\varphi_2(\omega,x)$$

is realized by the relation [14]

$$\Gamma_0^{\varepsilon}(x)\varphi^{\varepsilon}(\omega,x) = \Gamma\varphi(\omega) + \varepsilon\theta_{\eta}^{\varepsilon}(x)\varphi(\omega),$$

where the residual term $\theta_{\eta}^{\varepsilon}(x)\varphi(\omega)$ is uniformly bounded at x.

The limit operator is determined by the formula

$$\Gamma = \Pi\Gamma_1(x)R_0\Gamma_1(x)\Pi + \Pi\Gamma_2(x)\Pi.$$

and applying Theorem 6.3 from [14], we obtain the necessary convergence of processes, i.e., the statement of the Theorem 1. □

Next, we are ready to study the asymptotic properties of the dynamical system, in particular, using the approaches proposed in [14,18].

Theorem 2. *When the balance condition (3) and Levy approximation conditions* ***L1–L3*** *are satisfied, weak convergence in the sense of generator convergence is valid*

$$(u^{\varepsilon}(t), \eta^{\varepsilon}(t)) \to \left(u^0(t), \eta^0(t)\right), \varepsilon \to 0$$

The limit process is determined by the generator

$$\boldsymbol{L}\varphi(w,v) = \hat{C}(u)\varphi'(w,) + \Gamma\varphi(x) \tag{4}$$

The averaged function has the form

$$\hat{C}(u) = \int_X \Pi(dx)C(u,x)$$

Proof. As it was established in the monograph [14], the generator of the two-component Markov process $\left(u^{\varepsilon}(t), x\left(t/\varepsilon^2\right)\right), t \geq 0$, can be written in the form

$$\boldsymbol{L}^{\varepsilon}(x)\varphi(\omega,x) = \varepsilon^{-2}\boldsymbol{Q}\varphi(\omega,x) + \Gamma^{\varepsilon}(x)\varphi(\omega,x) + \\ + C(x)\varphi(\omega,x) + \theta_{\omega}^{\varepsilon}\varphi(\omega,x)$$

where

$$C(x)\varphi(\omega,x) = C(u,x)\varphi'_{\omega}(\omega,x)$$

and $\|\theta_{\omega}^{\varepsilon}(x)\varphi(\omega,x)\| \to 0$ at $\varepsilon \to 0$.

The generator $\boldsymbol{L}^{\varepsilon}(x)$ in the case of an impulse perturbation process of admits an asymptotic representation [14]

$$\boldsymbol{L}^{\varepsilon}(x)\varphi(\omega,x) = \varepsilon^{-2}\boldsymbol{Q}\varphi(\omega,x) + \varepsilon^{-1}\Gamma_1(x)\varphi(\omega,x) + \\ +\Gamma_2(x)\varphi(\omega,x) + C(x)\varphi(\omega,x) + \hat{\theta}^{\varepsilon}_{\omega}\varphi(\omega,x)$$

where

$$\hat{\theta}^{\varepsilon}_{\omega}(x) = \Gamma^{\varepsilon} + \theta^{\varepsilon}_{\omega}(x),$$

$$\Gamma_1(x)\varphi(\omega) = a_1(x)\varphi\prime(\omega),$$

$$\Gamma_2(x)\varphi(\omega) = (a_2(x) - a_0(x))\varphi\prime(\omega) + \frac{1}{2}(b(x) - b_0(x))\varphi''(\omega) + \\ +\int_R[\varphi(\omega+v) - \varphi(v)]\Gamma_0(dv,x)\ .$$

Remainder term $\left\|\hat{\theta}^{\varepsilon}_{\omega}(x)\varphi(\omega,x)\right\| \to 0$ at $\varepsilon \to 0$.

We apply the truncated operator of the form:

$$\boldsymbol{L}^{\varepsilon}_0(x)\varphi = \varepsilon^{-2}\boldsymbol{Q}\varphi + \varepsilon^{-1}\Gamma_1(x)\varphi + \Gamma_2(x)\varphi + C(x)\varphi \tag{5}$$

Taking into account the fulfillment of the balance condition (3), we will solve the singular perturbation problem for the truncated operator (5) on the test functions

$$\varphi^{\varepsilon}(\omega,x) = \varphi(\omega) + \varepsilon\varphi_1(\omega,x) + \varepsilon^2\varphi_2(\omega,x)$$

using relation

$$\boldsymbol{L}^{\varepsilon}_0(x)\varphi^{\varepsilon}(\omega,x) = \boldsymbol{L}\varphi(\omega) + \varepsilon^2\theta^{\varepsilon}_{\omega}(x)\varphi(\omega) \tag{6}$$

where remainder term $\theta^{\varepsilon}_{\omega}(x)$ is uniformly bounded at x.

The limit operator **L** is given by the formula

$$\boldsymbol{L} = \Pi[C(x) + \Gamma_1(x)R_0\Gamma_1(x) + \Gamma_2(x)]\Pi \tag{7}$$

Indeed, in order to fulfill equality (6), it is necessary that the coefficients with the same powers of ε on the left and right sides are equal. For this purpose, we calculate:

$$\boldsymbol{L}^{\varepsilon}_0(x)\varphi^{\varepsilon}(\omega,x) = \varepsilon^{-2}\boldsymbol{Q}(x)\varphi(\omega) + \\ +\varepsilon^{-1}[\boldsymbol{Q}\varphi_1(\omega,x) + \Gamma_1(x)\varphi(\omega)] + \\ +[\boldsymbol{Q}\varphi_2(\omega,x) + \Gamma_1(x)\varphi_1(\omega,x) + \\ +\Gamma_2(x)\varphi + C(x)\varphi(\omega)] + \\ +\varepsilon[\Gamma_1(x)\varphi_2(\omega,x) + \Gamma_2(x)\varphi_1(\omega,x) + C(x)\varphi_1(\omega,x)] + \\ +\varepsilon^2[\Gamma_2(x)\varphi_2(\omega,x) + C(x)\varphi_2(\omega,x)]$$

Since

$$\boldsymbol{Q}\varphi(\omega) = 0 \Leftrightarrow \varphi(\omega) \in N_Q,$$

Then $\varphi(\omega)$ not depends from x.

The balance condition (3) is a condition for the solvability of the equation

$$\boldsymbol{Q}\varphi_1(\omega,x) + \Gamma_1(x)\varphi(\omega) = 0$$

Then,

$$\varphi_1(\omega,x) = R_0\Gamma_1(x)\varphi(\omega),$$

and

$$\boldsymbol{Q}\varphi_2(\omega,x) + \Gamma_1(x)\varphi_1(\omega,x) + \\ +\Gamma_2(x)\varphi(\omega) + C(x)\varphi(\omega) = \boldsymbol{L}\varphi(\omega).$$

or

$$\boldsymbol{Q}\varphi_2(\omega,x) = [\boldsymbol{L} - \Gamma_1(x)R_0\Gamma_1(x) - \Gamma_2(x) - C(x)]\varphi(\omega)$$

The solvability condition of the last equation gives the limit operator $\boldsymbol{L}$ in the form (7). Applying Theorem 6.3 from [14], we finally obtain the statement of the Theorem 2. □

5. Modeling and Software Implementation

In our formulation, the average marginal model of information warfare is outlined as follows: this model serves as a comprehensive framework for analyzing various facets of information warfare, encompassing a wide array of factors and dynamics. By examining the average marginal effects across different dimensions, we gain insights into the nuanced interplay between variables and their impact on the broader landscape of information threats. This model offers a structured approach to dissecting the complexities inherent in information warfare scenarios, enabling a deeper understanding of its underlying mechanisms and potential mitigation strategies. The proposed model formulation can be presented by Equation (4) with the matrix in the form (2) appearing and the assumptions described during the construction of the model, in particular (A) and (B), are fulfilled.

For the software implementation, the decision was made to utilize the Python language due to its versatility and extensive libraries. The visualization aspect was executed using the Matplotlib 3.6.2 library package, known for its effectiveness in creating clear and insightful graphical representations. Opting against pre-existing solutions for calculations provided us with greater flexibility in obtaining precise numerical solutions tailored to our specific requirements. Despite eschewing off-the-shelf solutions for fundamental calculations, Python (JetBrains PyCharm Community Edition 2023.3) offers a wealth of libraries that expedite the coding process for tasks such as generating numerical series according to the normal distribution law or handling operations with arrays and tabular data. Leveraging these libraries not only streamlines development but also ensures reliability and efficiency in our implementation.

Consider an example for six types of information threats. This scenario can apply to situations where some propaganda is distributed across different, for example, TV channels. Input parameters:

$$N_0 = 500000,\ N_1(0) = 0,\ N_2(0) = 0,\ N_3(0) = 0,\ N_4(0) = 0,\ N_5(0) = 0,\ N_6(0) = 0,$$
$$\alpha_1 = 0.000012,\ \alpha_2 = 0.000014,\ \alpha_3 = 0.000011,\ \alpha_4 = 0.000015,\ \alpha_5 = 0.000013,\ \alpha_6 = 0.000011,$$
$$\beta_1 = 0.00000001,\ \beta_2 = 0.00000002,\ \beta_3 = 0.000000009,\ \beta_4 = 0.00000002,\ \beta_5 = 0.00000002,\ \beta_6 = 0.00000001.$$

Also, the following parameters were used to characterize the diffusion process: $\theta = 1$, $\sigma = 5$. These values are used to calculate asymptotic standard deviation as part of the Euler-Maruyama method to get a numerical solution. By changing those values, we can increase or decrease the diffusion process's deviation.

The characteristics of the jumps process are following $\lambda = 5$, $\sigma = 5$. Here, we are using a discrete jumps process, so λ is intensity of jumps and σ used to describe variability.

In this case, input parameters for the outer channel rely on TV channel ratings or the amount of auditory.

As depicted in Figure 1, the fourth type of information threat emerges as the predominant leader, attributed to its notably higher initial coefficients. However, it is noteworthy that during the initial stages, the leadership position was transiently held by other types of information threats, primarily due to the influence of random shifts. In real-world scenarios, such shifts may manifest as unexpected downtime or the dissemination of compromising information by certain TV channels.

It is pertinent to highlight that the magnitude of random disturbances, often referred to as "strength," can be manipulated for experimental purposes. For instance, by augmenting parameters such as θ and σ to values of 4 and 8 respectively, the ensuing results are markedly affected. Notably (Figure 2), deviations exhibit a substantial increase under these augmented conditions.

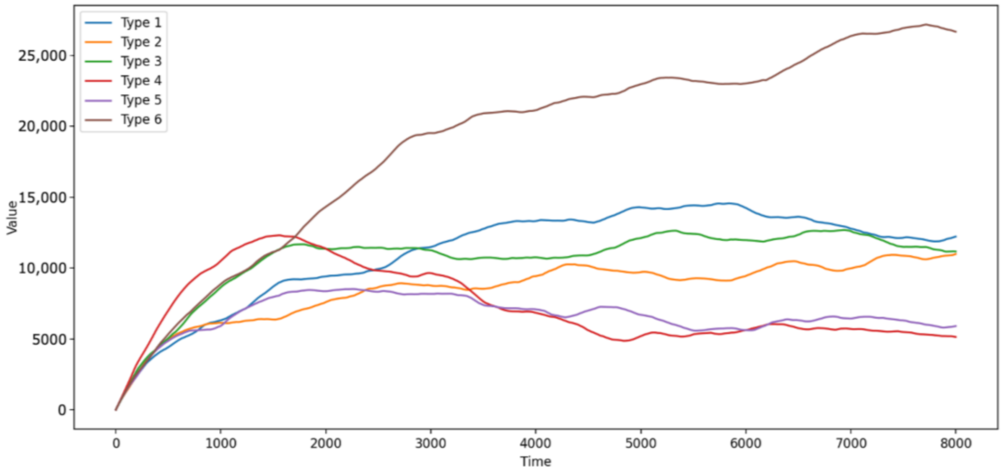

Figure 1. Visualization of information warfare (shift, diffusion plus jumps).

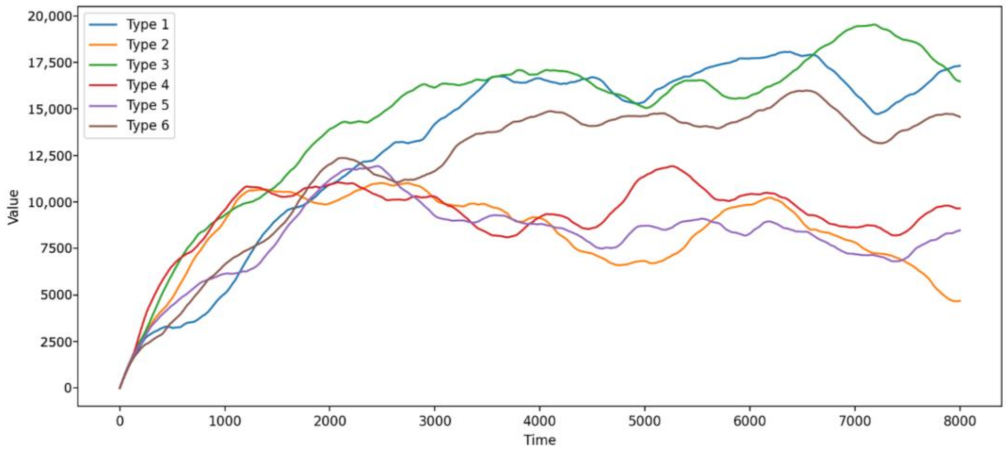

Figure 2. Visualization of information warfare at higher random deviations.

As depicted in the graph (Figure 1), it is evident that the fourth type of information threat no longer maintains its status as the predominant contender. This observation underscores a critical insight: even when the combined significance of the α_n and β_n coefficients appears substantial, it may not guarantee decisive influence under certain conditions. The emergence of this outcome highlights the potential impact of robust random processes. These processes, characterized by their unpredictability and strength, possess the capacity to exert substantial influence on the dynamics of information dissemination. Consequently, even factors traditionally perceived as influential may be overshadowed or mitigated by the effects of these potent random processes. This realization underscores the importance of accounting for stochastic elements in modeling and analyzing information dissemination dynamics, as they can significantly alter expected outcomes and challenge conventional interpretations of influence and dominance.

In addition, we also consider the case where the same input parameters apply to all types of information threats.

$$N_0 = 500000,\ N_1(0) = 0,\ N_2(0) = 0,\ N_3(0) = 0,\ N_4(0) = 0,\ N_5(0) = 0,\ N_6(0) = 0,$$
$$\alpha_1 = 0.000012,\ \alpha_2 = 0.000012,\ \alpha_3 = 0.000012,\ \alpha_4 = 0.000012,\ \alpha_5 = 0.000012,\ \alpha_6 = 0.000012,$$
$$\beta_1 = 0.00000001,\ \beta_2 = 0.00000001,\ \beta_3 = 0.00000001,\ \beta_4 = 0.00000001,\ \beta_5 = 0.00000001,\ \beta_6 = 0.00000001$$

Upon reflection of the scenario outlined above, it becomes evident that one might anticipate an equal distribution of adherents across each type of information threat. However, due to the interplay of diffusion processes and sudden jumps, the resulting output becomes inherently unpredictable. The diffusion process, characterized by the gradual spread of information over time, interacts with stochastic jumps, which represent sudden, significant shifts in the number of adherents for specific ideas. As a result of these dynamic forces, the distribution of adherents among different types of information threats deviates from a uniform pattern, leading to unpredictable outcomes. This unpredictability is illustrated in Figure 3, where the fluctuating nature of the distribution highlights the complex dynamics at play in information dissemination.

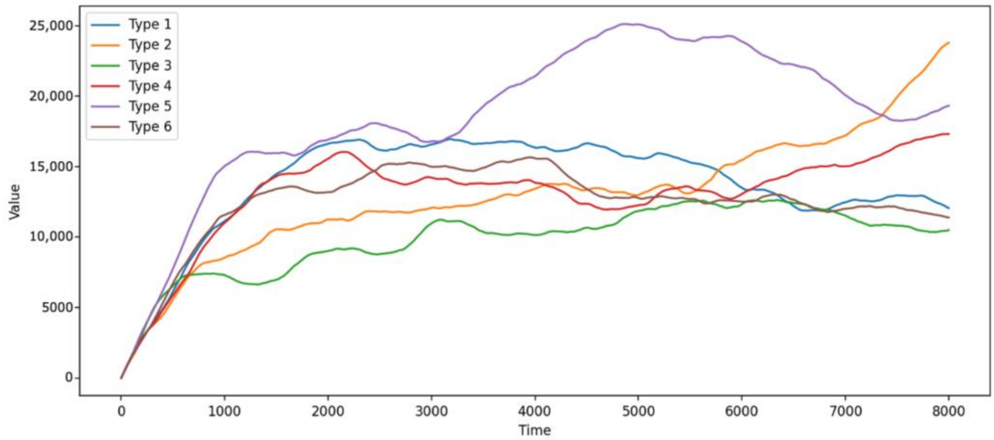

Figure 3. Displaying a case with the same input parameters for each information threat.

Given the stochastic nature of the process, it is anticipated that subsequent simulations will yield varying results, even in the absence of alterations to the input parameters. This randomness stems from the inherent unpredictability of certain factors influencing the simulation, such as initial conditions or external variables. As a consequence, each simulation run may produce distinct outcomes, reflecting the inherent variability of the system being modeled. Therefore, the expectation of observing different results in successive simulations, even when input parameters remain unchanged, is a natural consequence of the inherent randomness inherent in the simulation process (Figure 4).

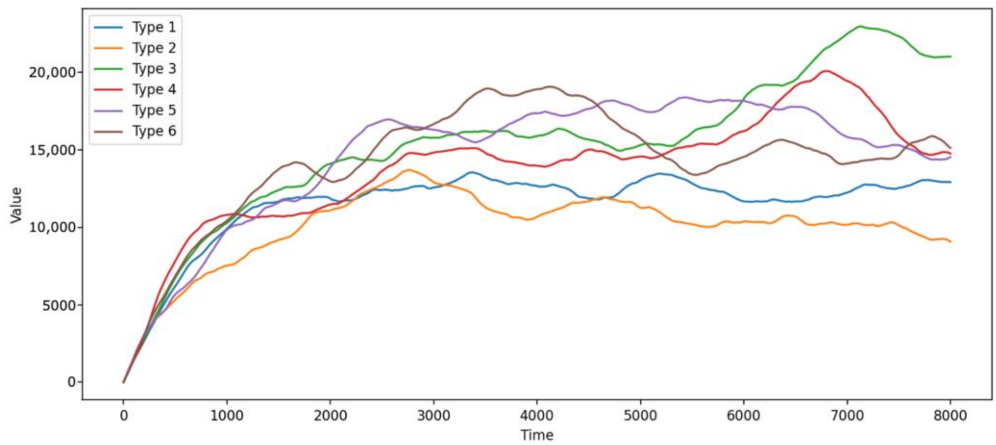

Figure 4. Displaying a case with the same input parameters for each information threat, second simulation.

6. Conclusions

This article presents a comprehensive examination of the complexities involved in developing a model to address n-types of information threats while considering environmental variability. The primary aim is to assess the velocity of information threat dissemination and integrate it into the development of countermeasure strategies. By employing the Poisson approximation scheme, the model can differentiate between deterministic shifts and jumps in the limit process, thereby accommodating real-world nuances in information propagation.

Key findings from this study include the proposal of a multidimensional model that surpasses the classical approach in terms of generality and accuracy [22–24]. Moreover, explicit constructions of the limit generators of the impulse process and the dynamical system were achieved, facilitating a deeper understanding of the model's behavior. Furthermore, the developed software solution streamlines calculations, enhancing the model's practical applicability.

The proposed model offers significant advantages, notably in conducting precise and comprehensive analyses of information threat spread. Enabling accurate measurement of spread rates it empowers decision-makers to formulate more effective response strategies. Additionally, the model's ability to analyze temporal changes in information threat spread facilitates trend identification and prediction of future threats.

In conclusion, the developed model represents a significant advancement in understanding and mitigating information threats. Its ability to account for environmental variability and accurately analyze information spread dynamics makes it a valuable tool for policymakers and security practitioners in devising proactive and effective response strategies. Further research and experimentation will continue to refine and validate the model's performance in diverse contexts and scenarios.

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