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## On the Nature of the Tsallis–Fourier Transform

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**Abstract:** By recourse to tempered ultradistributions, we show here that the effect of a  $q$ -Fourier transform ( $q$ FT) is to map equivalence classes of functions into other classes in a one-to-one fashion. This suggests that Tsallis'  $q$ -statistics may revolve around equivalence classes of distributions and not individual ones, as orthodox statistics does. We solve here the  $q$ FT's non-invertibility issue, but discover a problem that remains open.

**Keywords:**  $q$ -Fourier transform; tempered ultradistributions

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### 1. Introduction

Non-extensive statistical mechanics (NEXT) [1–3], a well-known generalization of the Boltzmann–Gibbs (BG) one, is used in many scientific and technological endeavors. NEXT's central concept is that of a non-additive (though extensive [4]) entropic information measure characterized by the real index  $q$  (with  $q = 1$  recovering the standard BG entropy). Applications include cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spin glasses [8], turbulence in the heliosphere [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker–Planck equation's studies [16], EEG's [17], complex signals [18], Vlasov–Poisson equations [19], *etc.*

So-called  $q$ -Fourier transforms ( $q$ FT) were developed by Tsallis *et al.* [20]. They constitute a central piece in Tsallis'  $q$ -machinery. However (see [21] and the references therein),  $q$ FT seems not to be an invertible transformation.

The imaginary  $q$ -exponential function, a protagonist of Tsallis' statistics, is defined as:

$$e_q(ix) = [1 + i(1 - q)x]^{\frac{1}{1-q}}, \tag{1}$$

with:

$$e_q(ix) \rightarrow \exp(ix) \text{ whenever } q \rightarrow 1. \tag{2}$$

However, the function:

$$e_q^b(ix) = [1 + i(1 - q)x]^{\frac{1}{1-q}+b}, \tag{3}$$

with  $-\infty < b < 0$ ,  $b$  a real number, also fulfills:

$$e_q^b(ix) \rightarrow \exp(ix) \text{ whenever } q \rightarrow 1. \tag{4}$$

Accordingly, there is a class of functions  $e_q^b(ix)$ , labeled by  $b$ , that tend to the ordinary exponential in the limit  $q \rightarrow 1$ . This fact profoundly affects the workings of the  $q$ -Fourier transform.

The same happens with the  $q$ -logarithm defined as:

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \rightarrow \ln(x) \text{ whenever } q \rightarrow 1 \tag{5}$$

The function:

$$\ln_q^c(x) = \frac{x^{\frac{1-q}{1+c(1-q)}} - 1}{1 - q} \rightarrow \ln(x) \text{ whenever } q \rightarrow 1 \tag{6}$$

where  $-\infty < c < 0$  is a real number. Moreover:

$$\ln_q^b[e_q^b(ix)] = ix. \tag{7}$$

Recall that Schwartz space  $\mathcal{S}$  is the space of functions, all of whose derivatives are rapidly decreasing [22]. This space has the important property that the Fourier transform is an automorphism on this space. This property enables one, by duality, to define the Fourier transform for elements in the dual space of  $\mathcal{S}$ , called the space of tempered distributions [22].

In the present communication, we reconcile the Tsallis *et al.* developments [20] with the non-invertibility issue and show, by recourse to tempered ultradistributions (a generalization and extension to the complex plane of Schwartz' tempered distributions) that the  $q$ FT does indeed map, in a one-to-one fashion, classes of functions into other classes, not isolated functional instances. Thus, such an issue can be resolved by appealing to a higher order of mathematical perspective. Section 2 recapitulates the findings of our work in [21], related to an extension of the Tsallis *et al.* environment [23], which becomes just a particular case (real line) of a more encompassing theory (complex plane) [21]. Section 3, the core of our presentation, specializes the developments of [21] for an important situation. We analyze there the particular instantiation of the theory that leads to the scenario investigated by both Tsallis *et al.* [20]. The results thus obtained, our main contribution here, are illustrated in Section 5 by an important example. Finally, the mathematical problem that remains open is discussed in Section 5. Conclusions are drawn in Section 6.

## 2. Reviewing an Alternative qFT Definition

In [21] (see also [23]), we introduced an alternative qFT-definition that, by generalizing and extending the original one, overcomes the non-invertibility problem afflicting the one of [20]. We briefly review that alternative version in this section. Our protagonists in such an endeavor were tempered ultradistributions [22–25], which constitute a generalization of the distribution set for which the test functions are members of a Schwartz space  $\mathcal{S}$ , a function space in which its members possess derivatives that are rapidly decreasing.  $\mathcal{S}$  exhibits a notable property: the Fourier transform is an automorphism on  $\mathcal{S}$ , a property that allows, by duality, defining the Fourier transform for elements in the dual space of  $\mathcal{S}$ . This dual is the space of tempered distributions [22].

In physics, it is not uncommon to face functions that grow exponentially in space or time. In such circumstances, Schwartz’ space of tempered distributions is too restrictive. One needs ultradistributions to generate appropriate answers [26]. They are continuous linear functionals defined on the space of entire functions rapidly decreasing on straight lines parallel to the real axis [26]. Now, following [23], we appeal to the Heaviside step function  $H$ :

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{8}$$

These step functions allow us to properly define the q-Fourier transform in a different fashion as that of Tsallis *et al.* [20] by recourse [23] to the space  $\mathcal{U}$  of tempered ultradistributions (see [24])

As stated above, a tempered ultradistribution is a continuous linear functional defined on the space  $\mathcal{H}_1$  of entire functions rapidly decreasing on straight lines parallel to the real axis. Let  $\Omega$  be the space of functions of the real variable  $x$  that are parametrized by a real parameter  $q$ :

$$\Omega = \{f_q(x)/f_q(x) \in \Omega^+ \cap \Omega^-\}, \tag{9}$$

where:

$$\begin{aligned} \Omega^+ = \left\{ f_q(x)/f_q|_{\mathbb{R}^+}(x) \{1 + i(1 - q')kx[f_q|_{\mathbb{R}^+}(x)]^{(q'-1)}\}^{\frac{1}{1-q'}} \in \mathcal{L}^1[\mathbb{R}^+]; \right. \\ \left. f_q(x) \geq 0; |f_q(x)| \leq |x|^p g(x)e^{ax}; p, a \in \mathbb{R}^+; k \in \mathbb{C}; \Im(k) \geq 0 \right. \\ \left. 1 \leq q, q' < 2 \right\}, \end{aligned} \tag{10}$$

and:

$$\begin{aligned} \Omega^- = \left\{ f_q(x)/f_q|_{\mathbb{R}^-}(x) \{1 + i(1 - q')kx[f_q|_{\mathbb{R}^-}(x)]^{(q'-1)}\}^{\frac{1}{1-q'}} \in \mathcal{L}^1[\mathbb{R}^-]; \right. \\ \left. f_q(x) \geq 0; |f_q(x)| \leq |x|^p g(x)e^{ax}; p, a \in \mathbb{R}^+; k \in \mathbb{C}; \Im(k) \leq 0 \right. \\ \left. 1 \leq q, q' < 2 \right\}, \end{aligned} \tag{11}$$

Here,  $\mathcal{L}^1$  is the space of functions integrable in the Lebesgue sense,  $g(x)$  is bounded, continuous and positive-definite,  $f_q|_{\mathbb{R}^+}(x)$  is the restriction of  $f_q(x)$  to  $\mathbb{R}^+$  and  $f_q|_{\mathbb{R}^-}(x)$  is the restriction of  $f_q(x)$  to  $\mathbb{R}^-$ .

Our q-Fourier transform is now defined ( $\Im(k)$  is the imaginary part of  $k$ ) as:

$$F : \Omega \longrightarrow \mathcal{U} \text{ (the space of tempered ultradistributions),} \tag{12}$$

where:

$$F(f_q)(k, q') \equiv F(k, q', q), \tag{13}$$

with:

$$F(k, q', q) = [H(q' - 1) - H(q' - 2)] \times \left\{ H[\Im(k)] \int_0^\infty f_q(x) \{1 + i(1 - q')kx[f_q(x)]^{(q'-1)}\}^{\frac{1}{1-q'}} dx - H[-\Im(k)] \int_{-\infty}^0 f_q(x) \{1 + i(1 - q')kx[f_q(x)]^{(q'-1)}\}^{\frac{1}{1-q'}} dx \right\}. \tag{14}$$

The inverse transformation is ([21,23]):

$$f_q(x) = \frac{1}{2\pi} \oint_{\Gamma} \left[ \lim_{\epsilon \rightarrow 0^+} \int_1^2 F(k, q', q) \delta(q' - 1 - \epsilon) dq' \right] e^{-ikx} dk. \tag{15}$$

As has been proven in [21,23],  $F$  is one-to-one from  $\Omega$  to  $\mathcal{U}$

On the real axis:

$$F(k, q') = [H(q' - 1) - H(q' - 2)] \times \int_{-\infty}^\infty f_q(x) \{1 + i(1 - q')kx[f_q(x)]^{(q'-1)}\}^{\frac{1}{1-q'}} dx, \tag{16}$$

for the real transform and:

$$f_q(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[ \lim_{\epsilon \rightarrow 0^+} \int_1^2 F(k, q', q) \delta(q' - 1 - \epsilon) dq' \right] e^{-ikx} dk, \tag{17}$$

for its inverse.

### 3. q-FT in the Limit $q' \rightarrow q$

Our main result is to be presented now, by consideration of the limit  $q' \rightarrow q$ .

This leads to the (restricted) scenario in which the Tsallis *et al.* non-invertibility issue raises its head.

Define the restricted (*i.e.*, to the  $q' = q$  situation) transform, *i.e.*, the Tsallis *et al.* one,

$$F_T : \Omega \longrightarrow \mathcal{U} \tag{18}$$

as:

$$F_T(f_q)(k) = \lim_{q' \rightarrow q} F(f_q)(k, q') = F(f_q)(k, q')|_{q'=q} \tag{19}$$

Thus, according to Equation (14),

$$F_T(k, q) = [H(q - 1) - H(q - 2)] \times \left\{ H[\Im(k)] \int_0^\infty f_q(x) \{1 + i(1 - q)kx[f_q(x)]^{(q-1)}\}^{\frac{1}{1-q}} dx - \right.$$

$$H[-\Im(k)] \int_{-\infty}^0 f_q(x) \{1 + i(1 - q)kx[f_q(x)]^{(q-1)}\}^{\frac{1}{1-q}} dx \} \tag{20}$$

It is seen in [21] that  $F_T$  is not one-to-one from  $\Omega$  to  $\mathcal{U}$ .

The problem is best understood if we introduce a particularly important set  $\Lambda_{f_q}$ , crucial for our considerations. Let  $\Lambda_{f_q}$  be given by:

$$\Lambda_{f_q} = \{g_q \in \Omega / F_T(g_q)(k) = F_T(f_q)(k)\}, \tag{21}$$

and:

$$\Lambda = \{\Lambda_{f_q} / f_q \in \Omega\}. \tag{22}$$

We define the equivalence relation:

$$g_q(x) \sim f_q(x) \iff g_q \in \Lambda_{f_q} \tag{23}$$

and, subsequently, the special the version of the Tsallis *et al.* q-Fourier transform  $F_{UTS}$ : [20]

$$F_{UTS} : \Lambda \longrightarrow \mathcal{U} \tag{24}$$

as:

$$F_{UTS}(\Lambda_{f_q})(k) = F_T(f_q)(k). \tag{25}$$

We see now that  $F_{UTS}$  is an application from equivalence classes into equivalence classes and, as a consequence, one-to-one from  $\Lambda$  into  $\mathcal{U}$ !

We realize now that the Tsallis *et al.* qFT is actually a (one-to-one) set-to-set transformation, which solves the non-invertibility issue that occupies our attention in this work.

### 4. Illustration

Illustrating our theory, we reconsider an important example. We focus our attention on the so-called Hilhorst function (see the pertinent details in [21] and the references therein):

$$f_q(x) = \begin{cases} (\frac{\lambda}{x})^\beta ; & x \in [a, b] ; 0 < a < b ; \lambda > 0 \\ 0 ; & x \text{ outside } [a, b], \end{cases} \tag{26}$$

with:

$$\lambda = \left[ \left( \frac{q-1}{2-q} \right) \left( a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}} \right) \right]^{1-q} \quad \beta = \frac{1}{q-1}. \tag{27}$$

In [23], we evaluated the q-Fourier transform on this function and obtained:

$$F(k, q', q) = [H(q' - 1) - H(q' - 2)]H[\Im(k)] \times \left\{ \left\{ H(q' - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \times \frac{(q' - 1)\lambda^\beta}{(2 - q')[(1 - q')ik\lambda^\beta]^{\frac{1}{q'-1}}} \right\} \times$$

$$\begin{aligned}
 & \left\{ a^{\frac{q'-2}{q'-1}} F \left( \frac{1}{q'-1}, \frac{2-q'}{(q'-1)[1-\beta(q'-1)]}, \frac{1}{q'-1} + \frac{\beta(2-q')}{1-\beta(q'-1)}; \right. \right. \\
 & \quad \left. \left. \frac{1}{(q'-1)ik\lambda^{\beta(q'-1)}a^{1-\beta(q'-1)}} \right) - \right. \\
 & \left. b^{\frac{q'-2}{q'-1}} F \left( \frac{1}{q'-1}, \frac{2-q'}{(q'-1)[1-\beta(q'-1)]}, \frac{1}{q'-1} + \frac{\beta(2-q')}{1-\beta(q'-1)}; \right. \right. \\
 & \quad \left. \left. \frac{1}{(q'-1)ik\lambda^{\beta(q'-1)}b^{1-\beta(q'-1)}} \right) \right\} + \\
 & \quad \left\{ H \left[ q' - \left( 1 + \frac{1}{\beta} \right) \right] - H(q'-2) \right\} \frac{\lambda^\beta}{\beta-1} \times \\
 & \quad \left\{ a^{1-\beta} F \left( \frac{1}{q'-1}, \frac{\beta-1}{\beta(q'-1)-1}, \frac{\beta q'-2}{\beta(q'-1)-1}; \right. \right. \\
 & \quad \left. \left. (q'-1)ik\lambda^{\beta(q'-1)}a^{1-\beta(q'-1)} \right) - \right. \\
 & \quad \left. b^{1-\beta} F \left( \frac{1}{q'-1}, \frac{\beta-1}{\beta(q'-1)-1}, \frac{\beta q'-2}{\beta(q'-1)-1}; \right. \right. \\
 & \quad \left. \left. (q'-1)ik\lambda^{\beta(q'-1)}b^{1-\beta(q'-1)} \right) \right\}. \tag{28}
 \end{aligned}$$

Taking  $q' = q$  in Equation (28), we have for  $F_{UTS}$ :

$$F_{UTS}(\Lambda_{f_q})(k) = H[\mathfrak{S}(k)] [H(q-1) - H(q-2)] [1 + (1-q)ik\lambda]^{\frac{1}{1-q}}, \tag{29}$$

and, on the real axis,

$$\begin{aligned}
 F_{UTS}(\Lambda_{f_q})(k) &= [H(q-1) - H(q-2)] [1 + (1-q)i(k+i0)\lambda]^{\frac{1}{1-q}} = \\
 & [H(q-1) - H(q-2)] [1 + (1-q)ik\lambda]^{\frac{1}{1-q}}. \tag{30}
 \end{aligned}$$

Now, from Equations (29) and (30), we see that the Tsallis *et al.* q-Fourier transform is one-to-one. However, it is a transformation from  $\Lambda$  into  $\mathcal{U}$ , a class-to-class one.

This fact reconciles the viewpoints of Tsallis *et al.* and those of [21], and this is achieved via a rigorous definition of the qFT and of its domain and image.

As a second example, we consider:

$$f(x) = H(x).$$

In this case, the equivalence class is made up of just one function:

$$F_{UTS}(k, q) = [H(q-1) - H(q-2)] H[\mathfrak{S}(k)] \int_0^\infty [1 + (1-q)ikx]^{\frac{1}{1-q}} dx. \tag{31}$$

Evaluating the integral, we have:

$$F_{UTS}(k, q) = [H(q-1) - H(q-2)] H[\mathfrak{S}(k)] \frac{\Gamma\left(\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)} [(1-q)ik]^{-1}, \tag{32}$$

and, finally,

$$F_{UTS}(k, q) = [H(q-1) - H(q-2)] \frac{i}{2-q} \frac{H[\mathfrak{S}(k)]}{k}. \tag{33}$$

### 5. An Open Problem

Let  $\sim$  be the equivalence relation defined by:

$$g_q \sim f_q \iff \lim_{q \rightarrow 1} g_q = \lim_{q \rightarrow 1} f_q, \tag{34}$$

where  $f_q, g_q \in \Omega$ . Let:

$$\Xi_{f_q} = \{g_q \in \Omega / g_q \sim f_q\}, \tag{35}$$

and let:

$$\Xi = \{\Xi_{f_q} / f_q \in \Omega\}. \tag{36}$$

If  $f_q = f_{1q} + (q - 1)f_{2q}$ ,  $g_q = g_{1q} + (q - 1)g_{2q}$  and  $\lim_{q \rightarrow 1} f_{1q} = \lim_{q \rightarrow 1} g_{1q}$ , then  $f_q, g_q \in \Xi_{f_q}$ . However, since  $f_{2q}, g_{2q} \in \Omega$  are different in general, one does not have  $f_q \sim g_q$ . As a simple example of this, take  $f_{1q}(x) = f_{2q}(x) = x^{q-1}$  and  $g_{1q} = g_{2q} = x^{2(q-1)}$ . Accordingly,  $\Lambda_{f_q} \subseteq \Xi_{f_q}$ .

One would, of course, be interested in finding out which are the mathematical properties of the functions  $f_q$  and  $g_q$  that generate the belonging  $g_q \in \Lambda_{f_q}$ . In other words, what are the mathematical properties that the functions  $f_q$  and  $g_q$  need to have so that  $F_T(f_q) = F_T(g_q)$ . Note that one has:

$$F_T(f_q) = F_T(g_q) \implies F(f) = F(g) \implies f = g. \tag{37}$$

That is  $\lim_{q \rightarrow 1} f_q = \lim_{q \rightarrow 1} g_q$ . This is a problem that we were unable to solve and that we would like to be considered by the mathematical community

### 6. Conclusions

We have shown here an important original result: the q-generalization advanced by Tsallis *et al.* in [20] is to be properly regarded as a transformation between classes of equivalence and, thus, one-to-one, a finding of this paper that solves the qFT's non-invertibility issue [21].

Our present findings may indicate that Tsallis' q-statistics revolves around equivalence classes of distributions and not individual ones, as orthodox statistics does. In Section 5, we have seen, however, that an open problem remains that should be addressed in the future.

### Author Contributions

A. Plastino and Mario C. Rocca contributed equally to this work.

### Conflicts of Interest

The authors declare no conflict of interest.

### References

1. Gell-Mann, M.; Tsallis, C. (Eds.) *Nonextensive Entropy: Interdisciplinary Applications*; Oxford University Press: New York, NY, USA, 2004.
2. Tsallis, C. *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*; Springer: New York, NY, USA, 2009.

3. Plastino, A.R.; Plastino, A. Tsallis Entropy, Erhenfest Theorem and Information Theory. *Phys. Lett. A* **1993**, *177*, 177–179.
4. Tsallis, C.; Gell-Mann, M.; Sato, Y. Asymptotically scale-invariant occupancy of phase space makes the entropy Sq extensive. *Proc. Natl. Acad. Sci. USA* **2005**, *102*, 15377–15382.
5. Douglas, P.; Bergamini, S.; Renzoni, F. Tunable Tsallis Distributions in Dissipative Optical Lattices. *Phys. Rev. Lett.* **2006**, *96*, 110601, doi:10.1103/PhysRevLett.96.110601.
6. Liu, B.; Goree, J. Superdiffusion and Non-Gaussian Statistics in a Driven-Dissipative 2D Dusty Plasma. *Phys. Rev. Lett.* **2008**, *100*, doi:10.1103/PhysRevLett.100.055003.
7. DeVoe, R.G. Power-Law Distributions for a Trapped Ion Interacting with a Classical Buffer Gas. *Phys. Rev. Lett.* **2009**, *102*, 063001, doi:10.1103/PhysRevLett.102.063001.
8. Pickup, R.M.; Cywinski, R.; Pappas, C.; Farago, B.; Fouquet, P. Generalized Spin-Glass Relaxation. *Phys. Rev. Lett.* **2009**, *102*, 097202, doi:10.1103/PhysRevLett.102.097202.
9. Burlaga, L.F.; Ness, N.F. Compressible turbulence observed in the helioheath by Voyager 2. *Astrophys. J.* **2009**, *703*, 311.
10. Caruso, F.; Pluchino, A.; Latora, V.; Vinciguerra, S.; Rapisarda, A. Analysis of self-organized criticality in the Olami-Feder-Christensen model and in real earthquakes. *Phys. Rev. E* **2007**, *75*, 055101(R), doi:10.1103/PhysRevE.75.055101.
11. The CMS Collaboration. Transverse-Momentum and Pseudorapidity Distributions of Charged Hadrons in pp Collisions at s=7 TeV. *Phys. Rev. Lett.* **2010**, *105*, 022002, doi:10.1103/PhysRevLett.105.022002.
12. Adare, A.; Afanasiev, S.; Aidala, C.; Ajitanand, N.N.; Akiba, Y.; Al-Bataineh, H.; Alexander, J.; Aoki, K.; Aphecetche, L.; Armendariz, R.; *et al.* Measurement of neutral mesons in p+p collisions at s=200 GeV and scaling properties of hadron. *Phys. Rev. D* **2011**, *83*, 052004, doi:10.1103/PhysRevD.83.052004.
13. Lyra, M.L.; Tsallis, C. Nonextensivity and Multifractality in Low- Dimensional Dissipative Systems. *Phys. Rev. Lett.* **1998**, *80*, 53, doi:10.1103/PhysRevLett.80.53.
14. Borland, L. Option Pricing Formulas Based on a Non-Gaussian Stock Price Model. *Phys. Rev. Lett.* **2002**, *89*, 098701, doi:10.1103/PhysRevLett.89.098701.
15. Plastino, A.R.; Plastino, A. Tsallis Entropy and Stellar Polytropes. *Phys. Lett A* **1993**, *174*, 384–386.
16. Plastino, A.R.; Plastino, A. Non-extensive statistical mechanics and generalized Fokker-Planck equation. *Phys. A* **1995**, *222*, 347–354.
17. Capurro, A.; Diambra, L.; Lorenzo, D.; Macadar, O.; Martin, M.T.; Mostaccio, C.; Plastino, A. Tsallis entropy and cortical dynamics: The analysis of EEG signals. *Phys. A* **1998**, *257*, 149–155.
18. Martin, M.T.; Plastino, A.R.; Plastino, A. Tsallis-like information measures and the analysis of complex signals. *Phys. A* **2000**, *275*, 262–271.
19. Plastino, A.R.; Plastino, A. Non-extensive solutions to the Vlasov equation. *Braz. J. Phys.* **1999**, *29*, 79.
20. Umarov, S.; Tsallis, C.; Steinberg, S. On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics. *Milan J. Math.* **2008**, *76*, 307.

21. Plastino, A.; Rocca, M.C. Inversion of Tsallis' q-Fourier Transform and the complex-plane generalization. *Phys. A* **2012**, *391*, 4740–4747.
22. Sebastiao e Silva, J. Les fonctions analytiques comme ultra-distributions dans le calcul operationnel. *Math. Ann.* **1958**, *136*, 58–96.
23. Plastino, A.; Rocca, M.C. q-Fourier Transform and its inversion- problem. *Milan J. Math.* **2012**, *80*, 243–249.
24. Hasumi, M. Note on the n-dimensional tempered ultra-distributions. *Tôhoku Math. J.* **1961**, *13*, 94–104.
25. Schwartz, L. *Théorie des Distributions*; Hermann: Paris, France, 1966.
26. Bollini, C.G.; Escobar, T.; Rocca, M.C. Convolution of ultradistri-butions and Âfeld theory. *Int. J. Theor. Phys.* **2003**, *43*, doi: 10.1023/B:IJTP.0000048599.21501.93.

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