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Article

Convergence of the Quadrature-Differences Method for Singular Integro-Differential Equations on the Interval

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Abstract: In this paper, we propose and justify the quadrature-differences method for the full linear singular integro-differential equations with the Cauchy kernel on the interval $(-1,1)$. We consider equations of zero, positive and negative indices. It is shown that the method converges to an exact solution, and the error estimation depends on the sharpness of derivative approximations and on the smoothness of the coefficients and the right-hand side of the equation.

Keywords: singular integro-differential equations; quadrature-differences method

1. Introduction

In [1–4], the quadrature-differences methods for the various classes of the periodic singular integro-differential equations with Hilbert kernels were justified. The convergence of the methods was proven and the error estimates were obtained. Here, we propose and justify the same method for the full linear singular integro-differential equations with the Cauchy kernel on the interval $(-1, 1)$. Note that for the first order equations, this method was justified in [5].

It is known (see, e.g., [6,7]) that the theories of the singular integral equations in periodic (with the Hilbert kernel) and non-periodic (with the Cauchy kernel) cases differ greatly, due to the discontinuity of the contour in the latter case. Therefore, the calculation schemes and the justifications of the method in these cases have essential distinctions. Thus, if for the equations with Hilbert kernels, the same uniform grid is used both for the approximation of the derivatives and integrals and as collocation nodes, then, for the equations with the Cauchy kernel, we must use two different grids: the roots of the special polynomials. For the first class of equations, the problem is stated in Hölder space, and therefore, the usual technique of the compact approximation [8] for the justification is used. The rate of convergence grows with the growing of the smoothness of the coefficients and the right-hand side of the equation infinitely. For the second class of equations, the highest order derivative of the desired function has, in general, integrable singularities at the end points of the contour. Therefore, the problem is stated in the spaces of weighted quadratically integrable functions; the "second kind" [8] of theory of the approximation methods is used, and the rate of convergence is restricted by the order of smoothness of the desired function, coefficients and the right-hand side of the equation.

In the paper, equations of zero, positive and negative indices are considered. The convergence of the method is proven, and the rate of convergence is obtained.

2. Formulation of the Problem

Consider a linear singular integro-differential equation of the form:

$$\sum_{\nu=0}^{m}(a_\nu(t)x^{(\nu)}(t)+b_\nu(t)(Sx^{(\nu)})(t)+(Th_\nu x^{(\nu)})(t))=f(t), \quad -1<t<1, \quad m\geq 1 \tag{1}$$

with the initial conditions:

$$x^{(\nu)}(\xi_0)=0, \quad \nu=0,1,...,m-1, \quad -1\leq \xi_0\leq 1 \tag{2}$$

where $x(t)$ is a desired unknown and $a_\nu(t), b_\nu(t), h_\nu(t,\tau),\ \nu=0,1,...,m,\ f(t)$ are given continuous functions of their arguments, $t,\tau\in[-1,1]$; $b_m(t)$ is a polynomial of some order, $n_0\geq 0$ and singular integrals:

$$(Sx^{(\nu)})(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{x^{(\nu)}(\tau)d\tau}{\tau-t}, \qquad \nu=0,1,...,m$$

are to be interpreted as the Cauchy–Lebesgue principal value; and

$$(Th_\nu x^{(\nu)})(t)=\frac{1}{\pi}\int_{-1}^{1}h_\nu(t,\tau)x^{(\nu)}(\tau)d\tau, \quad \nu=0,1,...,m$$

are regular integrals.

First, we consider in detail a zero index equation $(\kappa=0)$ and then point out the changes in the calculation scheme and justification for the cases of positive $(\kappa>0)$ and negative $(\kappa<0)$ indices.

3. Calculation Scheme

Let us define, following Muskhelishvili [7], the index and the canonical function of the Equation (1). To do this, denote $\theta(t)=\pi^{-1}arg(a_m(t)+ib_m(t)), \quad t\in[-1,1]$, some continuous and one-valued branch of the multi-valued function $\pi^{-1}arg(a_m(t)+ib_m(t))$. Then, the canonical function of Equation (1) will be:

$$Z(t)=(1-t)^{\gamma_1}(1+t)^{\gamma_2}\exp(-\int_{-1}^{1}\frac{\theta(\tau)d\tau}{\tau-t}), \quad t\in(-1,1)$$

where $\gamma_1=\lambda_1-\theta(1), \quad \gamma_2=\lambda_2+\theta(-1)$ and $\lambda_1,\ \lambda_2$ are the integers subject to the condition, $\gamma_1,\gamma_2\in(-1,1)$. The integer $\kappa=-(\lambda_1+\lambda_2)$ is called the index of Equation (1), and the numbers, γ_1 and γ_2, determine the class of possible solutions of the problems, (1) and (2) (see [7,9]).

Now, we will define two weight-functions:

$$\rho(t) = Z(t)(a_m^2(t) + b_m^2(t))^{-1/2} \text{ and } \bar{\rho}(t) = Z^{-1}(t)(a_m^2(t) + b_m^2(t))^{-1/2},\ (a_m^2(t) + b_m^2(t))^{1/2} > 0$$

and two sequences of polynomials $\{\phi_n(t)\}_{n=o}^{\infty}$ and $\{\psi_n(t)\}_{n=o}^{\infty}$ with the following properties:

$$\int_{-1}^{1} \rho(\tau)\phi_k(\tau)\phi_l(\tau)d\tau = \sigma_k\delta_{k,l},\ k \geq l,\ \int_{-1}^{1} \bar{\rho}(\tau)\psi_k(\tau)\psi_l(\tau)d\tau = \zeta_k\delta_{k,l},\ k \geq l \tag{3}$$

$$a_m(t)\rho(t)\phi_{n+1}(t) + b_m(t)(S\rho\phi_{n+1})(t) = (-1)^{\kappa}((\sigma_{n+1}\beta_{n+1-\kappa})/(\zeta_{n+1-\kappa}\alpha_{n+1}))\psi_{n+1-\kappa}(t),\ n \geq \max\{n_0, \kappa\} \tag{4}$$

where $\alpha_{n+1} > 0$ and $\beta_{n+1-\kappa} > 0$ are the senior coefficients of the polynomials, $\phi_{n+1}(t)$ and $\psi_{n+1-\kappa}(t)$, correspondingly, and $\delta_{k,l}$ is the Kronecker symbol. The existence of the polynomials satisfying (3), due to the positiveness and integrability of the weight-functions, $\rho(t)$ and $\bar{\rho}(t)$, was shown in [10]. Moreover, it was shown there that each of the polynomials $\{\phi_n(t)\}_{n=o}^{\infty}$, $\{\psi_n(t)\}_{n=o}^{\infty}$ has just n real simple roots on the interval $(-1, 1)$. Identity (4), which plays the crucial role in the following account, was obtained by Elliott [6].

Let:

$$\{\tau_k \mid \phi_{n+1}(\tau_k) = 0,\ k = 0, 1, ..., n\} \tag{5}$$

$$\{t_j \mid \psi_{n+1-\kappa}(t_j) = 0,\ j = 0, 1, ..., n - \kappa\} \tag{6}$$

be the grids on $[-1,1]$. By:

$$\{\tau_k \mid k = -m_1, -m_1 + 1, ..., n + m_2\} \tag{7}$$

we denote the union of the the grid (5) with the nodes:

$$\tau_k = -1 + (\tau_0 + 1)(k + m_1)/(m_2 + 1),\ k = -m_1, -m_1 + 1, ..., -1$$

$$\tau_k = 1 + (1 - \tau_n)(k - n - m_2)/(m_2 + 1),\ k = n + 1, n + 2, ..., n + m_2$$

Here, m_1 and m_2 are two nonnegative integers:

$$m_1 = m_2 = m/2 \quad \text{for even} \quad m$$

$$m_1 = (m + 1)/2, \quad m_2 = (m - 1)/2 \quad \text{for odd} \quad m$$

We will seek an approximate solution of Equation (1) as a vector:

$$\mathbf{x}_n = (x_{-m_1}, ..., x_{n+m_2}) \tag{8}$$

of values of unknown function in the nodes of the grid (7). Derivatives and values of the unknown function in the nodes of the grids, (5) and (6), and for the initial condition in the point, ξ_0, we will approximate by any numerical formulae:

$$x^{(m)}(\tau_k) \sim [D_n^{(m)}\mathbf{x}_n]_{\tau_k},\ k = 0, 1, ..., n$$

$$x^{(\nu)}(t_j) \sim [D_n^{(\nu)}\mathbf{x}_n]_{t_j},\ j = 0, 1, ..., n - \kappa$$

$$x^{(\nu)}(\xi_0) \sim [D_n^{(\nu)}\mathbf{x}_n]_{\xi_o},\ \nu = 0, 1, ..., m - 1$$

which use only the nodes (7) and the components of the vector (8).

Singular integrals, $(Sx^{(\nu)})(t),\ \nu = 0, 1, ..., m-1$, are to be approximated by the quadratures. To do this, we will integrate polynomials:

$$(Q_{n-\kappa} D_n^{(\nu)} \mathbf{x}_n)(\tau) = \sum_{j=0}^{n-\kappa} [D_n^{(\nu)} \mathbf{x}_n]_{t_j} l_j(\tau),\ \nu = 0, 1, ..., m-1$$

$$l_j(\tau) = \frac{\psi_{n+1-\kappa}(\tau)}{(\tau - t_j)\psi'_{n+1-\kappa}(t_j)},\ j = 0, 1, ..., n-\kappa$$

$$(SQ_{n-\kappa} D_n^{(\nu)} \mathbf{x}_n)(t) = \sum_{j=0}^{n-\kappa} [D_n^{(\nu)} \mathbf{x}_n]_{t_j} (Sl_j)(t),\ \nu = 0, 1, ..., m-1 \tag{9}$$

$$(Sl_j)(t) = \frac{(S\psi_{n+1-\kappa})(t) - (S\psi_{n+1-\kappa})(t_j)}{(t - t_j)\psi'_{n+1-\kappa}(t_j)},\ j = 0, 1, ..., n-\kappa$$

To approximate regular integrals, $(Th_\nu x^{(\nu)})(t),\ \nu = 0, 1, ..., m-1$, we will integrate polynomials:

$$(Q_{n-\kappa} h_\nu D_n^{(\nu)} \mathbf{x}_n)(t, \tau) = \sum_{j=0}^{n-\kappa} [D_n^{(\nu)} \mathbf{x}_n]_{t_j} h_\nu(t, t_j) l_j(\tau),\ \nu = 0, 1, ..., m-1$$

$$(TQ_{n-\kappa} h_\nu D_n^{(\nu)} \mathbf{x}_n)(t) = \sum_{j=0}^{n-\kappa} [D_n^{(\nu)} \mathbf{x}_n]_{t_j} h_\nu(t, t_j) Tl_j,\ \nu = 0, 1, ..., m-1 \tag{10}$$

$$Tl_j = (S\psi_{n+1-\kappa})(t_j)/\psi'_{n+1-\kappa}(t_j),\ j = 0, 1, ..., n-\kappa$$

Coefficients of the quadrature Formulae (9) and (10) depend on the integrals, $(S\psi_{n+1-\kappa})(t)$, which, according to the relations ($[\frac{k-1}{2}]$ denotes the largest integer not exceeding $\frac{k-1}{2}$):

$$(S1)(t) = \frac{1}{\pi} \ln \left| \frac{1-t}{1+t} \right|$$

$$(S\tau^k)(t) = \frac{t^k}{\pi} \ln \left| \frac{1-t}{1+t} \right| + \frac{2}{\pi} \sum_{j=0}^{[\frac{k-1}{2}]} \frac{t^{k-(2j+1)}}{2j+1}, \quad k = 1, 2, ...$$

could be calculated explicitly for all fixed n.

To approximate the dominant part of Equation (1):

$$(Ux^{(m)})(t) = a_m(t) x^{(m)}(t) + b_m(t)(Sx^{(m)})(t)$$

we will apply the operator, U, to the polynomial:

$$(P_n \rho^{-1} D_n^{(m)} \mathbf{x}_n)(\tau) = \sum_{k=0}^{n} \rho^{-1}(\tau_k) [D_n^{(m)} \mathbf{x}_n]_{\tau_k} \bar{l}_k(\tau)$$

$$\bar{l}_k(\tau) = \frac{\phi_{n+1}(\tau)}{(\tau - \tau_k)\phi'_{n+1}(\tau_k)},\ k = 0, 1, ..., n$$

multiplied to the weight-function, $\rho(\tau)$,

$$(U\rho P_n\rho^{-1}D_n^{(m)}\mathbf{x}_n)(t) = \sum_{k=0}^{n}\rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k}(U\rho\bar{l}_k)(t) \tag{11}$$

where, using Equation (4),

$$(U\rho\bar{l}_k)(t) = (-1)^{\kappa}\frac{\sigma_{n+1}\beta_{n+1-\kappa}(\psi_{n+1-\kappa}(t) - \psi_{n+1-\kappa}(\tau_k))}{\zeta_{n+1-\kappa}\alpha_{n+1}(t-\tau_k)\phi'_{n+1}(\tau_k)}, \quad k = 0, 1, ..., n \tag{12}$$

To approximate the regular integral, $(Th_m x^{(m)})(t)$, we will integrate the polynomial:

$$(P_n\rho^{-1}h_m D_n^{(m)}\mathbf{x}_n)(t,\tau) = \sum_{k=o}^{n}\rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k}h_m(t,\tau_k)\bar{l}_k(\tau)$$

also multiplied to the weight-function, $\rho(\tau)$,

$$(T\rho P_n\rho^{-1}h_m D_n^{(m)}\mathbf{x}_n)(t) = \sum_{k=o}^{n}\rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k}h_m(t,\tau_k)T\rho\bar{l}_k \tag{13}$$

where $T\rho\bar{l}_k,\ k = 0, 1, ...n$, are coefficients of the Gauss-type quadrature formula, and for τ_k, which are not the roots of the polynomial, $b_m(t)$, the following relationship is valid:

$$T\rho\bar{l}_k = (-1)^{\kappa}\frac{\sigma_{n+1}\beta_{n+1-\kappa}\psi_{n+1-\kappa}(\tau_k)}{\zeta_{n+1-\kappa}\alpha_{n+1}b_m(\tau_k)\phi'_{n+1}(\tau_k)}$$

Substituting the numerical derivative formulae, the values of Quadratures (9)–(11), (13) and the right-hand side in the nodes of the grid (6) in Equation (1) and the numerical formulae for the point, ξ_0, in the initial conditions (2), we will obtain the system of linear algebraic equations:

$$\begin{gathered}\sum_{k=0}^{n}\rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k}(U\rho\bar{l}_k)(t_i) + \sum_{\nu=0}^{m-1}(a_\nu(t_i)[D_n^{(\nu)}\mathbf{x}_n]_{t_i}+ \\ +b_\nu(t_i)\sum_{j=0}^{n-\kappa}[D_n^{(\nu)}\mathbf{x}_n]_{t_j}(Sl_j)(t_i) + \sum_{j=0}^{n-\kappa}[D_n^{(\nu)}\mathbf{x}_n]_{t_j}h_\nu(t_i,t_j)Tl_j)+ \\ +\sum_{k=0}^{n}\rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k}h_m(t_i,\tau_k)T\rho\bar{l}_k = f(t_i), \quad i = 0, 1, ..., n-\kappa\end{gathered} \tag{14}$$

$$[D_n^{(\nu)}\mathbf{x}_n]_{\xi_0} = 0, \quad \nu = 0, 1, ..., m-1 \tag{15}$$

of the quadrature-differences method.

4. Justification

Let us denote by $W^m_{2,\rho}$ ($W^0_{2,\rho} = L_{2,\rho}$) the set of functions that have on $[-1,1]$ absolutely continuous $(m-1)$-order derivatives and quadratically integrable with the weight-function $\rho(\tau)$ m-order derivatives. Then, let us define the following spaces, $X, X_n;\ Y, Y_n;\ Z, Z_{n-\kappa}$:

$$X = \{x \in W^m_{2,\rho^{-1}} \mid x^{(\nu)}(\xi_0) = 0,\ \nu = 0,1,...,m-1\},\ Y = L_{2,\rho^{-1}},\ Z = L_{2,\bar{\rho}}$$

with the norms:

$$\|x\|_X = \left\{\int_{-1}^{1} \rho^{-1}(\tau) \mid x^{(m)}(\tau) \mid^2 d\tau\right\}^{1/2},\ x \in X$$

$$\|y\|_Y = \left\{\int_{-1}^{1} \rho^{-1}(\tau) \mid y(\tau) \mid^2 d\tau\right\}^{1/2},\ y \in Y$$

$$\|z\|_Z = \left\{\int_{-1}^{1} \bar{\rho}(\tau) \mid z(\tau) \mid^2 d\tau\right\}^{1/2},\ z \in Z$$

$X_n = \{\mathbf{x}_n\}$ - the set of $n+m+1$-components vectors of the form (8) satisfying the conditions

$$[D_n^{(\nu)}\mathbf{x}_n]_{\xi_0} = 0, \quad \nu = 0,1,...,m-1$$

$Y_n = \{\mathbf{y}_n\}$ - the set of $n+1$-components vectors, $Z_{n-\kappa} = \{\mathbf{z}_{n-\kappa}\}$, the set of $n+1-\kappa$-components vectors with the norms:

$$\|\mathbf{x}_n\|_{X_n} = \left\{\int_{-1}^{1} \rho(\tau) \mid\ (P_n\rho^{-1}\bar{D}_n^{(m)}\mathbf{x}_n)(\tau) \mid^2 d\tau\right\}^{1/2},\ \mathbf{x}_n \in X_n$$

$$\|\mathbf{y}_n\|_{Y_n} = \left\{\int_{-1}^{1} \rho(\tau) \mid\ (P_n\rho^{-1}\mathbf{y}_n)(\tau) \mid^2 d\tau\right\}^{1/2},\ \mathbf{y}_n \in Y_n$$

$$\|\mathbf{z}_{n-\kappa}\|_{Z_{n-\kappa}} = \left\{\int_{-1}^{1} \bar{\rho}(\tau) \mid (Q_{n-\kappa}\mathbf{z}_{n-\kappa})(\tau) \mid^2 d\tau\right\}^{1/2},\ \mathbf{z}_{n-\kappa} \in Z_{n-\kappa}$$

where operator $\bar{D}_n^{(m)} : X_n \to Y_n$ is defined by the formulae:

$$\bar{D}_n^{(0)}\mathbf{x}_n = \mathbf{x}_n,\ \bar{D}_n\mathbf{x}_n = \bar{D}_n^{(1)}\mathbf{x}_n,\ [\bar{D}_n^{(1)}\mathbf{x}_n]_{\tau_k} = \frac{x_k - x_{k-1}}{\tau_k - \tau_{k-1}}, \quad k = 0,1,...,n$$

$$[\bar{D}_n^{(\nu)}\mathbf{x}_n]_{\tau_k} = \begin{cases} [\bar{D}_n\bar{D}_n^{(\nu-1)}\mathbf{x}_n]_{\tau_{k+1}},\ k = -m_1 + \frac{\nu}{2},...,n+m_2-\frac{\nu}{2},\ \nu \text{ -- even} \\ [\bar{D}_n\bar{D}_n^{(\nu-1)}\mathbf{x}_n]_{\tau_k},\ k = -m_1 + \frac{\nu+1}{2},...,n+m_2-\frac{\nu-1}{2},\ \nu \text{ -- odd} \end{cases}$$
$$\nu = 1,2,...,m.$$

Here, and further, the product of the vector and function is equal to the vector, the components of which are the products of vector components and function values in the same nodes and:

$$(P_n\mathbf{y}_n)(\tau) = \sum_{k=0}^{n}[\mathbf{y}_n]_{\tau_k}\bar{l}_k(\tau),\ \bar{l}_k(\tau) = \frac{\phi_{n+1}(\tau)}{(\tau-\tau_k)\phi'_{n+1}(\tau_k)}, \quad k = 0,1,...,n$$

$$(Q_{n-\kappa}\mathbf{z}_{n-\kappa})(t)=\sum_{j=0}^{n-\kappa}[\mathbf{z}_{n-\kappa}]_{t_j}l_j(t),\ l_j(t)=\frac{\psi_{n+1-\kappa}(t)}{(t-t_j)\psi'_{n+1-\kappa}(t_j)},\quad j=0,1,...,n-\kappa$$

are Lagrange interpolative operators. We will need also the following operators:

$$p_n^m:\ X\to X_n,\ p_n^m x=(x(\tau_{-m_1}),x(\tau_{-m_1+1}),...,x(\tau_{n+m_2}))$$

$$p_n:\ Y\to Y_n,\ p_n y=(y(\tau_0),y(\tau_1),...,y(\tau_n))$$

$$q_{n-\kappa}:\ Z\to Z_{n-\kappa},\ q_{n-\kappa}z=(z(t_0),z(t_1),...,z(t_{n-\kappa}))$$

Proof of Theorem 1. It is known (see, e.g., [6,7,9]) that if the right-hand side of Equation (1) belongs to H_μ or $L_{2,\bar{\rho}}$, then the m-order derivative of the solution of the problems, (1) and (2), has the form $x^{*(m)}(t)=\rho(t)\omega(t)$, where $\omega(t)\in H_\mu$ or $\omega(t)\in L_{2,\rho}$ correspondently, i.e., $x^*(t)\in W^m_{2,\rho^{-1}}$. Thus, we will consider the problems, (1) and (2), as operator equation:

$$Kx\equiv UD^{(m)}x+Vx=f,\ K:X\to Z \tag{16}$$

where:

$$Uy=a_my+b_mSy,\ U:Y\to Z,\quad Vx=\sum_{\nu=0}^{m}A_\nu D^{(\nu)}x,\ V:X\to Z$$

$$A_\nu D^{(\nu)}x=a_\nu D^{(\nu)}x+b_\nu SD^{(\nu)}x+Th_\nu D^{(\nu)}x,\quad \nu=0,1,...,m-1$$

$$A_mD^{(m)}x=Th_mD^{(m)}x,\ D^{(\nu)}x=x^{(\nu)},\quad \nu=0,1,...,m-1$$

Here, as is shown in [6,7,9,11], $K:X\to Z$ is a linear bounded operator, $V:X\to Z$ is a compact operator and $U:Y\to Z$ is continuously invertible.

Let us consider η an arbitrary constant, which is not an eigenvalue of the problem:

$$D^{(m)}x+\eta\rho x=0,\ x^{(\nu)}(\xi_0)=0,\ \nu=0,1,...,m-1$$

and make a substitution:

$$z=U(D^{(m)}x+\eta\rho x) \tag{17}$$

in Equation (16). Due to the invertibility of the operator, $U:Y\to Z$:

$$x=GU^{-1}z,\quad D^{(m)}x=U^{-1}z-\eta\rho GU^{-1}z \tag{18}$$

where $G:Y\to X$ is the inverse to the operator:

$$Fx=D^{(m)}x+\eta\rho x,\quad F:X\to Y$$

Equation (16) will take the form:

$$Bz\equiv z+VGU^{-1}z-\eta U\rho GU^{-1}z=f,\ B:Z\to Z \tag{19}$$

being still equivalent to the original one. The equivalence here means that the solvability of one of them yields the solvability of another, and their solutions are joined by the relationships, (17) and (18).

Now, let us rewrite the system of Equations (14) and (15) as an operator equation:

$$K_{n-\kappa}\mathbf{x}_n \equiv U_{n-\kappa}D_n^{(m)}\mathbf{x}_n + V_{n-\kappa}\mathbf{x}_n = \mathbf{f}_{n-\kappa}, \quad K_{n-\kappa} : X_n \to Z_{n-\kappa} \tag{20}$$

where:

$$U_{n-\kappa}\mathbf{y}_n = q_{n-\kappa}U\rho P_n\rho^{-1}\mathbf{y}_n, \quad U_{n-\kappa} : Y_n \to Z_{n-\kappa}$$

$$V_{n-\kappa}\mathbf{x}_n = q_{n-\kappa}\sum_{\nu=0}^{m} A_{\nu n}D_n^{(\nu)}\mathbf{x}_n, \quad V_{n-\kappa} : X_n \to Z_{n-\kappa}$$

$$A_{\nu n}D_n^{(\nu)}\mathbf{x}_n = a_\nu Q_{n-\kappa}D_n^{(\nu)}\mathbf{x}_n + b_\nu SQ_{n-\kappa}D_n^{(\nu)}\mathbf{x}_n + TQ_{n-\kappa}h_\nu D_n^{(\nu)}\mathbf{x}_n, \quad \nu = 0, 1, ..., m-1$$

$$A_{mn}D_n^{(m)}\mathbf{x}_n = T\rho P_n\rho^{-1}h_m D_n^{(m)}\mathbf{x}_n, \ \mathbf{f}_{n-\kappa} = q_{n-\kappa}f$$

and make a substitution:

$$\mathbf{z}_{n-\kappa} = U_{n-\kappa}F_n\mathbf{x}_n \tag{21}$$

where:

$$[F_n\mathbf{x}_n]_{\tau_k} = [D_n^{(m)}\mathbf{x}_n]_{\tau_k} + \eta\rho(\tau_k)[\mathbf{x}_n]_{\tau_k}, \ k = 0, 1, ..., n, \quad F_n : X_n \to Y_n$$

The operator, $U_{n-\kappa} : Y_n \to Z_{n-\kappa}$, is invertible explicitly for all n, beginning from some $n_1, \ n_1 \geq \max\{n_0, \kappa\}$ (see, [6]) and:

$$\mathbf{x}_n = G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}, \quad D_n^{(m)}\mathbf{x}_n = U_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - \eta\rho G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}$$

where $G_n : Y_n \to X_n$ is the inverse to F_n. The invertibility of F_n for all n beginning from some $n_2, \ n_2 \geq n_1$ follows from Conditions (B.1), (B.2) of Theorem 1 and the choice of η (see [8]). Moreover, for any $y(t) = \rho(t)\omega(t), \ \omega(t) \in H_\mu$,

$$\|p_n^m Gy - G_np_ny\|_{X_n} \leq C\,\varepsilon_n(Gy) \tag{22}$$

Therefore, by the substitution (21), we will get an equation:

$$B_{n-\kappa}\mathbf{z}_{n-\kappa} \equiv \mathbf{z}_{n-\kappa} + V_{n-\kappa}G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - \eta U_{n-\kappa}\rho G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} = \mathbf{f}_{n-\kappa} \tag{23}$$

$$B_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}$$

which is equivalent to Equation (20).

Now, to prove the unique solvability of Equation (23), we have to establish, according to Theorem 6.1 [8], the following:

(a) $\|Q_{n-\kappa}\mathbf{f}_{n-\kappa} - f\|_Z \to 0$ for $n \to \infty$;

(b) the sequence of operators $(B_{n-\kappa})$ approximates operator B compactly;

(c) $B : Z \to Z$ is invertible.

The validity of (a) follows immediately from the estimations [12]:

$$\|Q_{n-\kappa}\mathbf{f}_{n-\kappa} - f\|_Z \leq CE_{n-\kappa}(f) \tag{24}$$

$$E_{n-\kappa}(f) \leq C(n-\kappa)^{-\mu}, \quad f(t) \in H_\mu, \quad n > \kappa \tag{25}$$

where $E_{n-\kappa}(f)$ is the best uniform approximation of the function, $f(t)$, by the polynomials of order not higher than $n-\kappa$ on $[-1,1]$.

To check (b), we will show first that the sequence $(B_{n-\kappa})$ approximates the operator, B, with respect to $Q_{n-\kappa}$. For arbitrary $\mathbf{z}_{n-\kappa} \in Z_{n-\kappa}$, we will write:

$$\|Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \leq \|Q_{n-\kappa}V_{n-\kappa}G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - \tag{26}$$

$$-VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z + \mid \eta \mid \|Q_{n-\kappa}U_{n-\kappa}\rho G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - U\rho GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z$$

and estimate each summand of the right-hand side independently.

To estimate the first summand, we will use the partial uniform best approximation, $E^\tau_n(h)$ $(E^t_n(h))$, of the function, $h(t,\tau)$, by the variable, τ (t):

$$E^\tau_n(h) = \|E_n(h)\|_Z, \; (E^t_n(h) = \|E_n(h)\|_Z)$$

Here, inside the norm symbol, we take first the best approximation by the variable, τ (t), and then take the norm by the other variable. Using the boundedness of the operator, $S : Z \to Z$ [11], the equivalence:

$$U^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} = p_nU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} \tag{27}$$

and Estimations (24) and (22), we will obtain:

$$\|Q_{n-\kappa}V_{n-\kappa}G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \leq$$

$$\leq C(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^\tau_n(\rho^{-1}h_mD^{(m)}GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^t_{n-\kappa}(h_m) +$$

$$+\sum_{\nu=0}^{m-1}(E_{n-\kappa}(a_\nu D^{(\nu)}GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E_{n-\kappa}(b_\nu SQ_{n-\kappa}D^{(\nu)}_nG_np_nU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) +$$

$$+E_{n-\kappa}(D^{(\nu)}GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^\tau_{n-\kappa}(h_\nu D^{(\nu)}GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^t_{n-\kappa}(h_\nu)))$$

For the second summand, using once more Equations (27), (22) and (24), and the boundedness of the operator, $U : Y \to Z$, we will have:

$$\mid \eta \mid \|Q_{n-\kappa}U_{n-\kappa}\rho G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - U\rho GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \leq$$

$$\leq C(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}))$$

Finally, using Conditions (A.1), (A.4) of Theorem 1 and Estimation (25), we will obtain:

$$\|Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \leq$$

$$\leq C(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + (n-\kappa)^{-\gamma}), \; \gamma = \min\{\mu, 1+\gamma_1, 1+\gamma_2\}$$

which means the approximation of the operator, B, by the sequence of the operators $(B_{n-\kappa})$ with respect to $Q_{n-\kappa}$.

Let us assume, now, that the sequence $(\mathbf{z}_{n-\kappa})$, $\mathbf{z}_{n-\kappa} \in Z_{n-\kappa}$, is bounded $\|\mathbf{z}_{n-\kappa}\|_{Z_{n-\kappa}} \leq 1$. As the functions, $Q_{n-\kappa}V_{n-\kappa}G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa}$ and $\eta Q_{n-\kappa}U_{n-\kappa}\rho G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa}$, are polynomials and the derivatives of the functions, $VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}$ and $\eta U\rho GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}$ are bounded in Z, then, according to the Riesz theorem [13], the functions:

$$Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa} = Q_{n-\kappa}V_{n-\kappa}G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}-$$

$$-\eta Q_{n-\kappa}U_{n-\kappa}\rho G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} + \eta U\rho GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}$$

form a compact sequence in Z, and thus, Condition (b) is valid. The validity of Condition (c) follows from Condition (A.5) of Theorem 1 and the equivalence of Equations (16) and (19).

Therefore, according to Theorem 6.1 [8], for all n, beginning from some $n_3,\ n_3 \geq n_2$, operators $B_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}$ and, thus, operators $K_{n-\kappa} : X_n \to Z_{n-\kappa}$ are invertible, and their inverses are bounded collectively; and, the approximate solutions $\mathbf{x}_n^* = G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}^*$ of the system of Equations (14) and (15) converge to the exact solution $x^* = GU^{-1}z^*$ of Problems (1) and (2) with a rate:

$$\|\mathbf{x}_n^* - p_n^m x^*\|_{X_n} \leq C\|q_{n-\kappa}Kx^* - K_{n-\kappa}p_n^m x^*\|_{Z_{n-\kappa}} \leq \tag{28}$$

$$\leq C(\|q_{n-\kappa}UD^{(m)}x^* - U_{n-\kappa}D_n^{(m)}p_n^m x^*\|_{Z_{n-\kappa}} + \sum_{\nu=0}^{m-1}(\|q_{n-\kappa}(a_\nu D^{(\nu)}x^* - a_\nu Q_{n-\kappa}D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}}+$$

$$+\|q_{n-\kappa}(b_\nu SD^{(\nu)}x^* - b_\nu SQ_{n-\kappa}D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}} + \|q_{n-\kappa}(Th_\nu D^{(\nu)}x^* - TQ_{n-\kappa}h_\nu D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}})+$$

$$+\|q_{n-\kappa}(Th_mD^{(m)}x^* - T\rho P_n\rho^{-1}h_mD_n^{(m)}p_n^m x^*)\|_{Z_{n-\kappa}})$$

Using once more the boundedness of the operators, $U : Y \to Z,\ S : Z \to Z$, Estimation (24), Hölder inequality and the error estimate of the Gauss-type quadrature formula, we will find:

$$\|q_{n-\kappa}UD^{(m)}x^* - U_{n-\kappa}D_n^{(m)}p_n^m x^*\|_{Z_{n-\kappa}} \leq C(E_{n-\kappa}(UD^{(m)}x^*) + E_n(\rho^{-1}D^{(m)}x^*) + \varepsilon_n(x^*)),$$

$$\|q_{n-\kappa}(a_\nu D^{(\nu)}x^* - a_\nu Q_{n-\kappa}D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}} \leq C\varepsilon_n(x^*)$$

$$\|q_{n-\kappa}(b_\nu SD^{(\nu)}x^* - b_\nu SQ_{n-\kappa}D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}} \leq C(E_{n-\kappa}(b_\nu SD^{(\nu)}x^*) + E_{n-\kappa}(D^{(\nu)}x^*)+$$

$$+\varepsilon_n(x^*) + E_{n-\kappa}(b_\nu SQ_{n-\kappa}D_n^{(\nu)}p_n^m x^*))$$

$$\|q_{n-\kappa}(Th_\nu D^{(\nu)}x^* - TQ_{n-\kappa}h_\nu D_n^{(\nu)}p_n^m x^*)\|_{Z_{n-\kappa}} \leq C(E_{n-\kappa}^t(h_\nu)+$$

$$+E_{n-\kappa}^\tau(h_\nu) + \varepsilon_n(x^*)), \quad \nu = 0, 1, ...m-1$$

$$\|q_{n-\kappa}(Th_mD^{(m)}x^* - T\rho P_n\rho^{-1}h_mD_n^{(m)}p_n^m x^*)\|_{Z_{n-\kappa}} \leq C(E_{n-\kappa}^t(h_m) + E_n^\tau(\rho^{-1}h_mD^{(m)}x^*) + \varepsilon_n(x^*))$$

Thus, taking into account the smoothness of the functions in the right-hand side, Estimation (25) and the obvious inequality (here, C depends on γ and n_3), $(n-\kappa)^{-\gamma} \leq Cn^{-\gamma}$, we will obtain the requested estimation:

$$\|\mathbf{x}_n^* - p_n^m x^*\|_{X_n} \leq C(n^{-\gamma} + \varepsilon_n(x^*)),\ \gamma = \min\{\mu, 1+\gamma_1, 1+\gamma_2\}$$

□

5. Equations of Non-Zero Indices

It is known [7] that for the unique solvability of Problems (1) and (2), in the case when $\kappa > 0$, the equations:

$$\int_{-1}^{1} \tau^j x^{(m)}(\tau) d\tau = 0, \quad j = 0, 1, ..., \kappa - 1 \tag{29}$$

should be added. Therefore, the equations:

$$\sum_{k=0}^{n} \rho^{-1}(\tau_k)[D_n^{(m)}\mathbf{x}_n]_{\tau_k} \int_{-1}^{1} \tau^j \rho(\tau)\bar{l}_k(\tau) d\tau = 0, \quad j = 0, 1, ..., \kappa - 1 \tag{30}$$

should be added to the system of Equations (14) and (15). The justification of the method in this case is similar to the justification in the $\kappa = 0$ case, except the definitions of the spaces, X and X_n, where Conditions (29) and (30) should be added.

Theorem 1. *Let for $\kappa > 0$ Problems (1), (2) and (29) and the calculation scheme, (5)–(15), (30), of the method satisfy Conditions (A.1)–(A.5), (B.1), (B.2) of Theorem 1. Then, for n large enough, the system of Equations (14), (15), (30) is uniquely solvable, and approximate solutions $\mathbf{x}_n^*$ converge to the exact solution, $x^*(\tau) \in X$, of Problems (1), (2), (29) with the error estimation:*

$$\|\mathbf{x}_n^* - p_n^m x^*\|_{X_n} \leq C(n^{-\gamma} + \varepsilon_n(x^*))$$

The case, when $\kappa < 0$, is more complicated, because the operator $U : Y \to Z$, and therefore, the operator, $K : X \to Z$, in this case is, in general, non-invertible; and, Condition (A.5) of Theorem 1 will not be satisfied. We may assume, instead, only the solvability of the concrete equation with the fixed coefficients and the right-hand side. Moreover, the system of Equations (14) and (15) in this case will contain $n + m + 1$ unknown variables, but consist of $n + m + 1 - \kappa$ equations. Therefore, it will be overdetermined and, thus, in general, unsolvable. This means that the previously used proof cannot be applied here. Nevertheless, we may reduce this case to a general one by a simple technique first used by V.V. Ivanov [14] and later by many authors (see, e.g., [15–17]).

Instead of Equation (1), we will consider the equation:

$$UD^{(m)}x + Vx + w = f \tag{31}$$

containing in the left-hand side polynomial:

$$w(t) = \sum_{j=1}^{-\kappa} \chi_j t^{j-1}$$

with the coefficients $\chi_j,\ j = 1, ..., -\kappa$, which ought to be determined. Equations (1) and (31) are closely connected. Indeed, if $x^*(\tau)$ is the solution of Problems (1) and (2), then the couple $(x^*, w),\ w(t) \equiv 0$ will be the solution of Problems (31) and (2). On the other hand, if Problems (1) and (2) will be solvable for only one fixed right-hand side, then corresponding Problems (31) and (2) will be solvable for any right-hand side, $f(t) \in Z$, because to satisfy the conditions of solvability (see [7,9]), one needs to find out only the coefficients: $\chi_j, \quad j = 1, ..., -\kappa$ of the polynomial, $w(t)$, satisfying the equations:

$$\int_{-1}^{1} \bar{\rho}(t) t^{j-1} (f(t) - (Vx)(t) - w(t)) dt = 0, \quad j = 1, ..., -\kappa$$

and thus, the proof of Theorem 1 will be valid also for this case.

The system of Equations (14) and (15) also should be slightly changed. We will add the summands, $w_n(t_j),\ j = 0, 1, ..., n - \kappa$: the values of the approximating polynomial

$$w_n(t) = \sum_{j=1}^{-\kappa} \chi_{jn} t^{j-1}$$

in the nodes of the grid (6) to the left-hand sides of the equations of System (14). In the operator form, the system of equations will take the following form:

$$U_{n-\kappa} D_n^{(m)} \mathbf{x}_n + V_{n-\kappa}\mathbf{x}_n + \mathbf{w}_{n-\kappa} = \mathbf{f}_{n-\kappa}, \quad \mathbf{w}_{n-\kappa} = (w_n(t_0), ..., w_n(t_{n-\kappa})) \tag{32}$$

Now, the number of the unknown variables is $n + m + 1 - \kappa$, so it is equal to the number of equations. These changes now allow us to use the proof, like the one of Theorem 1.

Theorem 2. *Let, for $\kappa < 0$, Problems (1), (2) and the calculation scheme, (5)–(13), (32), of the method satisfy Conditions (A.1)–(A.4), (B.1), (B.2) of Theorem 1. Let us assume, also, that Problems (1) and (2) have a unique solution, $x^*(t)$. Then, for n, large enough, the system of Equation (32) is uniquely solvable, and the approximate solutions: $\bar{\mathbf{x}}_n^* = (\mathbf{x}_n^*, \chi_{1n}^*, ..., \chi_{-\kappa n}^*)$ converge to the exact solution $\bar{x}^* = (x^*, 0)$ of Equation (31) with the error estimation:*

$$\|\bar{\mathbf{x}}_n^* - \bar{p}_n^m \bar{x}^*\|_{\bar{X}_n} = \|\mathbf{x}_n^* - p_n^m x^*\|_{X_n} + \max_{1\le j\le -\kappa} \mid \chi_{jn} \mid \le C(n^{-\gamma} + \varepsilon_n(x^*))$$

$$\bar{p}_n^m \bar{x}^* = (p_n^m x^*, \chi_1^*, ..., \chi_{-\kappa}^*),\ \bar{X}_n = X_n \times R_{-\kappa},\ \|\bar{\mathbf{x}}_n\|_{\bar{X}_n} = \|\mathbf{x}_n\|_{X_n} + \max_{1\le j\le -\kappa} \mid \chi_{jn} \mid$$

The proof of Theorem 3 is, in general, similar to the proof of Theorem 1, so we will give it briefly, paying attention only to the major differences.

Proof of Theorem 3. Let us rewrite Equation (31) in operator form:

$$\bar{K}\bar{x} \equiv \bar{U}(D^{(m)}x, w) + Vx = f,\ \bar{K} : \bar{X} \to Z \tag{33}$$

where:

$$\bar{X} = \{\bar{x} \mid \bar{x} = (x, w), x \in X\},\ \|\bar{x}\|_{\bar{X}} = \|x\|_X + \max_{1\le j\le -\kappa} \mid \chi_j \mid$$

$$\bar{Y} = \{\bar{y} \mid \bar{y} = (y, w), y \in Y\},\ \|\bar{y}\|_{\bar{Y}} = \|y\|_Y + \max_{1\le j\le -\kappa} \mid \chi_j \mid$$

$$\bar{U}(y, w) = Uy + w,\ \bar{U} : \bar{Y} \to Z$$

The operator, $\bar{U}$, is invertible and:

$$\bar{U}^{-1}z = (U^{-1}(z - w), w)$$

where $w(t)$ is a polynomial, the coefficients of which can be found from the equations:

$$\int_{-1}^{1} \bar{\rho}(t) t^{j-1}(z(t) - w(t))dt = 0,\ j = 1, ..., -\kappa$$

Thus, the substitution:

$$z = \bar{U}(D^{(m)}x + \eta\rho x, w) \tag{34}$$

allows us to reduce Equation (33) to the equivalent equation:

$$\bar{B}z \equiv z + VG\bar{U}^{-1}z - \eta U\rho G\bar{U}^{-1}z = f,\ \bar{B} : Z \to Z \tag{35}$$

Let us rewrite Equation (32) in the same way:

$$\bar{K}_{n-\kappa}\bar{\mathbf{x}}_n \equiv \bar{U}_{n-\kappa}(D_n^{(m)}\mathbf{x}_n, \mathbf{w}_{n-\kappa}) + V_{n-\kappa}\mathbf{x}_n = \mathbf{f}_{n-\kappa},\ \bar{K}_{n-\kappa} : \bar{X}_n \to Z_{n-\kappa} \tag{36}$$

where:

$$\bar{X}_n = X_n \times R_{-\kappa},\ \|\bar{\mathbf{x}}_n\|_{\bar{X}_n} = \|\mathbf{x}_n\|_{X_n} + \max_{1\le j\le -\kappa} |\ \chi_{jn}\ |$$

$$\bar{Y}_n = Y_n \times R_{-\kappa},\ \|\bar{\mathbf{y}}_n\|_{\bar{Y}_n} = \|\mathbf{y}_n\|_{Y_n} + \max_{1\le j\le -\kappa} |\ \chi_{jn}\ |$$

$$\bar{U}_{n-\kappa}(\mathbf{y}_n, \mathbf{w}_{n-\kappa}) = U_{n-\kappa}\mathbf{y}_n + \mathbf{w}_{n-\kappa}, \quad \bar{U}_{n-\kappa} : \bar{Y}_n \to Z_{n-\kappa}$$

with $\mathbf{w}_{n-\kappa} = (w_n(t_0), ..., w_n(t_{n-\kappa}))$—a vector of the values of the polynomial, $w_n(t)$, the coefficients of which can be found from the equations:

$$\int_{-1}^{1} \bar{\rho}(t)t^{j-1}((Q_{n-\kappa}\mathbf{z}_{n-\kappa})(t) - w_n(t))dt = 0, \quad j = 1, ..., -\kappa \tag{37}$$

Now, we will use the substitution:

$$\mathbf{z}_{n-\kappa} = \bar{U}_{n-\kappa}(D_n^{(m)}\mathbf{x}_n + \eta\rho\mathbf{x}_n, \mathbf{w}_{n-\kappa})$$

which allows us to reduce Equation (36) to the equivalent equation:

$$\bar{B}_{n-\kappa}\mathbf{z}_{n-\kappa} \equiv \mathbf{z}_{n-\kappa} + V_{n-\kappa}G_n\bar{U}_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - \eta U_{n-\kappa}\rho G_n\bar{U}_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} = \mathbf{f}_{n-\kappa},\ \bar{B}_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}$$

Besides, according to the proof of Theorem 1, we have to check that conditions (a)–(c) are satisfied. Condition (a) may be checked like the one in the proof of Theorem 1. In order to check (b), we had previously to calculate $\mathbf{w}_{n-\kappa}$ for the chosen $\mathbf{z}_{n-\kappa}$, according to Formula (37) and, then, follow the proof of Theorem 1, taking $\mathbf{z}_{n-\kappa} - \mathbf{w}_{n-\kappa}$ instead of $\mathbf{z}_{n-\kappa}$. The validity of Condition (c) follows from the invertibility of the operator, $\bar{K}$. Indeed, for the given right-hand side of Equation (35), we will obtain the right-hand side of Equation (33). Then, due to the invertibility of $\bar{K}$, we will find the couple (x, w) and, via Equation (34), will obtain z.

The error estimation:

$$\|\bar{\mathbf{x}}_n^* - \bar{p}_n^m\bar{x}^*\|_{\bar{X}_n} = \|\mathbf{x}_n^* - p_n^m x^*\|_{X_n} + \max_{1\le j\le -\kappa} |\ \chi_{jn}\ | \le C\|q_{n-\kappa}\bar{K}\bar{x}^* - \bar{K}_{n-\kappa}\bar{p}_n^m\bar{x}^*\|_{Z_{n-\kappa}} =$$

$$= C\|q_{n-\kappa}Kx^* - K_{n-\kappa}p_n^m x^*\|_{Z_{n-\kappa}} \le C(n^{-\gamma} + \varepsilon_n(x^*))$$

obtained just as in the proof of Theorem 1, finishes the proof in this case. □

Remark 1. Theorems 1–3 might be extended to the case of the mostly general boundary conditions [8]:

$$u_\nu(x) \equiv \sum_{i=0}^{m-1} \int_{-1}^{1} x^{(i)}(\tau)d\zeta_{i\nu}(\tau) = 0, \quad \nu = 0, 1, ..., m-1$$

where $\zeta_{i\nu}(\tau),\ i,\nu = 0,1,...,m-1$ are given functions of the bounded variation and integrals are interpreted as Stieltjes ones. These boundary conditions might be approximated by any difference conditions:

$$u_{\nu n}(\mathbf{x}_n) = 0, \quad \nu = 0,1,...,m-1$$

satisfying $u_{\nu n}(p_n^m x) \to u_\nu(x)$ for $n \to \infty$ for any $x \in X$. Theorems 1–3 will remain valid, but the value of $\max\limits_{0\le\nu\le m-1} \mid [D_n^{(\nu)} p_n^m x^*]_{\xi_0} \mid$ in the definition of $\varepsilon_n(x^*)$ should be substituted by the value of $\max\limits_{0\le\nu\le m-1} \mid u_{\nu n}(p_n^m x^*) \mid$.

Remark 2. Condition (A.3) of Theorem 1 is only sufficient and, as is shown in [17–19], can be reduced.

Conflicts of Interest

The author declares no conflict of interest.

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