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Inverse Eigenvalue Problems for Two Special Acyclic Matrices

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Abstract: In this paper, we study two inverse eigenvalue problems (IEPs) of constructing two special acyclic matrices. The first problem involves the reconstruction of matrices whose graph is a path, from given information on one eigenvector of the required matrix and one eigenvalue of each of its leading principal submatrices. The second problem involves reconstruction of matrices whose graph is a broom, the eigen data being the maximum and minimum eigenvalues of each of the leading principal submatrices of the required matrix. In order to solve the problems, we use the recurrence relations among leading principal minors and the property of simplicity of the extremal eigenvalues of acyclic matrices.

Keywords: inverse eigenvalue problem; leading principal minors; graph of a matrix

MSC: 65F18, 05C50

1. Introduction

The problems of reconstruction of specially structured matrices from a prescribed set of eigen data are collectively known as inverse eigenvalue problems (IEPs). The level of difficulty of an IEP depends on the structure of the matrices which are to be reconstructed and on the type of eigen information available. M.T. Chu in [1] gave a detailed characterization of inverse eigenvalue problems. A few special types of inverse eigenvalue problems have been studied in [2–8]. Inverse problems for matrices with prescribed graphs have been studied in [9–14]. Inverse eigenvalue problems arise in a number of applications such as control theory, pole assignment problems, system identification, structural analysis, mass spring vibrations, circuit theory, mechanical system simulation and graph theory [1,12,15,16].

In this paper, we study two IEPs, namely IEPP (inverse eigenvalue problem for matrices whose graph is a path) and IEPB (inverse eigenvalue problem for matrices whose graph is a broom). Similar problems were studied in [5], for arrow matrices. The usual process of solving such problems involves the use of recurrence relations among the leading principal minors of $\lambda I - A$ where A is the required matrix. However, we have included graphs in our analysis by bringing in the requirement of constructing matrices which are described by graphs. In particular, we have considered paths and brooms. Thus, in addition to recurrence relations among leading principal minors, we have used spectral properties of acyclic matrices to solve the problems. Particularly, the strict interlacing of the eigenvalues in IEPB could be proved because of the fact that the minimal and maximal eigenvalues of an acyclic matrix are simple.

The paper is organized as follows : In Section 2, we discuss some preliminary concepts and clarify the notations used in the paper. In Section 3, we define the inverse problems to be studied, namely IEPP and IEPB. Section 4 deals with the analysis of IEPP, the main result being presented as Theorem 4. Section 5 deals with the analysis of IEPB, the main result being presented as Theorem 9. In Section 6, we present some numerical examples to illustrate the solutions of IEPP and IEPB.

2. Preliminary Concepts

Let V be a finite set and let P be the set of all subsets of V which have two distinct elements. Let $E \subset P$. Then $G = (V, E)$ is said to be a *graph* with vertex set V and edge set E. To avoid confusion, the vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. Our choice of P implies that the graphs under consideration are free of multiple edges or loops and are undirected. If $u, v \in V$ and $\{u, v\} \in E$, then we say that uv is an *edge* and u and v are then called *adjacent* vertices. The degree of a vertex u is the number of edges which are incident on u. A vertex of degree one is called a *pendant vertex*. A *path* P of G is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that consecutive vertices are adjacent. The path on n vertices is denoted by P_n. A graph is said to be *connected* if there exists a path between every pair of its vertices. A *cycle* is a connected graph in which each vertex is adjacent to exactly two other vertices. A connected graph without any cycles is called a *tree*.

Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G(A)$, has vertex set $V(G) = \{1, 2, 3, \ldots, n\}$ and edge set $\{ij : i \neq j, a_{ij} \neq 0\}$. For a graph G with n vertices, $S(G)$ denotes the set of all $n \times n$ symmetric matrices which have G as their graph. A matrix whose graph is a tree is called an *acyclic* matrix. Some simple examples of acyclic matrices are the matrices whose graphs are paths or brooms (Figure 1).

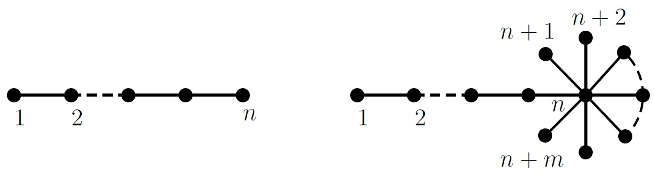

Figure 1. Path P_n and Broom B_{n+m}.

Throughout this paper, we shall use the following notation :

1. Matrix of a path P_n will be a tridiagonal matrix with non zero off-diagonal entries :

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & \ldots & 0 & 0 \\ b_1 & a_2 & b_2 & \ldots & 0 & 0 \\ 0 & b_2 & a_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \ldots & b_{n-1} & a_n \end{pmatrix}_{n \times n},$$

where the b_is are non-zero.

2. Matrix of a broom B_{n+m} will be of the following form :

$$A_{n+m} = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & \ddots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & b_{n-1} & a_n & b_n & b_{n+1} & \dots & b_{n+m-1} \\ 0 & 0 & 0 & \dots & 0 & b_n & a_{n+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & b_{n+1} & 0 & a_{n+2} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_{n+m-1} & 0 & 0 & \dots & a_{n+m} \end{pmatrix}_{(n+m)\times(n+m)},$$

where the b_is are non zero.

3. A_i will denote the ith leading principal submatrix of the required matrix (A_n or A_{n+m}).
4. $P_i(\lambda) = det(\lambda I_i - A_i)$, $i = 1, 2, \dots, n$ (respectively $i = 1, 2, \dots n+m$) *i.e.*, the ith leading principal minor of $\lambda I_n - A_n$ (respectively $\lambda I_{n+m} - A_{n+m}$), I_i being the identity matrix of order i. For the sake of writing the recurrence relations with ease, we define $P_0(\lambda) = 1, b_0 = 0$.

3. IEPs to be Studied

In this paper we shall study the following two inverse eigenvalue problems :

IEPP Given n real numbers $\lambda_j, 1 \le j \le n$ and a real vector $X_n = (x_1, x_2, \dots, x_n)^T$ find an $n \times n$ matrix $A_n \in S(P_n)$ such that λ_j is an eigenvalue of A_j, $j = 1, 2, \dots, n$ and (λ_n, X_n) is an eigenpair of A_n.

IEPB Given $2n + 2m - 1$ real numbers $\lambda_1^{(j)}, 1 \le j \le n+m$ and $\lambda_j^{(j)}, 2 \le j \le n+m$, find an $(n+m) \times (n+m)$ matrix $A_{n+m} \in S(B_{n+m})$ such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are respectively the minimal and maximal eigenvalues of A_j, $j = 1, 2, \dots, n+m$.

4. Solution of IEPP

The following Lemma gives the relation between successive leading principal minors of $\lambda I_n - A_n$:

Lemma 1. *The sequence* $\{P_j(\lambda) = det(\lambda I_j - A_j)\}_{j=1}^n$ *of characteristic polynomials of* A_j *satisfies the following recurrence relations :*

1. $P_1(\lambda) = (\lambda - a_1)$
2. $P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda), 2 \le j \le n.$

Here A_j *denotes the jth leading principal submatrix of* A_n*, the matrix corresponding to the path on n vertices.*

Lemma 2. *For any* $\lambda \in \mathbb{R}$ *and* $1 \le j \le n$, $P_j(\lambda)$ *and* $P_{j+1}(\lambda)$ *cannot be simultaneously zero.*

Proof. If $P_1(\lambda) = 0 = P_2(\lambda)$, then $(\lambda - a_2)P_1(\lambda) - b_1^2 = 0$, which implies $b_1 = 0$, but this contradicts the restriction on A_n that $b_1 \neq 0$. Once again, for $2 < j \le n$, if $P_{j-1}(\lambda) = 0 = P_j(\lambda)$, then the recurrence relation (ii) from Lemma 1, $(\lambda - a_{j+1})P_j(\lambda) - b_j^2 P_{j-1}(\lambda) = 0$, which gives $P_{j-1}(\lambda) = 0$. This will in turn imply that $P_{j-2}(\lambda) = 0$. Thus, we will end up with $P_2(\lambda) = 0$, implying that $b_1 = 0$ which is a contradiction. □

Lemma 3. *If* $X = (x_1, x_2, \dots, x_n)^T$ *is an eigenvector of* A_n *corresponding to an eigenvalue* λ*, then* $x_1 \neq 0$ *and the components of this eigenvector are given by*

$$x_j = \frac{P_{j-1}(\lambda)}{\prod_{i=1}^{j-1} b_i} x_1, j = 2, 3, \dots, n.$$

Proof. Since (λ, X) is an eigenpair of A_n, we have $A_n X = \lambda X$. Comparing the first $n-1$ rows of this matrix equation on both sides, we have

$$(a_1 - \lambda)x_1 + b_1 x_2 = 0, \tag{1}$$

$$b_{j-1}x_{j-1} + (a_j - \lambda)x_j + b_j x_{j+1} = 0, j = 2, \ldots, n-1. \tag{2}$$

By the second recurrence relation from Lemma 1,

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda), j = 2, 3, \ldots, n. \tag{3}$$

We define the quantities $v_1, v_2, \ldots, v_n$ as

$$v_1 = x_1, v_j = x_j \prod_{i=1}^{j-1} b_i, 2 \le j \le n.$$

Multiplying Equation (2) by $\prod_{i=1}^{j-1} b_i$, we get

$$\begin{aligned} &b_{j-1}x_{j-1}\prod_{i=1}^{j-1} b_i + (a_j - \lambda)x_j \prod_{i=1}^{j-1} b_i + b_j x_{j+1} \prod_{i=1}^{j-1} b_i = 0 \\ \Rightarrow &b_{j-1}^2 v_{j-1} + (a_j - \lambda)v_j + v_{j+1} = 0, \end{aligned}$$

which gives

$$v_{j+1} = (\lambda - a_j)v_j - b_{j-1}^2 v_{j-1}, j = 2, 3, \ldots, n-1. \tag{4}$$

Now, from Equation (1), we have $v_2 = (\lambda - a_1)x_1 = x_1 P_1(\lambda)$. Again from Equation (4),

$$v_3 = (\lambda - a_2)v_2 - b_1^2 v_1 = x_1\{(\lambda - a_2)P_1(\lambda) - b_1^2\} = x_1 P_2(\lambda).$$

Proceeding this way, we see that $v_{j+1} = x_1 P_j(\lambda), j = 1, 2, \ldots, n-1$ which can also be written as $v_j = x_1 P_{j-1}(\lambda), j = 2, 3, \ldots, n$. This further implies that

$$x_j = \frac{P_{j-1}(\lambda)}{\prod_{i=1}^{j-1} b_i} x_1, j = 2, 3, \ldots, n. \tag{5}$$

Since X is an eigenvector, $X \neq 0$. If $x_1 = 0$, then from Equation (5), we see that all the other components of X become zero. Thus, $x_1 \neq 0$. □

Theorem 4. *The **IEPP** has a unique solution if and only if $x_j \neq 0$ for all $j = 1, 2, \ldots, n$. The unique solution is given by*

$$a_1 = \lambda_1, a_j = \lambda_j - \frac{b_{j-1}^2 P_{j-2}(\lambda_j)}{P_{j-1}(\lambda_j)}, j = 2, \ldots, n$$

$$b_1 = \frac{x_1}{x_2} P_1(\lambda_n) \text{ and } b_{j-1} = \frac{x_1 P_{j-1}(\lambda_n)}{x_j \prod_{i=1}^{j-2} b_i}, j = 3, 4, \ldots, n.$$

Proof. Let $x_j \neq 0$ for all $j = 1, 2, \ldots, n$. As per the conditions of IEPP, λ_j is an eigenvalue of A_j for each $j = 1, 2, \ldots, n$. Thus, $P_1(\lambda_1) = 0 \Rightarrow a_1 = \lambda_1$.

$$\begin{aligned} &P_j(\lambda_j) = 0 \\ \Rightarrow &(\lambda_j - a_j)P_{j-1}(\lambda_j) - b_{j-1}^2 P_{j-2}(\lambda_j) = 0 \\ \Rightarrow &a_j = \lambda_j - \frac{b_{j-1}^2 P_{j-2}(\lambda_j)}{P_{j-1}(\lambda_j)}, \end{aligned} \tag{6}$$

which gives the expression for calculating a_j. The expression is valid as $P_{j-1}(\lambda_j) \neq 0$, because by Lemma 2, $P_{j-1}(\lambda_j)$ and $P_j(\lambda_j)$ cannot be simultaneously zero.

Now, since (λ_n, X) is an eigenpair of A_n, so by Equation (5), $x_j = \dfrac{P_{j-1}(\lambda_n)}{\prod_{i=1}^{j-1} b_i} x_1$, which implies that

$$b_1 = \frac{x_1}{x_2} P_1(\lambda_n) \text{ and } b_{j-1} = \frac{x_1 P_{j-1}(\lambda_n)}{x_j \prod_{i=1}^{j-2} b_i}. \tag{7}$$

Since $x_j \neq 0$ hence it follows that, $P_{j-1}(\lambda_n) \neq 0$. Hence the above expression for b_{j-1} is valid and $b_{j-1} \neq 0$ for all $j = 2, 3, \ldots, n$. Successive use of Equations (6) and (7) will give us the values of a_j and b_{j-1} for $j = 1, 2, \ldots, n$.

Conversely, if there exists a unique solution for IEPP, then since X is an eigenvector of A_n, so by Lemma 3, $x_1 \neq 0$. The existence of a solution implies that $b_{j-1} \neq 0$ for $j = 2, 3, \ldots, n$. It then follows from the expressions in Equation (7) that $x_j \neq 0$ for $j = 2, 3, \ldots, n$. □

5. Solution of IEPB

Lemma 5. *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeros and λ_{min} and λ_{max} be the minimal and maximal zero of P respectively.*

- *If $\mu < \lambda_{min}$, then $(-1)^n P(\mu) > 0$.*
- *If $\mu > \lambda_{max}$, then $P(\mu) > 0$.*

The proof immediately follows after expressing the polynomial as a product of its linear factors.

Lemma 6. *If T is a tree, then the minimal and maximal eigenvalues of any matrix $A \in S(T)$ are simple i.e., of multiplicity one. [Corollary 6 of Theorem 2 in [17]]*

In other words, this Lemma says that the minimal and maximal eigenvalues of an acyclic matrix are simple. Again, since for each j, the leading principal submatrix A_j corresponds to a tree so by Lemma 6 the minimal and maximal eigenvalues of A_j must be simple *i.e.*, in particular $\lambda_1^{(j)} \neq \lambda_j^{(j)}$.

Lemma 7. *The sequence $\{P_j(\lambda) = det(\lambda I_j - A_j)\}_{j=1}^{n+m}$ of characteristic polynomials of A_j satisfies the following recurrence relations :*

1. $P_1(\lambda) = (\lambda - a_1)$.
2. $P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 P_{j-2}(\lambda), 2 \leq j \leq n+1$.
3. $P_{n+j}(\lambda) = (\lambda - a_{n+j})P_{n+j-1}(\lambda) - b_{n+j-1}^2 P_{n-1}(\lambda) \displaystyle\prod_{i=1}^{j-1}(\lambda - a_{n+i}), 2 \leq j \leq m$.

Lemma 8. *For any $\lambda \in \mathbb{R}$ and $1 \leq j \leq n$, $P_j(\lambda)$ and $P_{j+1}(\lambda)$ cannot be simultaneously zero.*

Proof. Same as Lemma 2. □

By Cauchy's interlacing theorem ([14,18]), the eigenvalues of a symmetric matrix and those of any of its principal submatrix interlace each other. Thus, $\lambda_1^{(j)}$'s and $\lambda_j^{(j)}$'s must satisfy :

$$\lambda_1^{(n+m)} \leq \lambda_1^{(n+m-1)} \leq \ldots \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(3)} \leq \ldots \leq \lambda_{n+m-1}^{(n+m-1)} \leq \lambda_{n+m}^{(n+m)}.$$

Each diagonal element a_i is also a 1×1 principal submatrix of A. Hence $\lambda_1^j \leq a_i \leq \lambda_j^{(j)}, 1 \leq i \leq j$. Since $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j, so $P_j(\lambda_1^{(j)}) = 0$ and $P_j(\lambda_j^{(j)}) = 0$. We need to solve these equations successively using the recurrence relations in Lemma 1. For $j = 1$, $P_1(\lambda_1^{(1)}) = 0 \Rightarrow a_1 = \lambda_1^{(1)}$. For $j = 2$, $P_2(\lambda_1^{(2)}) = 0, P_2(\lambda_2^{(2)}) = 0$ which imply that

$$a_2 = \frac{\lambda_2^{(2)} P_1(\lambda_2^{(2)}) - \lambda_1^{(2)} P_1(\lambda_1^{(2)})}{P_1(\lambda_2^{(2)}) - P_1(\lambda_1^{(2)})}, b_1^2 = \frac{(\lambda_2^{(2)} - \lambda_1^{(2)}) P_1(\lambda_1^{(2)}) P_1(\lambda_2^{(2)})}{P_1(\lambda_1^{(2)}) - P_1(\lambda_2^{(2)})}.$$

a_2 will always exist as the denominator in the above expression for a_2 can never be zero. We have $\lambda_1^{(2)} \neq \lambda_2^{(2)}$ and so if $P_1(\lambda_2^{(2)}) = P_1(\lambda_1^{(2)})$, then by Rolle's theorem $\exists\, c \in (\lambda_1^{(2)}, \lambda_2^{(2)})$ such that $P_1'(c) = 0 \Rightarrow 1 = 0$, which is not possible. Thus, $P_1(\lambda_2^{(2)}) - P_1(\lambda_1^{(2)}) \neq 0$. Also, by Lemma 5, $(-1)^1 P_1(\lambda_1^{(2)}) \geq 0$ and so the expression for b_1^2 is non-negative and so we can get real values of b_1.

Now for $3 \leq j \leq n$, we have

$$P_j(\lambda_1^{(j)}) = 0, P_j(\lambda_j^{(j)}) = 0,$$

which gives

$$a_j P_{j-1}(\lambda_1^{(j)}) + b_{j-1}^2 P_{j-2}(\lambda_1^{(j)}) - \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) = 0,$$

$$a_j P_{j-1}(\lambda_j^{(j)}) + b_{j-1}^2 P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) = 0.$$

Let D_j denote the determinant of the coefficient matrix of the above system of linear equations in a_j and b_{j-1}^2. Then $D_j = P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})$. If $D_j \neq 0$, then the system will have a unique solution, given by

$$\begin{aligned} a_j &= \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})}{D_j}, \\ b_{j-1}^2 &= \frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{D_j}. \end{aligned} \tag{8}$$

We claim that the expression for b_{j-1}^2 in RHS is non negative. This follows from Lemma 5. Since $\lambda_1^{(j)} \leq \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} \leq \lambda_j^{(j)}$, so by Lemma 5,

$$(-1)^{j-1} D_j = (-1)^{(j-1)} P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) + (-1)^{j-2} P_{j-2}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)}) \geq 0.$$

In addition, by Lemma 2, $P_j(\lambda_1^{(j)})$ and $P_{j-1}(\lambda_1^{(j)})$ cannot be simultaneously zero. Thus, $P_{j-1}(\lambda_1^{(j)}) \neq 0$. Similarly, $P_{j-1}(\lambda_j^{(j)}) \neq 0$. This implies that $\lambda_1^{(j)} \neq \lambda_1^{(j-1)}$ and $\lambda_j^{(j)} \neq \lambda_{j-1}^{j-1}$. Thus, we can get non-zero real values of b_{j-1} from Equation (8) if and only if $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$ for all $j = 2, 3, \ldots, n$.

Now, if $D_j = 0$, then $(-1)^{j-1} D_j = 0$ *i.e.*, $(-1)^{(j-1)} P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) + (-1)^{j-2} P_{j-2}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)}) = 0$. Since both the terms in this sum are non negative, we must have $P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) = 0$ and $P_{j-2}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)}) = 0$. However, from Lemma 2, $P_{j-1}(\lambda)$ and $P_{j-2}(\lambda)$ cannot be simultaneously zero. In addition, $P_j(\lambda)$ and $P_{j-1}(\lambda)$ cannot be simultaneously zero.

Thus, the only possibility is that $P_{j-2}(\lambda_1^{(j)}) = P_{j-2}(\lambda_j^{(j)}) = 0$. However, this will then imply that $a_j = \lambda_1^{(j)} = \lambda_j^{(j)}$, which is not possible as by Lemma 6 $\lambda_1^{(j)} \neq \lambda_j^{(j)}$. Thus, $D_j \neq 0$ for all $j = 2, 3, \ldots, n$.

Again, $\lambda_1^{(n+j)}$ and $\lambda_{n+j}^{(n+j)}$ are the eigenvalues of A_{n+j} and so $P_{n+j}(\lambda_1^{(n+j)}) = 0$ and $P_{n+j}(\lambda_{n+j}^{(n+j)}) = 0$. Hence,

$$\begin{aligned}(\lambda_1^{(n+j)} - a_{n+j})P_{n+j-1}(\lambda_1^{(n+j)}) - b_{n+j-1}^2 P_{n-1}(\lambda_1^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = 0,\\ (\lambda_{n+j}^{(n+j)} - a_{n+j})P_{n+j-1}(\lambda_{n+j}^{(n+j)}) - b_{n+j-1}^2 P_{n-1}(\lambda_{n+j}^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = 0.\end{aligned} \tag{9}$$

so we get a system of equations linear in a_{n+j} and b_{n+j-1}^2.

$$\begin{aligned}a_{n+j}P_{n+j-1}(\lambda_1^{(n+j)}) + b_{n+j-1}^2 P_{n-1}(\lambda_1^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = \lambda_1^{(n+j)} P_{n+j-1}(\lambda_1^{(n+j)}),\\ a_{n+j}P_{n+j-1}(\lambda_{n+j}^{(n+j)}) + b_{n+j-1}^2 P_{n-1}(\lambda_{n+j}^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = \lambda_{n+j}^{(n+j)} P_{n+j-1}(\lambda_{n+j}^{(n+j)}).\end{aligned} \tag{10}$$

We first investigate the conditions under which the coefficient matrix of the above system is singular. By Cauchy's interlacing property, we have

$$\lambda_1^{(n+j)} \leq \lambda_1^{(n+j-1)} \leq \ldots \leq \lambda_1^{(n+1)} \leq a_{n+i} \leq \lambda_{n+1}^{(n+1)} \leq \ldots \lambda_{n+j-1}^{(n+j-1)} \leq \lambda_{n+j}^{(n+j)}, \text{ for all } i = 1, 2, \ldots, m. \tag{11}$$

Thus, $\prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) \geq 0$ and $(-1)^{j-1}\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) \geq 0$. Let D_{n+j} be the determinant of the coefficient matrix of Equation (10). Then,

$$\begin{aligned}(-1)^{n+j-1}D_{n+j} =& (-1)^{n+j-1}P_{n+j-1}(\lambda_1^{(n+j)})P_{n-1}(\lambda_{n+j}^{(n+j)})\prod_{i+1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}),\\ &+ (-1)^{n+j-2}P_{n+j-1}(\lambda_{n+j}^{(n+j)})P_{n-1}(\lambda_1^{(n+j)})\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}).\end{aligned}$$

As a consequence of Lemma 5, both the products in the LHS are non-negative and so $(-1)^{n+j-1}D_{n+j} \geq 0$ for all $j = 1, 2, \ldots, m$. Thus, D_{n+j} will vanish if and only if $(-1)^{n+j-1}D_{n+j}$ will vanish *i.e.*, if and only if

$$P_{n+j-1}(\lambda_1^{(n+j)})P_{n-1}(\lambda_{n+j}^{(n+j)})\prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = 0$$

and

$$P_{n+j-1}(\lambda_{n+j}^{(n+j)})P_{n-1}(\lambda_1^{n+j})\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = 0.$$

If $P_{n-1}(\lambda_1^{(n+j)}) = 0$, then since $\lambda_1^{(n+j)} \leq \lambda_1^{(n+j-1)} \leq \ldots \leq \lambda_1^{(n)} \leq \lambda_1^{(n-1)}$ and $\lambda_1^{(n-1)}$ is the minimum possible zero of P_{n-1}, we get $\lambda_1^{n+j} = \lambda_1^{(n+j-1)} = \ldots = \lambda_1^{(n)} = \lambda_1^{(n-1)}$. Consequently, $P_{n-1}(\lambda_1^{(n+j)}) = P_n(\lambda_1^{(n+j)}) = 0$ but this contradicts Lemma 8, according to which $P_{n-1}(\lambda_1^{(n+j)})$ and $P_n(\lambda_1^{(n+j)})$ cannot be simultaneously zero. Hence, $P_{n-1}(\lambda_1^{(n+j)}) \neq 0$. Similarly, it can be shown that $P_{n-1}(\lambda_{n+j}^{(n+j)}) \neq 0$. Thus there are the following possibilities :

i. $P_{n+j-1}(\lambda_1^{(n+j)}) = 0$ and $P_{n+j-1}(\lambda_{n+j}^{(n+j)}) = 0$.

ii. $P_{n+j-1}(\lambda_1^{(n+j)}) = 0$ and $\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = 0$.

iii. $P_{n+j-1}(\lambda_{n+j}^{(n+j)}) = 0$ and $\prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = 0$.

iv. $\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = 0$ and $\prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = 0$.

If (i) happens, then, since $b_{n+j-1} \neq 0$, so from the equations in Equation (10), $\prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i}) = 0$ and $\prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i}) = 0$. This implies that $a_{n+i} = \lambda_1^{(n+j)}$ for some $i = 1, 2, \ldots, m$ and $a_{n+i} = \lambda_{n+j}^{(n+j)}$ for some $i = 1, 2, \ldots, m$. However, as per the inequality Equation (11), it then follows that $\lambda_1^{(n+j)} = \lambda_1^{(n+j-1)} = \ldots = \lambda_1^{(n+1)}$ and $\lambda_{n+1}^{(n+1)} = \lambda_{n+2}^{(n+2)} = \ldots = \lambda_{n+j}^{(n+j)}$. Since $P_{n+2}(\lambda_1^{(n+2)}) = 0$ and $P_{n+1}(\lambda_1^{(n+1)}) = 0$, so the above equalities imply that $P_{n+2}(\lambda_1^{(n+j)}) = 0$ and $P_{n+1}(\lambda_1^{(n+j)}) = 0$. Hence from the recurrence relation (3) of Lemma 7, we get

$$(\lambda_1^{(n+j)} - a_{n+1})P_{n+1}(\lambda_1^{(n+j)}) - b_{n+1}^2 P_{n-1}(\lambda_1^{(n+j)})(\lambda_1^{(n+j)} - a_{n+1}) = 0$$

which implies that $\lambda_1^{(n+j)} = a_{n+1}$. Similarly, it will follow that $\lambda_{n+j}^{(n+j)} = a_{n+1}$. However, $\lambda_1^{(n+j)} \leq a_{n+1} \leq \lambda_{n+j}^{(n+j)}$ and so $\lambda_1^{(n+j)} = \lambda_{n+j}^{(n+j)}$, but this is not possible as $\lambda_1^{(n+j)}$ and $\lambda_{n+j}^{(n+j)}$ are the minimal and maximal eigenvalues of the acyclic matrix A_{n+j} and by Lemma 6, the minimal and maximal eigenvalues of an acyclic matrix are simple. Hence (i) cannot hold. From the above arguments, it also follows that (iv) cannot hold.

If (ii) holds, then the augmented matrix of the system of Equation (10) will be of rank one and so the system will have infinite number of solutions. Similarly, if (iii) holds, then the system will have infinite number of solutions. However, if we put the additional constraint that $\lambda_1^{(n+j)} < \lambda_1^{(n+j-1)}$ and $\lambda_{n+j-1}^{(n+j-1)} < \lambda_{n+j}^{(n+j)}$ for all $j = 2, 3, \ldots, m$ then $P_{n+j-1}(\lambda_1^{(n+j)}) \neq 0$ and $P_{n+j-1}(\lambda_{n+j}^{(n+j)}) \neq 0$, so that (ii) and (iii) will not hold.

Thus, we see that $D_{n+j} \neq 0$ if and only if

$$\lambda_1^{(n+j)} < \lambda_1^{(n+j-1)} < \ldots < \lambda_1^{(n+1)} < \lambda_{n+1}^{(n+1)} < \ldots < \lambda_{n+j-1}^{(n+j-1)} < \lambda_{n+j}^{(n+j)}, \text{ for all } i = 1, 2, \ldots, m$$

Under this constraint, the unique solution of the system Equation (10) is given by

$$a_{n+j} = \frac{A_j - B_j}{D_{n+j}}, b_{n+j-1}^2 = \frac{(\lambda_{n+j}^{(n+j)} - \lambda_1^{(n+j)})P_{n+j-1}(\lambda_1^{(n+j)})P_{+j-1}(\lambda_{n+j}^{(n+j)})}{D_{n+j}}, \tag{12}$$

where $A_j = \lambda_1^{(n+j)} P_{n+j-1}(\lambda_1^{(n+j)}) P_{n-1}(\lambda_{n+j}^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_{n+j}^{(n+j)} - a_{n+i})$ and $B_j = \lambda_{n+j}^{(n+j)} P_{n+j-1}(\lambda_{n+j}^{(n+j)}) P_{n-1}(\lambda_1^{(n+j)}) \prod_{i=1}^{j-1}(\lambda_1^{(n+j)} - a_{n+i})$.

$$b_{n+j-1}^2 = \frac{(\lambda_{n+j}^{(n+j)} - \lambda_1^{(n+j)})P_{n+j-1}(\lambda_1^{(n+j)})P_{+j-1}(\lambda_{n+j}^{(n+j)})}{D_{n+j}}. \tag{13}$$

The above analysis of the IEP can be framed as the following theorem :

Theorem 9. *The **IEPB** has a solution if and only if*

$$\lambda_1^{(n+m)} < \lambda_1^{(n+m-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n+m-1}^{(n+m-1)} < \lambda_{n+m}^{(n+m)}$$

and the solution is given by

$$\begin{aligned}
a_1 &= \lambda_1^{(1)}, a_2 = \frac{\lambda_2^{(2)} P_1(\lambda_2^{(2)}) - \lambda_1^{(2)} P_1(\lambda_1^{(2)})}{P_1(\lambda_2^{(2)}) - P_1(\lambda_1^{(2)})}, b_1^2 = \frac{(\lambda_2^{(2)} - \lambda_1^{(2)}) P_1(\lambda_1^{(2)}) P_1(\lambda_2^{(2)})}{P_1(\lambda_1^{(2)}) - P_1(\lambda_1^{(2)})},\\
a_j &= \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})}{P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})}, j = 3, 4, \ldots, n+1,\\
b_{j-1}^2 &= \frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})}, j = 3, 4, \ldots, n+1,\\
a_{n+j} &= \frac{A_j - B_j}{D_{n+j}}, j = 1, 2, \ldots, m,\\
b_{n+j-1}^2 &= \frac{(\lambda_{n+j}^{(n+j)} - \lambda_1^{(n+j)}) P_{n+j-1}(\lambda_1^{(n+j)}) P_{+j-1}(\lambda_{n+j}^{(n+j)})}{D_{n+j}}, j = 2, 3, \ldots, m.
\end{aligned}$$

The solution is unique except for the signs of the non-zero off-diagonal entries.

6. Numerical Examples

We apply the results obtained in the previous section to solve the following :

Example 1. *Given 7 real numbers $\lambda_1 = 1, \lambda_2 = 5, \lambda_3 = -4, \lambda_4 = 3, \lambda_5 = 9, \lambda_6 = -8, \lambda_7 = -3$ and a real vector $X = (-2, 5, -7, 3, 1, 4, 8)^T$, find a matrix $A_7 \in S(P_7)$ such that λ_j is an eigenvalue of A_j for each $j = 1, 2, \ldots, 7$ and (λ_7, X) is an eigenpair of A_7.*

Solution Using Theorem 1, we obtain the following matrix as the solution :

$$A_7 = \begin{pmatrix}
1 & 1.6 & 0 & 0 & 0 & 0 & 0\\
1.6 & 4.36 & 4.8 & 0 & 0 & 0 & 0\\
0 & 4.8 & -1.0642 & -3.4832 & 0 & 0 & 0\\
0 & 0 & -3.4832 & 2.0515 & -39.5368 & 0 & 0\\
0 & 0 & 0 & -39.5368 & -347.5644 & 115.7937 & 0\\
0 & 0 & 0 & 0 & 115.7937 & -32.6379 & 0.3448\\
0 & 0 & 0 & 0 & 0 & 0.3448 & -3.1724
\end{pmatrix}$$

The eigenvalues of the all the leading principal submatrices are :
$\sigma(A_1) = \{\mathbf{1}\}$
$\sigma(A_2) = \{0.3600, \mathbf{5.0000}\}$
$\sigma(A_3) = \{\mathbf{-4.0000}, 0.8364, 7.4594\}$
$\sigma(A_4) = \{-5.2369, 0.5809, \mathbf{3.0000}, 8.0033\}$
$\sigma(A_5) = \{-351.9801, -4.7789, 0.7321, 5.8098, \mathbf{9.0000}\}$
$\sigma(A_6) = \{-389.1678, \mathbf{-8.0000}, -3.1028, 0.8636, 7.4210, 18.1310\}$
$\sigma(A_7) = \{-389.1678, -8.0084, -3.2694, \mathbf{-3.0000}, 0.8638, 7.4211, 18.1333\}$

Example 2. *Given 13 real numbers $0.5, 1, -1.4, 2, -2.2, 3, -3.8, 4.7, -4.4, 5, -6, 6, 7$, rearrange and label them as $\lambda_1^{(j)}, 1 \leq j \leq 7$ and $\lambda_j^{(j)}, 2 \leq j \leq 7$ and find a matrix $A_{4+3} \in S(B_{4+3})$ such that λ_1^j and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j, the jth leading principal sub matrix of A_{4+3}.*

Solution Using Theorem 9, we rearrange the numbers in the following way

$$\lambda_1^{(7)} < \lambda_1^{(6)} < \lambda_1^{(5)} < \lambda_1^{(4)} < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \lambda_4^{(4)} < \lambda_5^{(5)} < \lambda_6^{(6)} < \lambda_7^{(7)}$$

i.e.,

$$-6 < -4.4 < -3.8 < -2 < -1.4 < 0.5 < 1 < 2 < 3 < 4.7 < 5 < 6 < 7.$$

Then, using the expressions for a_j, b_{j-1}^2, a_{n+j} and b_{n+j-1}^2 we get

$$A_{4+3} = \begin{pmatrix} 1.0000 & 0.7071 & 0 & 0 & 0 & 0 & 0 \\ 0.7071 & 1.5000 & 1.9380 & 0 & 0 & 0 & 0 \\ 0 & 1.9380 & -0.0047 & 2.1580 & 0 & 0 & 0 \\ 0 & 0 & 2.1580 & 3.3615 & 1.8991 & 3.1884 & 3.7247 \\ 0 & 0 & 0 & 1.8991 & -3.1612 & 0 & 0 \\ 0 & 0 & 0 & 3.1884 & 0 & -1.5823 & 0 \\ 0 & 0 & 0 & 3.7247 & 0 & 0 & -3.1925 \end{pmatrix}.$$

Here we have taken all the b_is as positive. We can take some of the b_is as negative also. In fact, we can construct 2^6 such matrices for the above problem, the only difference being in the signs of the non-zero off-diagonal entries.

We compute the spectra of all the leading principal submatrices of A_{4+3} to verify the the conditions of the **IEPB** are satisfied. The minimal and maximal eigenvalues of each principal submatrix are shown in bold.

$\sigma(A_7) = \{\mathbf{-6.0000}, -3.1677, -2.1477, -1.4356, 0.8594, 2.8124, \mathbf{7.0000}\}$
$\sigma(A_6) = \{\mathbf{-4.4000}, -2.6174, -1.4386, 0.8449, 2.7245, \mathbf{6.0000}\}$
$\sigma(A_5) = \{\mathbf{-3.8000}, -1.8025, 0.7966, 2.5016, \mathbf{5.0000}\}$
$\sigma(A_4) = \{\mathbf{-2.0000}, 0.7714, 2.3854, \mathbf{4.7000}\}$
$\sigma(A_3) = \{\mathbf{-1.4000}, 0.8953, \mathbf{3.0000}\}$
$\sigma(A_2) = \{\mathbf{0.5000}, \mathbf{2.0000}\}$
$\sigma(A_1) = \{\mathbf{1}\}$

7. Conclusions

The inverse eigenvalue problems discussed in this paper require the construction of specially structured matrices from mixed eigendata. The results obtained here provide an efficient way to construct such matrices from given set of some of the eigenvalues of leading principal submatrices of the required matrix.

The problems IEPP and IEPB are significant in the sense that they are partially described inverse eigenvalue problems *i.e.*, they require the construction of matrices from partial information of eigenvalues and eigenvectors. Such partially described problems may occur in computations involving a complicated physical system where it is often difficult to obtain the entire spectrum. Many times, only the minimal and maximal eigenvalues are known in advance. Thus, the study of inverse problems having such prescribed eigen structure are significant. It would be interesting to consider such IEPs for other acyclic matrices as well.

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References

1. Chu, M.T. Inverse eigenvalue problems. *SIAM Rev.* **1998**, *40*, 1–39.
2. Ghanbari, K. m-functions and inverse generalized eigenvalue problem. *Inverse Prob.* **2001**, *17*, 211–217.
3. Zhang, Y. On the general algebraic inverse eigenvalue problems. *J. Comput. Math.* **2004**, *22*, 567–580.
4. Ehlay, S.; Gladwell, G.M.; Golub, G.H.; Ram,Y.M. On some eigenvector-eigenvalue relations. *SIAM J. Matrix Anal. Appl.* **1999**, *20*, 563–574.
5. Peng, J.; Hu, X.-Y.; Zhang, L. Two inverse eigenvalue problems for a special kind of matrices. *Linear Algebra Appl.* **2006**, *416*, 336–347.

6. Ghanbari, K.; Parvizpour, F. Generalized inverse eigenvalue problem with mixed eigen data. *Linear Algebra Appl.* **2012**, *437*, 2056–2063.
7. Pivovarchik, V.; Rozhenko, N.; Tretter, C. Dirichlet-Neumann inverse spectral problem for a star graph of Stieltjes strings. *Linear Algebra Appl.* **2013**, *439*, 2263–2292.
8. Pivovarchik, V.; Tretter, C. Location and multiples of eigenvalues for a star graph of Stieltjes strings. *J. Differ. Equ. Appl.* **2015**, *21*, 383–402.
9. Duarte, A.L. Construction of acyclic matrices from spectral data. *Linear Algebra Appl.* **1989**, *113* , 173–182.
10. Duarte, A.L.; Johnson, C.R. On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree. *Math. Inequal. Appl.* **2002**, *5*, 175–180.
11. Nair, R.; Shader, B.L. Acyclic matrices with a small number of distinct eigenvalues. *Linear Algebra Appl.* **2013**, *438*, 4075–4089.
12. Monfared, K.H.; Shader, B.L. Construction of matrices with a given graph and prescribed interlaced spectral data. *Linear Algebra Appl.* **2013**, *438*, 4348–4358.
13. Sen, M.; Sharma, D. Generalized inverse eigenvalue problem for matrices whose graph is a path. *Linear Algebra Appl.* **2014**, *446*, 224–236.
14. Hogben, L. Spectral graph theory and the inverse eigenvalue problem of a graph. *Electron. J. Linear Algebra* **2005**, *14*, 12–31.
15. Nylen, P.; Uhlig, F. Inverse Eigenvalue Problems Associated With Spring-Mass Systems. *Linear Algebra Appl.* **1997**, *254*, 409–425.
16. Gladwell, G.M.L. *Inverse Problems in Vibration*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2004.
17. Johnson, C.R.; Duarte, A.L.; Saiago, C.M. The Parter Wiener theorem: Refinement and generalization. *SIAM J. Matrix Anal. Appl.* **2003**, *25*, 352–361.
18. Horn, R.; Johnson, C.R. *Matrix Analysis*; Cambridge University Press: New York, NY, USA, 1985.