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Letter

The Complement of Binary Klein Quadric as a Combinatorial Grassmannian

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Abstract: Given a hyperbolic quadric of PG$(5, 2)$, there are 28 points off this quadric and 56 lines skew to it. It is shown that the $(28_6, 56_3)$-configuration formed by these points and lines is isomorphic to the combinatorial Grassmannian of type $G_2(8)$. It is also pointed out that a set of seven points of $G_2(8)$ whose labels share a mark corresponds to a Conwell heptad of PG$(5, 2)$. Gradual removal of Conwell heptads from the $(28_6, 56_3)$-configuration yields a nested sequence of binomial configurations identical with part of that found to be associated with Cayley-Dickson algebras (arXiv:1405.6888).

Keywords: combinatorial Grassmannian; binary Klein quadric; Conwell heptad; three-qubit Pauli group

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Let $\mathcal{Q}^+(5, 2)$ be a hyperbolic quadric in a five-dimensional projective space PG$(5, 2)$. As it is well known (see, e.g., [1,2]), there are 28 points lying off this quadric as well as 56 lines skew (or, external) to it. If the equation of the quadric is taken in a canonical form $\mathcal{Q}_0 : x_1x_2 + x_3x_4 + x_5x_6 = 0$, then the 28 off-quadric points are those listed in Table 1 and the 56 external lines are those given in Table 2. In Table 2, the "+" symbol indicates which point lies on a given line; for example, line 1 consists of points 1, 4 and 9. As it is obvious from this table, each line has three points and through each point there are six lines; hence, these points and lines form a $(28_6, 56_3)$-configuration.

Next, a combinatorial Grassmannian $G_k(|X|)$ (see, e.g., [3,4]), where k is a positive integer and X is a finite set, $|X| = N$, is a point-line incidence structure whose points are all k-element subsets of X

and whose lines are all $(k+1)$-element subsets of X, incidence being inclusion. Obviously, $G_k(N)$ is a $\left(\binom{N}{k}_{N-k}, \binom{N}{k+1}_{k+1}\right)$-configuration; hence, $G_2(8)$ is another $(28_6, 56_3)$-configuration.

It is straightforward to see that the two $(28_6, 56_3)$-configurations are, in fact, isomorphic. To this end, one simply employs the bijection between the 28 off-quadric points and the 28 points of $G_2(8)$ shown in Table 3 (here, by a slight abuse of notation, $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$) and verifies step by step that each of the above-listed 56 lines of PG(5, 2) is also a line of $G_2(8)$; thus, line 1 of PG(5, 2) corresponds to the line $\{1, 4, 6\}$ of $G_2(8)$, line 2 to the line $\{1, 2, 4\}$, line 3 to $\{1, 3, 4\}$, *etc*.

Table 1. The 28 points lying off the quadric $\mathcal{Q}_0$.

No.	x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	1	0	0	0
2	1	1	0	0	1	0
3	1	1	0	0	0	1
4	1	1	0	1	0	0
5	1	1	1	0	1	0
6	1	1	1	0	0	1
7	1	1	0	1	1	0
8	1	1	0	1	0	1
9	0	0	1	1	0	0
10	0	0	1	1	1	0
11	0	0	1	1	0	1
12	0	1	1	1	0	0
13	1	0	1	1	1	0
14	1	0	1	1	0	1
15	0	0	0	0	1	1
16	1	0	0	0	1	1
17	0	0	1	0	1	1
18	0	0	0	1	1	1
19	0	1	1	0	1	1
20	0	1	0	1	1	1
21	1	1	1	1	1	1
22	1	1	0	0	0	0
23	1	0	1	1	0	0
24	0	1	1	1	1	0
25	0	1	1	1	0	1
26	0	1	0	0	1	1
27	1	0	1	0	1	1
28	1	0	0	1	1	1

Table 2. The 56 lines having no points in common with the quadric $\mathcal{Q}_0$.

No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
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56																					+		+			+		

Table 3. A bijection between the 28 off-quadric points and the 28 points of $G_2(8)$.

off-$\mathcal{Q}_0$	$G_2(8)$	off-$\mathcal{Q}_0$	$G_2(8)$
1	$\{1,4\}$	15	$\{2,3\}$
2	$\{3,5\}$	16	$\{4,7\}$
3	$\{2,5\}$	17	$\{5,6\}$
4	$\{4,6\}$	18	$\{1,5\}$
5	$\{2,6\}$	19	$\{1,7\}$
6	$\{3,6\}$	20	$\{6,7\}$
7	$\{1,2\}$	21	$\{4,5\}$
8	$\{1,3\}$	22	$\{7,8\}$
9	$\{1,6\}$	23	$\{5,8\}$
10	$\{2,4\}$	24	$\{3,8\}$
11	$\{3,4\}$	25	$\{2,8\}$
12	$\{5,7\}$	26	$\{4,8\}$
13	$\{3,7\}$	27	$\{1,8\}$
14	$\{2,7\}$	28	$\{6,8\}$

This isomorphism entails a very interesting property related to so-called Conwell heptads [5]. Given a $\mathcal{Q}^+(5,2)$ of PG$(5,2)$, a Conwell heptad (in the modern language also known as a maximal exterior set of $\mathcal{Q}^+(5,2)$, see, e.g., [6]) is a set of seven off-quadric points such that each line joining two distinct points of the heptad is skew to the $\mathcal{Q}^+(5,2)$. There are altogether eight such heptads: any two of them have a unique point in common and each of the 28 points off the quadric is contained in two heptads. The points in Table 1 are arranged in such a way that the last seven of them represent a Conwell heptad, as it is obvious from the bottom part of Table 2. From Table 3 we read off that this particular heptad corresponds to those seven points of $G_2(8)$ whose representatives have mark "8" in common. Clearly, the remaining seven heptads correspond to those septuples of points of $G_2(8)$ that share one of the remaining seven marks each. Finally, we observe that by removing from our off-quadric $(28_6, 56_3)$-configuration the seven points of a Conwell heptad and all the 21 lines defined by pairs of them one gets a $(21_5, 35_3)$-configuration isomorphic to $G_2(7)$; gradual removal of additional heptads and the corresponding lines yields a remarkable nested sequence of configurations displayed in Table 4. Interestingly enough, this nested sequence of binomial configurations is identical with part of that found to be associated with Cayley-Dickson algebras [7]. Moreover, given the fact that PG$(5,2)$ is the natural embedding space for the symplectic polar space $W(5,2)$ that geometrizes the structure of the three-qubit Pauli group [8,9], this particular sequence of configurations may lead to further intriguing insights into the physical relevance of this group.

Table 4. A nested sequence of configurations located in the complement of a hyperbolic quadric of PG$(5, 2)$.

# of Heptads Removed	Configuration	CG	Remark
0	$(28_6, 56_3)$	$G_2(8)$	
1	$(21_5, 35_3)$	$G_2(7)$	
2	$(15_4, 20_3)$	$G_2(6)$	Cayley-Salmon
3	$(10_3, 10_3)$	$G_2(5)$	Desargues
4	$(6_2, 4_3)$	$G_2(4)$	Pasch
5	$(3_1, 1_3)$	$G_2(3)$	single line
6	$(1_0, 0_3)$	$G_2(2)$	single point
7			empty set

To conclude this letter, there are a few facts that deserve a special mention. First, the fact that the complement of $\mathcal{Q}^+(5, 2)$ is isomorphic to the combinatorial Grassmannian $G_2(8)$ can be implicitly be traced down even in the original paper of Conwell [5]. As mentioned above, the complement contains eight heptads and each point of the complement can be identified with the (unordered) pair of heptads through it; also the "grassmannian" rule of forming lines on the complement remains valid. After this observation is made, the combinatorial characterization of heptads becomes evident: these are the maximal cliques of the (binary) collinearity. (Clearly, Conwell himself could not formulate his characterization in this combinatorial language.) Second, the fact that removing a complete graph K_7 from $G_2(8)$ one obtains $G_2(7)$, and so on, was shown in a more general ("$G_{(n+1)}$ minus K_n") setting in [10] (see also [11]). Finally, it is worth pointing out that the group of automorphisms of the $(28_6, 56_3)$-configuration is isomorphic to $S_8 \cong SL_4(2)$:2 (which is the group of collineations and correlations of PG$(3, 2)$, also isomorphic—via the Klein correspondence—to the group of all collineations of PG$(5, 2)$ preserving a hyperbolic quadric).

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Conflicts of Interest

The author declares no conflict of interest.

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