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Article

Reformulated First Zagreb Index of Some Graph Operations

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Abstract: The reformulated Zagreb indices of a graph are obtained from the classical Zagreb indices by replacing vertex degrees with edge degrees, where the degree of an edge is taken as the sum of degrees of the end vertices of the edge minus 2. In this paper, we study the behavior of the reformulated first Zagreb index and apply our results to different chemically interesting molecular graphs and nano-structures.

Keywords: topological index; vertex degree; Zagreb indices; reformulated Zagreb indices; graph operations

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1. Introduction

Suppose $\sum$ denotes the class of all graphs, then a function $T : \sum \rightarrow \mathbf{R}^+$ is known as a topological index if for every graph H isomorphic to G, $T(G) = T(H)$. Different topological indices are found to be useful in isomer discrimination, structure-property relationship, structure-activity relationship, pharmaceutical drug design, *etc.* in chemistry, biochemistry and nanotechnology. Suppose G is a simple connected graph and $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G. Let, for any vertex $v \in V(G)$, $d_G(v)$ denotes its degree, that is the number of neighbors of v. Let n vertices of G be denoted by $v_1, v_2, ..., v_n$. If the edges of G are (v_1, v_2), $(v_2, v_3)....(v_{n-1}, v_n)$, (v_{n-1}, v_n) then the graph is called a path graph and is denoted by P_n. The wheel graph, denoted by W_n, is obtained by adding a new vertex to the cycle C_n and connects this new vertex to each vertex of C_n. Similarly, a fan graph F_n, is obtained by adding a new vertex to the path graph P_n and connects this new vertex to each vertex of P_n. The first and second Zagreb indices of a graph, denoted by $M_1(G)$ and $M_2(G)$ are among the oldest, most popular and most extensively studied vertex-degree-based topological indices. These indices were introduced Gutman and Trinajstić in a paper in 1972 [1] to study the structure-dependency of the total π-electron energy (ε) and are respectively defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

These indices are extensively studied in (chemical) graph theory. Interested readers are referred to [2,3] for some recent reviews on the topic. Milićević *et al.* [4] reformulated the Zagreb indices in terms of edge degrees instead of vertex degrees, where the degree of an edge $e = uv$ is defined as $d(e) = d(u) + d(v) - 2$. Thus, the reformulated first and second Zagreb indices of a graph G are defined as

$$EM_1(G) = \sum_{e \in E(G)} d(e)^2 \text{ and } EM_2(G) = \sum_{e \sim f} d(e)d(f)$$

where $e \sim f$ means that the edges e and f share a common vertex in G, *i.e.*, they are adjacent. Different mathematical properties of reformulated Zagreb indices have been studied in [5]. In [6], Ilić *et al.*, establish further mathematical properties of the reformulated Zagreb indices. In [7], bounds for the reformulated first Zagreb index of graphs with connectivity at most k are obtained. De [8] found some upper and lower bounds of these indices in terms of some other graph invariants and also derived reformulatrd Zagreb indices of a class of dendrimers [9]. Ji *et al.* [10,11] computed these indices for acyclic, unicyclic, bicyclic and tricyclic graphs.

Graph operations play a very important role in mathematical chemistry, since some chemically interesting graphs can be obtained from some simpler graphs by different graph operations. In [12], Khalifeh *et al.*, derived some exact expressions for computing first and second Zagreb indices of some graph operations. Ashrafi *et al.* [13] derived explicit expressions for Zagreb coindices of different graph operations. Das *et al.* [14] derived some upper bounds for multiplicative Zagreb indices of different graph operations. Azari and Iranmanesh [15], presented explicit formulae for computing the eccentric-distance sum of different graph operations. In [16] and [17], the present authors obtained some bounds and exact formulae for the connective eccentric index and for F-index of different graph operations. There are several other papers concerning topological indices of different graph operations. For more results on topological indices of different graph operations, interested readers are referred to the papers [18–22].

In this paper we present some exact expressions for the reformulated first Zagreb index of different graph operations such as join, Cartesian product, composition, corona product, splice and link of graphs. Also we apply our results by specializing components of these graph operations to compute the reformulated first Zagreb index for some important classes of molecular graphs and nano-structures.

2. Main Results

In this section, we study the reformulated first Zagreb index under join, Cartesian product, composition, corona product, link and splice of graphs. All these operations are binary, and the join and Cartesian product of graphs are commutative operations, whereas the composition and corona product operations are noncommutative. All the graphs considered here are connected, finite and simple. Let G_1 and G_2 be two simple connected graphs, so that their vertex sets and edge sets are represented as $V(G_i)$ and $E(G_i)$ respectively, for $i \in \{1, 2\}$.

2.1. The Join of Graphs

The join $G_1 + G_2$ is the union $G_1 \cup G_2$ together with all the edges joining each vertex of $V(G_1)$ to each vertex of $V(G_2)$. The degree of a vertex v of $G_1 + G_2$ is given by

$$d_{G_1+G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, v \in V(G_2) \end{cases}$$

In the following theorem we compute the reformulated first Zagreb index of the join of two graphs.

Theorem 1. *The reformulated first Zagreb index of $G_1 + G_2$ is given by*

$$\begin{aligned} EM_1(G_1 + G_2) &= EM_1(G_1) + EM_1(G_2) + 5|V(G_1)|M_1(G_2) + 5|V(G_2)|M_1(G_1) \\ &+|V(G_1)||V(G_2)|(|V(G_1)| + |V(G_2)| - 2)^2 + 8|E(G_1)||E(G_2)| \\ &+4(|V(G_1)| + |V(G_2)| - 2)(|V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|) \\ &+4|V(G_1)|^2|E(G_2)| + 4|V(G_2)|^2|E(G_1)| - 8|V(G_1)||E(G_2)| \\ &-8|V(G_2)||E(G_1)|. \end{aligned}$$

Proof of Theorem 1. We have

$$\begin{aligned} EM_1(G_1 + G_2) &= \sum_{uv \in E(G_1+G_2)} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\ &= \sum_{uv \in E(G_1)} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 + \sum_{uv \in E(G_2)} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\ &+ \sum_{uv \in \{uv : u \in V(G_1), v \in V(G_2)\}} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\ &= J_1 + J_2 + J_3 \end{aligned}$$

where J_1, J_2 and J_3 denote the sums of the above terms in order. Next we calculate J_1, J_2 and J_3 separately.

$$\begin{aligned}
J_1 &= \sum_{uv\in E(G_1)} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\
&= \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) + 2|V(G_2)| - 2)^2 \\
&= \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 + 4|V(G_2)| \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2) \\
&\quad +4 \sum_{uv\in E(G_1)} |V(G_2)|^2 \\
&= EM_1(G_1) + 4|V(G_2)|^2|E(G_1)| + 4|V(G_2)|(M_1(G_1) - 2|E(G_1))|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_2 &= \sum_{uv\in E(G_2)} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\
&= \sum_{uv\in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) + 2|V(G_1)| - 2)^2 \\
&= \sum_{uv\in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 + 4|V(G_1)| \sum_{uv\in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2) \\
&\quad +4 \sum_{uv\in E(G_2)} |V(G_1)|^2 \\
&= EM_1(G_2) + 4|V(G_1)|^2|E(G_2)| + 4|V(G_1)|(M_1(G_2) - 2|E(G_2)|).
\end{aligned}$$

Finally,

$$\begin{aligned}
J_3 &= \sum_{uv\in\{uv:u\in V(G_1),v\in V(G_2)\}} (d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2)^2 \\
&= \sum_{u\in V(G_1),v\in V(G_2)} (d_{G_1}(u) + |V(G_2)| + d_{G_2}(v) + |V(G_1)| - 2)^2 \\
&= \sum_{u\in V(G_1),v\in V(G_2)} \{d_{G_1}(u)^2 + d_{G_2}(v)^2 + 2d_{G_1}(u)d_{G_2}(v) + (|V(G_1)| + |V(G_2)| - 2)^2 \\
&\quad +2(d_{G_1}(u) + d_{G_2}(v))(|V(G_1)| + |V(G_2)| - 2)\} \\
&= |V(G_2)|M_1(G_1) + |V(G_1)|M_1(G_2) + 8|E(G_1)||E(G_2)| + |V(G_1)||V(G_2)|(|V(G_1)| \\
&\quad +|V(G_2)| - 2)^2 + 4(|V(G_1)| + |V(G_2)| - 2)(|V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|).
\end{aligned}$$

Adding J_1, J_2 and J_3, we get the desired result. □

Example 1. *The complete bipartite graph $K_{p,q}$ is defined as join of $\bar{K}_p$ and $\bar{K}_q$, so that, from Theorem 1, its reformulated first Zagreb index is given by $EM_1(K_{p,q}) = pq(p + q - 2)^2$.*

The suspension of a graph G is defined as $G + K_1$. Thus the reformulated first Zagreb index of a graph is calculated as follows.

Corollary 2. *The suspension of a graph G is given by*

$$EM_1(G + K_1) = EM_1(G) + 5M_1(G) + n^3 - 2n^2 + n - 8m + 4nm.$$

Example 2. *The fan graph F_n on $(n + 1)$ vertices is the suspension of P_n. So, using Corollary 2, its reformulated first Zagreb index is calculated as*

$$EM_1(P_n + K_1) = n^3 + 2n^2 + 13n - 32.$$

Example 3. *The wheel graph W_n on $(n + 1)$ vertices is the suspension of C_n. So, its reformulated first Zagreb index is given by*

$$EM_1(C_n + K_1) = n^3 + 2n^2 + 17n.$$

Example 4. *The Dutch windmill graph or flower graph is the suspension of m copies of K_2, denoted by mK_2. So its reformulated first Zagreb index is calculated as*

$$EM_1(K_1 + mK_2) = 8m^3 + 4m.$$

2.2. The Cartesian Product of Graphs

Let G_1 and G_2 be two connected graphs. The Cartesian product of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$. Any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if $[u_p = u_q \in V(G_1)$ and $v_r v_s \in E(G_2)]$ or $[v_r = v_s \in V(G_2)$ and $u_p u_q \in E(G_1)]$ and $r, s = 1, 2, ..., |V(G_2)|$.

In the following theorem we obtain the reformulated first Zagreb index of the Cartesian product of two graphs.

Theorem 3. *The reformulated first Zagreb index of $G_1 \times G_2$ is given by*

$$EM_1(G_1 \times G_2) = |V(G_1)|EM_1(G_2) + |V(G_2)|EM_1(G_1) + 12|E(G_1)|M_1(G_2)$$

$$+12|E(G_2)|M_1(G_1) - 32|E(G_1)||E(G_2)|.$$

Proof of Theorem 3. We have,

$$\begin{aligned}
EM_1(G_1 \times G_2) &= \sum_{(a,x)(b,y) \in E(G_1 \times G_2)} \left(d_{G_1 \times G_2}(a,x) + d_{G_1 \times G_2}(b,y) - 2\right)^2 \\
&= \sum_{(a,x)(a,y), xy \in E(G_2)} \left(d_{G_1 \times G_2}(a,x) + d_{G_1 \times G_2}(a,y) - 2\right)^2 \\
&\quad + \sum_{(a,x)(b,x), ab \in E(G_1)} \left(d_{G_1 \times G_2}(a,x) + d_{G_1 \times G_2}(b,x) - 2\right)^2 \\
&= A_1 + A_2
\end{aligned}$$

where A_1 and A_2 denote the sums of the above terms in order. Next we calculate A_1 and A_2 separately one by one. Now,

$$\begin{aligned}
A_1 &= \sum_{(a,x)(a,y),xy\in E(G_2)} (d_{G_1\times G_2}(a,x)+d_{G_1\times G_2}(a,y)-2)^2 \\
&= \sum_{a\in V(G_1)}\sum_{xy\in E(G_2)} (2d_{G_1}(a)+d_{G_2}(x)+d_{G_2}(y)-2)^2 \\
&= 4\sum_{xy\in E(G_2)}\sum_{a\in V(G_1)} d_{G_1}(a)^2+\sum_{a\in V(G_1)}\sum_{xy\in E(G_2)}(d_{G_2}(x)+d_{G_2}(y)-2)^2 \\
&\quad +4\sum_{a\in V(G_1)} d_{G_1}(a)\sum_{xy\in E(G_2)}(d_{G_2}(x)+d_{G_2}(y)-2) \\
&= |V(G_1)|EM_1(G_2)+8|E(G_1)|M_1(G_2)+4|E(G_2)|M_1(G_1)-16|E(G_1)||E(G_2)|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_2 &= \sum_{(a,x)(b,x),ab\in E(G_1)} (d_{G_1\times G_2}(a,x)+d_{G_1\times G_2}(b,x)-2)^2 \\
&= \sum_{x\in V(G_2)}\sum_{ab\in E(G_1)} (2d_{G_2}(x)+d_{G_1}(a)+d_{G_1}(b)-2)^2 \\
&= 4\sum_{ab\in E(G_1)}\sum_{x\in V(G_2)} d_{G_2}(x)^2+\sum_{x\in V(G_2)}\sum_{ab\in E(G_1)}(d_{G_1}(a)+d_{G_1}(b)-2)^2 \\
&\quad +4\sum_{x\in V(G_2)} d_{G_2}(x)\sum_{ab\in E(G_1)}(d_{G_1}(a)+d_{G_1}(b)-2) \\
&= |V(G_2)|EM_1(G_1)+4|E(G_1)|M_1(G_2)+8|E(G_2)|M_1(G_1)-16|E(G_1)||E(G_2)|.
\end{aligned}$$

By adding A_1 and A_2, the desired result follows after simple calculation. □

Let $P_n(n \geq 1)$ and $C_n(n \geq 3)$ be path and cycle of order n respectively, then we have $EM_1(P_n) = 4n-10$, $EM_1(K_n) = 2n(n-1)(n-2)^2$ and $EM_1(C_n) = 4n$.

Example 5. *The Ladder graph L_n, made by n square and $2n+2$ vertices is the Cartesian product of P_2 and P_{n+1}, so the reformulated first Zagreb index of L_n is given by $EM_1(L_n) = 48n-36$.*

Example 6. *For a C_4-nanotorus $TC_4(m,n) = C_n \times C_m$, the reformulated first Zagreb index is given by $EM_1(TC_4(m,n)) = 72nm$.*

Example 7. *For a C_4-nanotube $TUC_4(m,n) = P_n \times C_m$, the reformulated first Zagreb index is given by $EM_1(TUC_4(m,n)) = 72nm-98m$.*

Example 8. *The reformulated first Zagreb index of the grids $(P_n \times P_m)$ is given by*

$$EM_1(P_n \times P_m) = 72mn-98m-98n+112.$$

Example 9. *The n-prism is defined as the Cartesian product of K_2 and C_n. The reformulated first Zagreb index of the n-prism is given by $EM_1(K_2 \times C_n) = 48n$.*

Example 10. *The rook's graph is defined as the Cartesian product of two complete graphs, say K_n and K_m. So, the reformulated first Zagreb index of the rook's graph is given by*

$$\begin{aligned} EM_1(K_n \times K_m) &= 2mn[(m-1)(m-2)^2 + (n-1)(n-2)^2 + 3(n-1)(m-1)^2 \\ &\quad +3(m-1)(n-1)^2 - 4(n-1)(m-1)]. \end{aligned}$$

2.3. Composition

This operation is also termed the lexicographic product. The composition of two graphs G_1 and G_2 is denoted by $G_1[G_2]$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G_1)$ or $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$. The vertex set of $G_1[G_2]$ is $V(G_1) \times V(G_2)$ and the degree of a vertex (a, b) of $G_1[G_2]$ is given by $d_{G_1[G_2]}(a, b) = n_2 d_{G_1}(a) + d_{G_2}(b)$. In the following theorem we compute the reformulated first Zagreb index of the composition of two graphs.

Theorem 4. *The reformulated first Zagreb index of $G_1[G_2]$ is given by*

$$\begin{aligned} EM_1(G_1[G_2]) &= |V(G_1)|EM_1(G_2) + |V(G_2)|^4 EM_1(G_1) + 4|V(G_2)|^2|E(G_2)|M_1(G_1) \\ &\quad +2|V(G_2)|\left(M_1(G_1) - 2|E(G_1)|\right)\left(4|V(G_2)||E(G_2)| + 2|V(G_2)|^2(|V(G_2)|-1)\right) \\ &\quad +4|V(G_2)|^2|E(G_1)|(|V(G_2)|-1)^2 + 16|V(G_2)||E(G_1)||E(G_2)|(|V(G_2)|-2) \\ &\quad +10|E(G_1)||V(G_2)|M_1(G_1) + 8|E(G_1)||E(G_2)|^2. \end{aligned}$$

Proof of Theorem 4. We have

$$\begin{aligned} EM_1(G_1[G_2]) &= \sum_{(a,x)(b,y)\in E(G_1[G_2])} \left(d_{G_1[G_2]}(a,x) + d_{G_1[G_2]}(b,y) - 2\right)^2 \\ &= \sum_{(a,x)(a,y)\in E(G_1[G_2]),xy\in E(G_2)} \left(d_{G_1[G_2]}(a,x) + d_{G_1[G_2]}(a,y) - 2\right)^2 \\ &\quad + \sum_{(a,x)(b,y)\in E(G_1[G_2]),ab\in E(G_1)} \left(d_{G_1[G_2]}(a,x) + d_{G_1[G_2]}(b,y) - 2\right)^2 \\ &= C_1 + C_2. \end{aligned}$$

where C_1 and C_2 denote the sums of the above terms in order. Next we calculate C_1 and C_2 separately.

Now,

$$\begin{aligned} C_1 &= \sum_{(a,x)(a,y)\in E(G_1[G_2]),xy\in E(G_2)} \left(d_{G_1[G_2]}(a,x) + d_{G_1[G_2]}(a,y) - 2\right)^2 \\ &= \sum_{a\in V(G_1)} \sum_{xy\in E(G_2)} (2|V(G_2)|d_{G_1}(a) + d_{G_2}(x) + d_{G_2}(y) - 2)^2 \\ &= 4|V(G_2)|^2 \sum_{xy\in E(G_2)} \sum_{a\in V(G_1)} d_{G_1}(a)^2 + \sum_{a\in V(G_1)} \sum_{xy\in E(G_2)} (d_{G_2}(x) + d_{G_2}(y) - 2)^2 \\ &\quad +4|V(G_2)| \sum_{a\in V(G_1)} d_{G_1}(a) \sum_{xy\in E(G_2)} (d_{G_2}(x) + d_{G_2}(y) - 2) - 8|V(G_2)| \sum_{xy\in E(G_2)} \sum_{a\in V(G_1)} d_{G_1}(a) \\ &= |V(G_1)|EM_1(G_2) + 4|V(G_2)|^2|E(G_2)|M_1(G_1) + 8|V(G_2)||E(G_1)|M_1(G_2) \\ &\quad -16|V(G_2)||E(G_1)||E(G_2)|. \end{aligned}$$

Also,

$$\begin{aligned}
C_2 &= \sum_{(a,x)(b,y)\in E(G_1[G_2])} \left(d_{G_1[G_2]}(a,x) + d_{G_1[G_2]}(b,y) - 2\right)^2 \\
&= \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} \sum_{ab\in E(G_1)} (|V(G_2)|d_{G_1}(a) + d_{G_2}(x) + |V(G_2)|d_{G_1}(b) + d_{G_2}(y) - 2)^2 \\
&= \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} \sum_{ab\in E(G_1)} (|V(G_2)|(d_{G_1}(a) + d_{G_1}(b) - 2) + d_{G_2}(x) + d_{G_2}(y) - 2 + 2|V(G_2)|)^2 \\
&= |V(G_2)|^2 \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} \sum_{ab\in E(G_1)} (d_{G_1}(a) + d_{G_1}(b) - 2)^2 \\
&\quad +2|V(G_2)| \sum_{ab\in E(G_1)} (d_{G_1}(a) + d_{G_1}(b) - 2) \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} (d_{G_2}(x) + d_{G_2}(y) + 2(|V(G_2)| - 1)) \\
&\quad +2|V(G_2)| \sum_{ab\in E(G_1)} \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} \left(d_{G_2}(x)^2 + d_{G_2}(y)^2 + 2d_{G_2}(x)d_{G_2}(y)\right) \\
&\quad +8|V(G_2)|(|V(G_2)| - 1) \sum_{ab\in E(G_1)} \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} 4(d_{G_2}(x) + d_{G_2}(y)) \\
&\quad +16|V(G_2)||E(G_1)||E(G_2)|(|V(G_2)| - 2) \\
&= 2|V(G_2)|\left(M_1(G_1) - 2|E(G_1)|\right)\left(4|V(G_2)||E(G_2)| + 2|V(G_2)|^2(|V(G_2)| - 1)\right) \\
&\quad +4|V(G_2)|^2|E(G_1)|(|V(G_2)| - 1)^2 + 2|V(G_2)||E(G_1)|M_1(G_2) + 8|E(G_1)||E(G_2)|^2 \\
&\quad +|V(G_2)|^4 EM_1(G_1) + 16|V(G_2)||E(G_1)||E(G_2)|(|V(G_2)| - 2).
\end{aligned}$$

Now combining C_1 and C_2, on simplification, the reformulated first Zagreb index of $G_1[G_2]$ is obtained as above. $\square$

Example 11. *The fence graph and closed fence graph are defined as $P_n[P_2]$ and $C_n[P_2]$. So, from Theorem 4, the reformulated first Zagreb index of these graphs are given by*
(i) $EM_1(P_n[P_2]) = 320n - 576$
(ii) $EM_1(C_n[P_2]) = 320n$.

2.4. Splice of Graphs

A splice of G_1 and G_2 was introduced by Doslić [23]. Let $y \in V(G_1)$ and $z \in V(G_2)$ be two given vertices of G_1 and G_2 respectively. The splice of two graphs G_1 and G_2 at the vertices y and z is denoted by $(G_1 \bullet G_2)(y, z)$ and is obtained by identifying the vertices y and z in the union of G_1 and G_2. The vertex set of $(G_1 \bullet G_2)(y, z)$ is given by $V((G_1 \bullet G_2)(y, z)) = [V(G_1) \backslash \{y\}] \cup [V(G_2) \backslash \{z\}] \cup v_{12}$, where we denote the vertex obtained by identifying y and z by v_{12}. From the construction of the splice of two graphs it is clear that

$$d_{G_1 \bullet G_2(y,z)}(v) = \begin{cases} d_{G_i}(v), & \text{for } v \in V(G_i) \text{ and } v \neq y, z \\ d_{G_1}(y) + d_{G_2}(z), & \text{for } v = v_{12} \end{cases}$$

Let $N(v)$ denotes the set of vertices which are the neighbors of the vertex v, so that $|N(v)| = d_G(v)$. Also let

$$\delta_G(v) = \sum_{u \in N(v)} d_G(u),$$

i.e., sum of degrees of the neighbor vertices of G. In the following theorem we obtain the reformulated first Zagreb index of the splice of two graphs.

Theorem 5. *The reformulated first Zagreb index of splice of graphs G_1 and G_2 is given by*

$$\begin{aligned} EM_1((G_1 \bullet G_2)(y, z)) &= EM_1(G_1) + EM_1(G_2) + 3d_{G_1}(y)^2 d_{G_2}(z) + 3d_{G_1}(y) d_{G_2}(z)^2 \\ &\quad + 2d_{G_1}(y)\delta_{G_2}(z) + 2d_{G_2}(z)\delta_{G_1}(y) - d_{G_1}(y) - d_{G_2}(z). \end{aligned}$$

Proof of Theorem 5. We have, from the definition of the reformulated first Zagreb index

$$
\begin{aligned}
EM_1((G_1 \bullet G_2)(y,z)) &= \sum_{(u,v)\in E((G_1\bullet G_2)(y,z))} \left(d_{(G_1\bullet G_2)(y,z)}(u) + d_{(G_1\bullet G_2)(y,z)}(v) - 2\right)^2 \\
&= \sum_{(u,v)\in E(G_1) uv\neq y} \left(d_{G_1}(u) + d_{G_1}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_2) uv\neq z} \left(d_{G_2}(u) + d_{G_2}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} \left(d_{G_1}(y) + d_{G_2}(z) + d_{G_1}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_2) u=z, v\in V(G_2)} \left(d_{G_1}(y) + d_{G_2}(z) + d_{G_2}(v) - 2\right)^2 \\
&= \sum_{(u,v)\in E(G_1) uv\neq y} \left(d_{G_1}(u) + d_{G_1}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_2) uv\neq z} \left(d_{G_2}(u) + d_{G_2}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} \left(d_{G_1}(y) + d_{G_1}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_2) u=z, v\in V(G_2)} \left(d_{G_2}(z) + d_{G_2}(v) - 2\right)^2 \\
&\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} d_{G_2}(z)^2 + \sum_{(u,v)\in E(G_2) u=z, v\in V(G_2)} d_{G_1}(y)^2 \\
&\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} 2d_{G_2}(z)\left(d_{G_1}(y) + d_{G_1}(v) - 2\right) \\
&\quad + \sum_{(u,v)\in E(G_2) u=z, v\in V(G_2)} 2d_{G_1}(y)\left(d_{G_2}(z) + d_{G_2}(v) - 2\right) \\
&= EM_1(G_1) + EM_1(G_2) + d_{G_1}(y)^2 d_{G_2}(z) + d_{G_1}(y) d_{G_2}(z)^2 \\
&\quad + 2d_{G_2}(z)\left(d_{G_1}(y)^2 + \delta_{G_1}(y) - 2d_{G_1}(y)\right) \\
&\quad + 2d_{G_1}(y)\left(d_{G_2}(z)^2 + \delta_{G_2}(z) - 2d_{G_2}(z)\right).
\end{aligned}
$$

From the above, after simple computation, the desired result follows. □

2.5. Link of Graphs

A link of G_1 and G_2 at the vertices y and z is denoted by $(G_1 \sim G_2)(y,z)$ and is obtained by joining the vertices y and z in the union of G_1 and G_2. From the construction of link graphs, it is clear that

$$
d_{(G_1\sim G_2)_{(y,z)}}(v) = \begin{cases} d_{G_i}(v), & v \in V(G_i), i = 1,2, \text{ and } v \neq y,z, \\ d_{G_i}(v) + 1, & v = y,z \end{cases}
$$

In the following theorem we obtain the reformulated first Zagreb index of the link of two graphs.

Theorem 6. *The reformulated first Zagreb index of link of graphs G_1 and G_2 is given by*

$$\begin{aligned} EM_1((G_1 \sim G_2)(y,z)) &= EM_1(G_1) + EM_1(G_2) + 3d_{G_1}(y)^2 + 3d_{G_2}(z)^2 + 2d_{G_1}(y)d_{G_2}(z) \\ &\quad +2\delta_{G_1}(y) + 3\delta_{G_2}(z) - 3d_{G_1}(y) - 3d_{G_2}(z) \end{aligned}$$

Proof of Theorem 6. From the definition of the reformulated first Zagreb index, we have

$$\begin{aligned} EM_1((G_1 \sim G_2)(y,z)) &= \sum_{(u,v)\in E((G_1\sim G_2)(y,z))} \left(d_{(G_1\sim G_2)(y,z)}(u) + d_{(G_1\sim G_2)(y,z)}(v) - 2\right)^2 \\ &= \sum_{(u,v)\in E(G_1) uv\neq y} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_2) uv\neq z} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} (1 + d_{G_1}(y) + d_{G_1}(v) - 2)^2 \\ &= \sum_{(u,v)\in E(G_1) uv\neq y} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_2) uv\neq z} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} (1 + d_{G_1}(y) + d_{G_1}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_2) u=z, v\in V(G_2)} (1 + d_{G_2}(z) + d_{G_2}(v) - 2)^2 \\ &\quad +\{(d_{G_1}(y) + 1) + (d_{G_2}(z) + 1) - 2\}^2 \\ &= \sum_{(u,v)\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 \\ &\quad + \sum_{(u,v)\in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 \\ &\quad +2 \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 \\ &\quad +2 \sum_{(u,v)\in E(G_2), u=z, v\in V(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 \\ &\quad +2 \sum_{(u,v)\in E(G_1), u=y, v\in V(G_1)} 1 + \sum_{(u,v)\in E(G_2), u=z, v\in V(G_2)} 1 \\ &\quad +(d_{G_2}(y) + d_{G_2}(z))^2 \\ &\quad = EM_1(G_1) + EM_1(G_2) + 2d_{G_1}(y)^2 + 2\delta_{G_1}(y) - 4d_{G_1}(y) + 2d_{G_2}(z)^2 \\ &\quad +2\delta_{G_2}(z) - 4d_{G_2}(z) + d_{G_1}(y) + d_{G_2}(z) + (d_{G_1}(y) + d_{G_2}(z))^2 \end{aligned}$$

From the above, we get the desired result after simple computation. □

2.6. Corona Product of Graphs

The corona product $G_1 \odot G_2$ of two graphs is obtained by taking one copy of G_1 and n_1 copies of G_2; and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G_1)|$. The corona product of G_1 and G_2 has a total of $(|V(G_1)||V(G_2)| + |V(G_1)|)$ vertices and $(|E(G_1)| + |V(G_1)||E(G_2)| + |V(G_1)||V(G_2)|)$ edges. Clearly, the corona product operation of two graphs is not commutative. Different topological indices of the corona product of two graphs have already been studied in [24,25]. Let the vertices of G_1 be denoted by $V(G_1) = \{u_1, u_2, ..., u_{|V(G_1)|}\}$ and the vertices of the i-th copy of G_2 are denoted by $V(G_2^i) = \{v_1^i, v_2^i, ..., v_{n_2}^i\}$ for $i = 1, 2, ..., |V(G_1)|$. Thus the vertex and edge sets of $G_1 \odot G_2$ are given by $V(G_1 \odot G_2) = V(G_1) \bigcup_{i=1,2,...,|V(G_1)|} V(G_{2,i})$ and $E(G_1 \odot G_2) = E(G_1) \bigcup_{i=1,2,...,|V(G_1)|} E(G_{2,i}) \bigcup \{u_i, v_j : u_i \in V(G_1), v_j^i \in V(G_{2,i})\}$. By definition, the degree of a vertex v of $G_1 \odot G_2$ is given by

$$d_{G_1 \odot G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, v \in V(G_1) \\ d_{G_{2,i}}(v) + 1, v \in V(G_{2,i}), i = 1, 2, ...|V(G_1)| \end{cases}$$

In the following, we compute the reformulated first Zagreb index of the corona product of two graphs.

Theorem 7. *The reformulated first Zagreb index of $G_1 \odot G_2$ is given by*

$$\begin{aligned} EM_1(G_1 \odot G_2) &= EM_1(G_1) + |V(G_1)|EM_1(G_2) + 5|V(G_2)|M_1(G_1) + 5|V(G_1)|M_1(G_2) \\ &\quad +4|V(G_1)||E(G_2)|(|V(G_2)| - 2) + 4|V(G_2)||E(G_1)|(2|V(G_2)| - 3) \\ &\quad +4|V(G_1)||V(G_2)|(|V(G_2)| - 1)^2 + 8|E(G_1)||E(G_2)|. \end{aligned}$$

Proof of Theorem 7. Let $|V(G_i)| = n_i$ $|E(G_i)| = e_i$, for $i \in \{1, 2\}$, then the edge set of $G_1 \odot G_2$ can be partitioned into three subsets

$E_1 = \{uv \in E(G_1 \odot G_2) : u, v \in V(G_1), i = 1, 2, ..., n_1\}$,

$E_2 = \{uv \in E(G_1 \odot G_2) : u \in V(G_1), v \in V(G_{2,i}), i = 1, 2, ..., n_1\}$, and

$E_3 = \{uv \in E(G_1 \odot G_2) : u \in V(G_1), v \in V(G_{2,i}), i = 1, 2, ..., n_1\}$.

To calculate the reformulated first Zagreb index of $G_1 \odot G_2$, we consider the following cases.

CASE 1. If $e \in E_1$ then $d_{G_1 \odot G_2}(e) = d_{G_1 \odot G_2}(u_i) + d_{G_1 \odot G_2}(v_i) - 2 = d_{G_1}(u_i) + d_{G_1}(v_i) + 2(n_2 - 1)$, for $i = 1, 2, ..., n_1$. So the contribution of these type of edges to the reformulated first Zagreb index of $G_1 \odot G_2$ is

$$\begin{aligned} Q_1 &= \sum_{e \in E_1} d_{G_1 \odot G_2}(e)^2 \\ &= \sum_{u_i v_i \in E(G_1)} (d_{G_1}(u_i) + d_{G_1}(v_i) + 2(n_2 - 1))^2 \\ &= \sum_{u_i v_i \in E(G_1)} \{(d_{G_1}(u_i) + d_{G_1}(v_i) - 2)^2 + 4{n_2}^2 + 4n_2(d_{G_1}(u_i) + d_{G_1}(v_i) - 2)\} \\ &= EM_1(G_1) + 4{n_2}^2 m_1 + 4n_2 M_1(G_1) - 8n_2 m_1. \end{aligned}$$

CASE 2. Let $e \in E_2$, then $d_{G_1 \odot G_2}(e) = d_{G_1 \odot G_2}(u_j) + d_{G_1 \odot G_2}(v_j) - 2 = d_{G_1}(u_i) + d_{G_2}(v_j)$, for $j = 1, 2, ..., n_2$. So, the contribution of these type of edges to the reformulated first Zagreb index of $G_1 \odot G_2$ is given by

$$\begin{aligned}
Q_2 &= \sum_{i=1}^{n_1} \sum_{u_j v_j \in E(G_2)} (d_{G_2}(u_j) + d_{G_2}(v_j))^2 \\
&= \sum_{i=1}^{n_1} \sum_{u_j v_j \in E(G_2)} \{(d_{G_2}(u_j) + d_{G_2}(v_j) - 2)^2 + 4(d_{G_2}(u_j) + d_{G_2}(v_j) - 2) + 4\} \\
&= \sum_{i=1}^{n_1} (EM_1(G_2) + 4M_1(G_2) - 8m_2 + 4m_2) \\
&= n_1 EM_1(G_2) + 4n_1 M_1(G_2) - 4n_1 m_2.
\end{aligned}$$

CASE 3. Let $e \in E_3$, then $d_{G_1 \odot G_2}(e) = d_{G_1 \odot G_2}(u_i) + d_{G_1 \odot G_2}(v_j) - 2 = d_{G_1}(u_i) + n_2 + d_{G_2}(v_j) + 1 - 2 = d_{G_1}(u_i) + d_{G_2}(v_j) + n_2 - 1$, for $i = 1, 2, ..., n_1$ and $j = 1, 2, ..., n_2$. So the contribution of these type of edges to the reformulated first Zagreb index of $G_1 \odot G_2$ is given by

$$\begin{aligned}
Q_3 &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (d_{G_1}(u_i) + d_{G_2}(v_j) + n_2 - 1)^2 \\
&= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \{d_{G_1}(u_i)^2 + d_{G_2}(v_j)^2 + 2(n_2 - 1)^2 + 2(n_2 - 1)d_{G_1}(u_i) + 2(n_2 - 1)d_{G_2}(v_j)\} \\
&\quad +2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d_{G_1}(u_i) d_{G_2}(u_j) \\
&= n_2 M_1(G_1) + n_1 M_1(G_2) + 4n_1 n_2 (n_2 - 1)^2 + 4n_2 m_1 (n_2 - 1) + 4n_1 m_2 (n_2 - 1) + 8m_1 m_2.
\end{aligned}$$

The reformulated Zagreb first index of $G_1 \odot G_2$ is obtained by adding Q_1, Q_2 and Q_3 then simplifying the expression. □

Corollary 8. *The bottleneck graph of a graph* G *is defined as the corona product of* K_2 *and* G*. Thus, its reformulated first Zagreb index is given by*
$EM_1(K_2 \odot G) = EM_1(G) + 10M_1(G) + 2n^2(n + 2) + 8m(n - 1)$.

A t-thorny graph is obtained by joining t thorns to each vertex of any given graph G. An edge $e = (u, v)$ of a graph G is called a thorn if either $d(u) = 1$ or $d(v) = 1$. A variety of topological indices of thorn graphs have been already studied by the researchers [26–29]. The t-thorny graph of G is obtained from corona product of G and the complement of the complete graph K_t. So, from Theorem 7, the following corollary follows.

Corollary 9. *The reformulated first Zagreb index of the* t*-thorny graph is given by*
$EM_1(G^t) = EM_1(G) + 5tM_1(G) + 4mt(2t - 3) + nt(t - 1)^2$.

Example 12. *The reformulated first Zagreb index of t-thorny path* $({P_n}^t)$ *and* t*-thorny cycle* $({C_n}^t)$ *are calculated as*

(i) $EM_1({P_n}^t) = nt^3 + 6nt^2 + 9nt - 8t^2 - 18t + 4n - 10$

(ii) $EM_1({C_n}^t) = nt^3 + 6nt^2 + 9nt + 4n$.

Next, we calculate the reformulated first Zagreb index of some particular bridge graphs. Let $G_1, G_2, ..., G_n$ be a set of finite pairwise disjoint graphs. The bridge graph with respect to the vertices $v_1, v_2, ..., v_n$, denoted by $B(G_1, G_2, ..., G_n; v_1, v_2, ..., v_n)$ is the graph obtained by connecting the vertices v_i and v_{i+1} of G_i and G_{i+1} by an edge for all $i = 1, 2, ..., (n-1)$. If $G_1 \cong G_2 \cong ... \cong G_n$ and $v_1 = v_2 = ... = v_n = v$, we define $G_n(G, v) = B(G, G, ..., G; v, v, ..., v)$. In particular, $B_n = G_n(P_3, v)$ and $T_{n,k} = G_n(C_k, u)$ are two special types of bridge graphs. Then, from the definition of the corona product of graphs, $B_n \cong P_n \odot \overline{K}_2$ and $T_{n,3} \cong P_n \odot K_2$. Using Theorem 7, the reformulated first Zagreb index of these bridge graphs are obtained as follows.

Example 13. *(i)* $EM_1(B_n) = 5n - 78$, *for* $n \geq 3$.
(ii) $EM_1(T_{m,3}) = 72m - 86$, *for* $m \geq 3$.
(iii) $EM_1(J_{n,m+1}) = nm(m^2 + 10m + 33) - 2m(4m + 13) + 4n - 10$, *for* $n \geq 3$ *and* $m \geq 3$.

3. Conclusions

In this paper, we have studied the reformulated first Zagreb index of different graph operations. Also, we applied our results to calculate the reformulated first Zagreb index of some classes of graphs by specializing the components of graph operations. Nevertheless, there are still many other graph operations and special classes of graphs that are not covered here. For further research, the second reformulated Zagreb index various graph operations can be computed.

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Author Contributions

All three authors have significant contribution to this paper and the final form of this paper is approved by all three authors.

Conflicts of Interest

The authors declare no conflict of interest.

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