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Article

Conformal Maps, Biharmonic Maps, and the Warped Product

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Abstract: In this paper we study some properties of conformal maps between equidimensional manifolds, we construct new example of non-harmonic biharmonic maps and we characterize the biharmonicity of some maps on the warped product manifolds.

Keywords: biharmonic map; conformal map; warped product

Mathematics Subject Classifications (2000): 31B30, 58E20, 58E30

1. Introduction.

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2} \int_K |d\phi|^2 dv_g \tag{1}$$

for any compact subset $K \subset M$. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau(\phi) = Tr_g \nabla d\phi = 0, \tag{2}$$

and $\tau(\phi)$ is called the tension field of ϕ. One can refer to [1–4] for background on harmonic maps. In the context of harmonic maps, the stress-energy tensor was studied in details by Baird and Eells in [5]. The stress-energy tensor for a map $\phi : (M^m, g) \longrightarrow (N^n, h)$ defined by

$$S(\phi) = e(\phi)g - \phi^* h$$

and the relation between $S(\phi)$ and $\tau(\phi)$ is given by

$$divS(\phi) = -h(\tau(\phi), d\phi).$$

The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional :

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g \tag{3}$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0, \tag{4}$$

where ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g\left(\nabla^\phi\right)^2\tau(\phi) = \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. In analogy with harmonic maps, Jiang In [6] has constructed for a map ϕ the stress bi-energy tensor defined by

$$S_2(\phi) = \left(\frac{-1}{2}\left|\tau(\phi)\right|^2 + divh\left(\tau(\phi), d\phi\right)\right)g - 2symh\left(\nabla\tau(\phi), d\phi\right),$$

where

$$symh\left(\nabla\tau(\phi), d\phi\right)(X,Y) = \frac{1}{2}\left\{h\left(\nabla_X\tau(\phi), d\phi\left(Y\right)\right) + h\left(\nabla_Y\tau(\phi), d\phi\left(X\right)\right)\right\},$$

for any $X, Y \in \Gamma(TM)$. The stress bi-energy tensor was also studied in [7] and those results could be useful when we study conformal maps. The stress bi-energy tensor of ϕ satisfies the following relationship

$$divS_2(\phi) = h\left(\tau_2(\phi), d\phi\right).$$

Clearly any harmonic map is biharmonic, therefore it is interesting to construct non-harmonic biharmonic maps. In [8] the authors found new examples of biharmonic maps by conformally deforming the domain metric of harmonic ones. While in [9] the author analyzed the behavior of the biharmonic equation under the conformal change the domain metric, she obtained metrics $\widetilde{g} = e^{2\gamma}$ such that the idendity map $Id : (M, g) \longrightarrow (M, \widetilde{g})$ is biharmonic non-harmonic. Moreover, in [10] the author gave some extensions of the result in [9] together with some further constructions of biharmonic maps. The author in [11] deform conformally the codomain metric in order to render a semi-conformal harmonic map biharmonic. In [12] the authors studied the case where $\phi : (M^n, g) \longrightarrow (N^n, h)$ is a conformal mapping between equidimensional manifolds where they show that a conformal mapping ϕ is biharmonic if and only if the gradient of its dilation satisfies a second order elliptic partial differential equation. We can refer the reader to [13], for a survey of biharmonic maps. In the first section of this paper, we present some properties for a conformal mapping $\phi : (M^n, g) \longrightarrow (N^n, h)$, we prove that the stress bi-energy tensor depend only on the dilation (Theorem 1) and we calculate the bitension field of ϕ (Theorem 2). In the last section we study the biharmonicity of some maps on the warped product (Theorem 4 and 5), with this setting we obtain new examples of biharmonic non-harmonic maps.

2. Some properties for conformal maps.

We study conformal maps between equidimensional manifolds of the same dimension $n \geq 3$. Note that by a result in [12], any such map can have no critical points and so is a local conformal diffeomorphism. Recall that a mapping $\phi : (M^n, g) \to (N^n, h)$ is called conformal if there exist a C^∞ function $\lambda : M \to \mathbb{R}^*_+$ such that for any $X, Y \in \Gamma(TM)$:

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X,Y).$$

The function λ is called the dilation for the map ϕ. The tension field and the stress energy tensor for a conformal map are given by (see [14]):

Proposition 1. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map of dilation λ, we have*

$$(i)\ \ divS(\phi) = (n-2)\lambda^2 d\ln\lambda, \tag{5}$$

$$(ii)\ \ divh(\tau(\phi), d\phi) = (2-n)\left(2\lambda^2\left|grad\ln\lambda\right|^2 + \lambda^2\Delta\ln\lambda\right). \tag{6}$$

$$(iii)\ \ \tau(\phi) = (2-n)d\phi(grad\ln\lambda). \tag{7}$$

$$(iv)\ \ \left|\tau(\phi)\right|^2 = (2-n)^2\lambda^2\left|grad\ln\lambda\right|^2. \tag{8}$$

Note that the conformal map $\phi : (M^n, g) \to (N^n, h)$ of dilation λ is harmonic if and only if $n = 2$ or the dilation λ is constant.

In the first, wa calculate the stress bi-energy tensor for a conformal map ϕ when we prove that $S_2(\phi)$ depend only the dilation.

Theorem 1. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then we have*

$$S_2(\phi) = (2-n)\,\lambda^2 \left\{ \left(\frac{n-2}{2} \left| grad \ln \lambda \right|^2 + \Delta \ln \lambda \right) g - 2\nabla d \ln \lambda \right\}, \tag{9}$$

and the trace of $S_2(\phi)$ is given by

$$TrS_2(\phi) = -\,(2-n)^2\,\lambda^2 \left\{ \frac{n}{2} \left| grad \ln \lambda \right|^2 + \Delta \ln \lambda \right\}. \tag{10}$$

To prove Theorem 1, we need the following Lemma :

Lemma 1. *Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, then for any function $f \in C^\infty(M)$ and for any $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned} h\left(\nabla_X d\phi\left(grad f\right), d\phi\left(Y\right)\right) &= \lambda^2 \left(X\left(\ln \lambda\right) Y\left(f\right) - Y\left(\ln \lambda\right) X\left(f\right)\right) \\ &\quad + \lambda^2 \nabla df\left(X, Y\right) + \lambda^2 d \ln \lambda\left(grad f\right) g\left(X, Y\right). \end{aligned} \tag{11}$$

Proof of Lemma 1. Let $f \in C^\infty(M)$, for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} h\left(\nabla_X d\phi\left(grad f\right), d\phi\left(Y\right)\right) &= X\left(\lambda^2 g\left(grad f, Y\right)\right) - h\left(d\phi\left(grad f\right), \nabla_X d\phi\left(Y\right)\right) \\ &= X\left(\lambda^2\right) g\left(grad f, Y\right) + \lambda^2 g\left(\nabla_X grad f, Y\right) + \lambda^2 g\left(grad f, \nabla_X Y\right) \\ &\quad - h\left(d\phi\left(grad f\right), \nabla d\phi\left(X, Y\right)\right) - h\left(d\phi\left(grad f\right), d\phi\left(\nabla_X Y\right)\right) \\ &= X\left(\lambda^2\right) g\left(grad f, Y\right) + \lambda^2 g\left(\nabla_X grad f, Y\right) + \lambda^2 g\left(grad f, \nabla_X Y\right) \\ &\quad - h\left(d\phi\left(grad f\right), \nabla d\phi\left(X, Y\right)\right) - \lambda^2 g\left(grad f, \nabla_X Y\right). \end{aligned}$$

Note that

$$g\left(\nabla_X grad f, Y\right) = \nabla df\left(X, Y\right),$$

then we obtain

$$h\left(\nabla_X d\phi\left(grad f\right), d\phi\left(Y\right)\right) = 2\lambda^2 X\left(\ln \lambda\right) Y\left(f\right) + \lambda^2 \nabla df\left(X, Y\right) - h\left(d\phi\left(grad f\right), \nabla d\phi\left(X, Y\right)\right).$$

By similary, we have

$$h\left(\nabla_Y d\phi\left(grad f\right), d\phi\left(X\right)\right) = 2\lambda^2 Y\left(\ln \lambda\right) X\left(f\right) + \lambda^2 \nabla df\left(X, Y\right) - h\left(d\phi\left(grad f\right), \nabla d\phi\left(X, Y\right)\right).$$

Then, we deduce that

$$\begin{aligned} h\left(\nabla_X d\phi\left(grad f\right), d\phi\left(Y\right)\right) &= h\left(d\phi\left(X\right), \nabla_Y d\phi\left(grad f\right)\right) \\ &\quad + 2\lambda^2 \left(X\left(\ln \lambda\right) Y\left(f\right) - Y\left(\ln \lambda\right) X\left(f\right)\right). \end{aligned} \tag{12}$$

For the term $h\left(d\phi\left(X\right),\nabla_{Y}d\phi\left(gradf\right)\right)$, we have

$$\begin{aligned}h\left(\nabla_{Y}d\phi\left(gradf\right),d\phi\left(X\right)\right) &= h\left(\nabla d\phi\left(gradf,Y\right),d\phi\left(X\right)\right)+\lambda^{2}g\left(\nabla_{Y}gradf,X\right)\\
&= h\left(\nabla_{gradf}d\phi\left(Y\right),d\phi\left(X\right)\right)-\lambda^{2}g\left(\nabla_{gradf}Y,X\right)\\
&+\lambda^{2}g\left(\nabla_{Y}gradf,X\right)\\
&= gradf\left(\lambda^{2}g\left(X,Y\right)\right)-h\left(\nabla_{gradf}d\phi\left(X\right),d\phi\left(Y\right)\right)\\
&-\lambda^{2}g\left(\nabla_{gradf}Y,X\right)+\lambda^{2}g\left(\nabla_{Y}gradf,X\right)\\
&= 2\lambda^{2}d\ln\lambda\left(gradf\right)g\left(X,Y\right)-h\left(\nabla d\phi\left(X,gradf\right),d\phi\left(Y\right)\right)\\
&+\lambda^{2}g\left(\nabla_{Y}gradf,X\right).\end{aligned}$$

We deduce that

$$\begin{aligned}h\left(\nabla_{Y}d\phi\left(gradf\right),d\phi\left(X\right)\right) &= -h\left(\nabla_{X}d\phi\left(gradf\right),d\phi\left(Y\right)\right)+2\lambda^{2}\nabla df\left(X,Y\right)\\
&+2\lambda^{2}d\ln\lambda\left(gradf\right)g\left(X,Y\right).\end{aligned} \tag{13}$$

Finally, if we replace (13) in (12), we obtain

$$\begin{aligned}h\left(\nabla_{X}d\phi\left(gradf\right),d\phi\left(Y\right)\right) &= \lambda^{2}\left(X\left(\ln\lambda\right)Y\left(f\right)-Y\left(\ln\lambda\right)X\left(f\right)\right)\\
&+\lambda^{2}\nabla df\left(X,Y\right)+\lambda^{2}d\ln\lambda\left(gradf\right)g\left(X,Y\right).\end{aligned}$$

This completes the proof of Lemma 1.

Remark 1. *Let $\phi:(M^{n},g)\to(N^{n},h)$ be a conformal map with dilation λ, then if we consider $f=\ln\lambda$, the equation (11) gives*

$$h\left(\nabla_{X}d\phi\left(grad\ln\lambda\right),d\phi\left(Y\right)\right)=\lambda^{2}\left(\nabla d\ln\lambda\left(X,Y\right)+\left|grad\ln\lambda\right|^{2}g\left(X,Y\right)\right).$$

Proof of Theorem 1. By definition, the stress bi-energy tensor is given by :

$$S_{2}(\phi)=\left(-\frac{1}{2}\left|\tau(\phi)\right|^{2}+divh\left(\tau(\phi),d\phi\right)\right)g-2symh\left(\nabla\tau(\phi),d\phi\right). \tag{14}$$

Using the equations (2) et (4) for the Proposition 1, we have

$$-\frac{1}{2}\left|\tau(\phi)\right|^{2}+divh\left(\tau(\phi),d\phi\right)=(2-n)\lambda^{2}\left(\frac{n+2}{2}\left|grad\ln\lambda\right|^{2}+\Delta\ln\lambda\right). \tag{15}$$

Calculate now $symh\left(\nabla\tau(\phi),d\phi\right)$, we have by definition for any $X,Y\in\Gamma\left(TM\right)$

$$\begin{aligned}symh\left(\nabla\tau(\phi),d\phi\right)(X,Y) &= \frac{1}{2}\left(h\left(\nabla_{X}\tau\left(\phi\right),d\phi\left(Y\right)\right)+h\left(\nabla_{Y}\tau\left(\phi\right),d\phi\left(X\right)\right)\right)\\
&= \frac{2-n}{2}\left(h\left(\nabla_{X}d\phi\left(grad\ln\lambda\right),d\phi\left(Y\right)\right)+h\left(\nabla_{Y}\left(grad\ln\lambda\right),d\phi\left(X\right)\right)\right).\end{aligned}$$

By Lemma 1, we have

$$h\left(\nabla_{X}d\phi\left(grad\ln\lambda\right),d\phi\left(Y\right)\right)=\lambda^{2}\left(\nabla d\ln\lambda\left(X,Y\right)+\left|grad\ln\lambda\right|^{2}g\left(X,Y\right)\right)$$

and

$$h\left(\nabla_{Y}d\phi\left(grad\ln\lambda\right),d\phi\left(X\right)\right)=\lambda^{2}\left(\nabla d\ln\lambda\left(X,Y\right)+\left|grad\ln\lambda\right|^{2}g\left(X,Y\right)\right),$$

then

$$symh\left(\nabla\tau(\phi),d\phi\right)(X,Y)=(2-n)\,\lambda^{2}\left(\nabla d\ln\lambda\,(X,Y)+|grad\ln\lambda|^{2}\,g\,(X,Y)\right). \tag{16}$$

If we substitute (15) and (16) in (14), we conclude that

$$S_2(\phi)=(2-n)\,\lambda^{2}\left\{\left(\frac{n-2}{2}\,|grad\ln\lambda|^{2}+\Delta\ln\lambda\right)g-2\nabla d\ln\lambda\right\}$$

Calculate now the trace of stress bi-energy tensor. Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame on M, we have

$$\begin{aligned}Tr_g S_2(\phi)&=S_2(\phi)(e_i,e_i)\\&=(2-n)\,\lambda^{2}\left(\frac{n-2}{2}\,|grad\ln\lambda|^{2}+\Delta\ln\lambda\right)g\,(e_i,e_i)\\&-2\,(2-n)\,\lambda^{2}\nabla d\ln\lambda\,(e_i,e_i)\\&=(2-n)\,n\lambda^{2}\left(\frac{n-2}{2}\,|grad\ln\lambda|^{2}+\Delta\ln\lambda\right)\\&-2\,(2-n)\,\lambda^{2}\,(\Delta\ln\lambda)\\&=(2-n)\,\lambda^{2}\left\{\frac{n\,(n-2)}{2}\,|grad\ln\lambda|^{2}+(n-2)\,\Delta\ln\lambda\right\}.\end{aligned}$$

Then

$$TrS_2(\phi)=-\,(2-n)^{2}\,\lambda^{2}\left\{\frac{n}{2}\,|grad\ln\lambda|^{2}+\Delta\ln\lambda\right\}.$$

By calculating the Laplacian of the function $\lambda^{\frac{n}{2}}$ and by using

$$\Delta\lambda^{\frac{n}{2}}=\frac{n}{2}\lambda^{\frac{n}{2}}\left(\frac{n}{2}\,|grad\ln\lambda|^{2}+\Delta\ln\lambda\right),$$

we obtain immediately the following corollary

Corollary 1. *Let $\phi:(M^n,g)\to(N^n,h)$, $(n\neq 2)$ to be a conformal map of dilation λ, then the trace of $S_2(\phi)$ is zero if and only if the function $\lambda^{\frac{n}{2}}$ is harmonic.*

The bi-tension field of the conformal map is given by

Theorem 2. *Let $\phi:(M^n,g)\to(N^n,h)$, $(n\geq 3)$ to be a conformal map of dilation λ, then bi-tension field of ϕ is given by :*

$$\tau_2(\phi)=(n-2)\,d\phi\,(H)$$

where

$$\begin{aligned}H=grad\Delta\ln\lambda-\frac{(n-6)}{2}grad\left(|grad\ln\lambda|^{2}\right)+2Ricci^{M}\,(grad\ln\lambda)\\-\left(2\,(\Delta\ln\lambda)+(n-2)\,|grad\ln\lambda|^{2}\right)grad\ln\lambda.\end{aligned} \tag{17}$$

Remark 2. *A. Balmus in [9] studied the case where $\phi=Id_M$, she obtained the biharmonicity of the identity map from (M,g) onto $\left(M,\lambda^2 g\right)$, this case was also studied in [15].*

To prove the Theorem 2, we need two Lemmas. In the first Lemma, we give a simple formula of the term $Tr_g\left(\nabla^{\phi}\right)^{2}d\phi\,(grad\gamma)$ for a conformal map $\phi:(M^n,g)\to(N^n,h)$ $(n\geq 3)$ of dilation λ and for any function $\gamma\in C^{\infty}(M)$.

Lemma 2. *Let $\phi:(M^n,g)\to(N^n,h)$ $(n\geq 3)$ to be a conformal map of dilation λ, then for any function $\gamma\in C^{\infty}(M)$, we have*

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^{2}d\phi\,(grad\gamma)=d\phi\,(grad\Delta\gamma)+4d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right)+d\phi\left(Ricci^{M}\,(grad\gamma)\right)\\+(\Delta\ln\lambda)\,d\phi\,(grad\gamma)-2\,(\Delta\gamma)\,d\phi\,(grad\ln\lambda)\\-(n-2)\,d\ln\lambda\,(grad\gamma)\,d\phi\,(grad\ln\lambda).\end{aligned} \tag{18}$$

Proof of Lemma 2. Let $\gamma \in C^{\infty}(M)$, by definition, we have

$$Tr_g \left(\nabla^{\phi}\right)^2 d\phi\left(grad\gamma\right) = \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) - \nabla^{\phi}_{\nabla_{e_i}e_i} d\phi\left(grad\gamma\right). \tag{19}$$

(Here henceforth we sum over repeated indices.) Let us start with the calculation of the term $\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right)$, we have

$$\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) = \nabla d\phi\left(e_i, grad\gamma\right) + d\phi\left(\nabla_{e_i} grad\gamma\right).$$

It is known that (see [16])

$$\nabla d\phi\left(e_i, grad\gamma\right) = e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right) + d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right) - e_i\left(\gamma\right) d\phi\left(grad\ln\lambda\right),$$

then

$$\begin{aligned}\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) = {} & e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right) + d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right) \\ & - e_i\left(\gamma\right) d\phi\left(grad\ln\lambda\right) + d\phi\left(\nabla_{e_i} grad\gamma\right).\end{aligned} \tag{20}$$

It follows that

$$\begin{aligned}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) = {} & \nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right)\right\} + \nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right)\right\} \\ & - \nabla^{\phi}_{e_i}\left\{e_i\left(\gamma\right) d\phi\left(grad\ln\lambda\right)\right\} + \nabla^{\phi}_{e_i} d\phi\left(\nabla_{e_i} grad\gamma\right).\end{aligned} \tag{21}$$

We will study term by term the right-hand of this expression. For the first term $\nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right)\right\}$, we have

$$\nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right)\right\} = e_i\left(\ln\lambda\right)\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) + e_i\left(e_i\left(\ln\lambda\right)\right) d\phi\left(grad\gamma\right).$$

By using the equation (20), we deduce that

$$\begin{aligned}\nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right)\right\} = {} & e_i\left(\ln\lambda\right) e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right) + e_i\left(\ln\lambda\right) d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right) \\ & - e_i\left(\ln\lambda\right) e_i\left(\gamma\right) d\phi\left(grad\ln\lambda\right) + e_i\left(\ln\lambda\right) d\phi\left(\nabla_{e_i} grad\gamma\right) \\ & + e_i\left(e_i\left(\ln\lambda\right)\right) d\phi\left(grad\gamma\right),\end{aligned}$$

then, we obtain

$$\begin{aligned}\nabla^{\phi}_{e_i}\left\{e_i\left(\ln\lambda\right) d\phi\left(grad\gamma\right)\right\} = {} & \left|grad\ln\lambda\right|^2 d\phi\left(grad\gamma\right) + d\phi\left(\nabla_{grad\ln\lambda} grad\gamma\right) \\ & + e_i\left(e_i\left(\ln\lambda\right)\right) d\phi\left(grad\gamma\right).\end{aligned} \tag{22}$$

For the second term $\nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right)\right\}$, a similar calculation gives

$$\begin{aligned}\nabla^{\phi}_{e_i}\left\{d\ln\lambda\left(grad\gamma\right) d\phi\left(e_i\right)\right\} & = d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i} d\phi\left(e_i\right) + e_i\left\{g\left(grad\ln\lambda, grad\gamma\right)\right\} d\phi\left(e_i\right) \\ & = d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i} d\phi\left(e_i\right) + g\left(\nabla_{e_i} grad\ln\lambda, grad\gamma\right) d\phi\left(e_i\right) \\ & \quad + g\left(grad\ln\lambda, \nabla_{e_i} grad\gamma\right) d\phi\left(e_i\right) \\ & = d\ln\lambda\left(grad\gamma\right)\nabla^{\phi}_{e_i} d\phi\left(e_i\right) + g\left(\nabla_{grad\gamma} grad\ln\lambda, e_i\right) d\phi\left(e_i\right) \\ & \quad + g\left(\nabla_{grad\ln\lambda} grad\gamma, e_i\right) d\phi\left(e_i\right),\end{aligned}$$

it follows that

$$\begin{aligned}\nabla^{\phi}_{e_i}\{d\ln\lambda\,(grad\gamma)\,d\phi\,(e_i)\} = {} & d\ln\lambda\,(grad\gamma)\,\nabla^{\phi}_{e_i}d\phi\,(e_i) + d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right)\\ & + d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right).\end{aligned} \tag{23}$$

For the third term $\nabla^{\phi}_{e_i}\{e_i(\gamma)\,d\phi\,(grad\ln\lambda)\}$, by using the same calculation method and the equation (20), we have

$$\begin{aligned}\nabla^{\phi}_{e_i}\{e_i(\gamma)\,d\phi\,(grad\ln\lambda)\} = {} & e_i(\gamma)\,\nabla^{\phi}_{e_i}d\phi\,(grad\ln\lambda) + e_i(e_i(\gamma))\,d\phi\,(grad\ln\lambda)\\ = {} & e_i(\gamma)\,e_i(\ln\lambda)\,d\phi\,(grad\ln\lambda) + e_i(\gamma)\,d\ln\lambda\,(grad\ln\lambda)\,d\phi\,(e_i)\\ & - e_i(\gamma)\,e_i(\ln\lambda)\,d\phi\,(grad\ln\lambda) + e_i(\gamma)\,d\phi\,(\nabla_{e_i}grad\ln\lambda)\\ & + e_i(e_i(\gamma))\,d\phi\,(grad\ln\lambda),\end{aligned}$$

which gives us

$$\begin{aligned}\nabla^{\phi}_{e_i}\{e_i(\gamma)\,d\phi\,(grad\ln\lambda)\} = {} & |grad\ln\lambda|^2\,d\phi\,(grad\gamma) + d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right)\\ & + e_i(e_i(\gamma))\,d\phi\,(grad\ln\lambda).\end{aligned} \tag{24}$$

Now let us look at the last term $\nabla^{\phi}_{e_i}d\phi\,(\nabla_{e_i}grad\gamma)$, a simple calculation gives

$$\begin{aligned}\nabla^{\phi}_{e_i}d\phi\,(\nabla_{e_i}grad\gamma) = {} & e_i(\ln\lambda)\,d\phi\,(\nabla_{e_i}grad\gamma) + d\ln\lambda\,(\nabla_{e_i}grad\gamma)\,d\phi\,(e_i)\\ & - g\,(e_i,\nabla_{e_i}grad\gamma)\,d\phi\,(grad\ln\lambda) + d\phi\,(\nabla_{e_i}\nabla_{e_i}grad\gamma)\\ = {} & 2d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) - (\Delta\gamma)\,d\phi\,(grad\ln\lambda)\\ & + d\phi\,(\nabla_{e_i}\nabla_{e_i}grad\gamma),\end{aligned}$$

then

$$\begin{aligned}\nabla^{\phi}_{e_i}d\phi\,(\nabla_{e_i}grad\gamma) = {} & d\phi\,(\nabla_{e_i}\nabla_{e_i}grad\gamma) + 2d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right)\\ & - (\Delta\gamma)\,d\phi\,(grad\ln\lambda).\end{aligned} \tag{25}$$

If we replace (22), (23), (24) and (25) in (21), we obtain

$$\begin{aligned}\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\,(grad\gamma) = {} & 4d\phi\left(\nabla_{grad\ln\lambda}grad\gamma\right) + e_i(e_i(\ln\lambda))\,d\phi\,(grad\gamma)\\ & + d\ln\lambda\,(grad\gamma)\,\nabla^{\phi}_{e_i}d\phi\,(e_i) - e_i(e_i(\gamma))\,d\phi\,(grad\ln\lambda)\\ & + d\phi\,(\nabla_{e_i}\nabla_{e_i}grad\gamma) - (\Delta\gamma)\,d\phi\,(grad\ln\lambda).\end{aligned} \tag{26}$$

To complete the proof, it remains to investigate the term $\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\,(grad\gamma)$, we have

$$\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\,(grad\gamma) = \nabla d\phi\,(\nabla_{e_i}e_i, grad\gamma) + d\phi\left(\nabla_{\nabla_{e_i}e_i}grad\gamma\right),$$

Therefore, by using the equation (20), we obtain

$$\begin{aligned}\nabla^{\phi}_{\nabla_{e_i}e_i}d\phi\,(grad\gamma) = {} & \nabla_{e_i}e_i(\ln\lambda)\,d\phi\,(grad\gamma) + d\ln\lambda\,(grad\gamma)\,d\phi\,(\nabla_{e_i}e_i)\\ & - \nabla_{e_i}e_i(\gamma)\,d\phi\,(grad\ln\lambda) + d\phi\left(\nabla_{\nabla_{e_i}e_i}grad\gamma\right).\end{aligned} \tag{27}$$

By substituting (26) and (27) in (19), we deduce

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\gamma\right) &= \nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} d\phi\left(grad\gamma\right) - \nabla^{\phi}_{\nabla_{e_i}e_i} d\phi\left(grad\gamma\right) \\ &= d\phi\left(Tr_g\nabla^2 grad\gamma\right) + 4d\phi\left(\nabla_{grad\ln\lambda} grad\gamma\right) \\ &+ \left(\Delta\ln\lambda\right) d\phi\left(grad\gamma\right) + d\ln\lambda\left(grad\gamma\right)\tau\left(\phi\right) \\ &- 2\left(\Delta\gamma\right) d\phi\left(grad\ln\lambda\right). \end{aligned}$$

Finally, using the fact that (see [11])

$$Tr_g\nabla^2 grad\gamma = grad\Delta\gamma + Ricci^M\left(grad\gamma\right)$$

and

$$\tau\left(\phi\right) = \left(2-n\right) d\phi\left(grad\ln\lambda\right),$$

we conclude that

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\gamma\right) &= d\phi\left(grad\Delta\gamma\right) + 4d\phi\left(\nabla_{grad\ln\lambda} grad\gamma\right) + d\phi\left(Ricci^M\left(grad\gamma\right)\right) \\ &+ \left(\Delta\ln\lambda\right) d\phi\left(grad\gamma\right) - 2\left(\Delta\gamma\right) d\phi\left(grad\ln\lambda\right) \\ &- \left(n-2\right) d\ln\lambda\left(grad\gamma\right) d\phi\left(grad\ln\lambda\right). \end{aligned}$$

This completes the proof of Lemma 2. Now, in the second Lemma, we will calculate $Tr_g R^N\left(d\phi\left(grad\gamma\right), d\phi\right) d\phi$ for a conformal maps $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ of dilation λ and for any function $\gamma \in C^{\infty}\left(M\right)$

Lemma 3. *Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ, then for any function $\gamma \in C^{\infty}\left(M\right)$, we have*

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\gamma\right), d\phi\right) d\phi &= d\phi\left(Ricci^M\left(grad\gamma\right)\right) - \left(n-2\right) d\phi\left(\nabla_{grad\gamma} grad\ln\lambda\right) \\ &- \left(\Delta\ln\lambda + \left(n-2\right)\left|grad\ln\lambda\right|^2\right) d\phi\left(grad\gamma\right) \\ &+ \left(n-2\right) d\ln\lambda\left(grad\gamma\right) d\phi\left(grad\ln\lambda\right) \end{aligned} \tag{28}$$

Proof of Lemma 3. Let $\gamma \in C^{\infty}\left(M\right)$, by definition we have

$$Tr_g R^N\left(d\phi\left(grad\gamma\right), d\phi\right) d\phi = R^N\left(d\phi\left(grad\gamma\right), d\phi\left(e_i\right)\right) d\phi\left(e_i\right) \tag{29}$$

but we know that (see [16])

$$\begin{aligned} Ric^N\left(d\phi\left(X\right), d\phi\left(Y\right)\right) &= Ric^M\left(X, Y\right) + \left(n-2\right) X\left(\ln\lambda\right) Y\left(\ln\lambda\right) \\ &- \left(n-2\right)\left|grad\ln\lambda\right|^2 g\left(X, Y\right) \\ &- \left(n-2\right)\nabla d\ln\lambda\left(X, Y\right) - \left(\Delta\ln\lambda\right) g\left(X, Y\right). \end{aligned}$$

Then

$$\begin{aligned} Ric^N\left(d\phi\left(grad\gamma\right), d\phi\left(e_i\right)\right) &= Ric^M\left(grad\gamma, e_i\right) + \left(n-2\right) grad\gamma\left(\ln\lambda\right) e_i\left(\ln\lambda\right) \\ &- \left(n-2\right)\left|grad\ln\lambda\right|^2 g\left(grad\gamma, e_i\right) \\ &- \left(n-2\right)\nabla d\ln\lambda\left(grad\gamma, e_i\right) - \left(\Delta\ln\lambda\right) g\left(grad\gamma, e_i\right) \end{aligned}$$

it follows that

$$\begin{aligned} Ric^N\left(d\phi\left(grad\gamma\right),d\phi\left(e_i\right)\right) = {} & Ric^M\left(grad\gamma,e_i\right)+(m-2)\,d\ln\lambda\left(grad\gamma\right)e_i\left(\ln\lambda\right) \\ & -(n-2)\left|grad\ln\lambda\right|^2 e_i\left(\gamma\right)-(n-2)\,\nabla d\ln\lambda\left(grad\gamma,e_i\right) \\ & -\left(\Delta\ln\lambda\right)e_i\left(\gamma\right). \end{aligned} \tag{30}$$

If we replace (30) in (29), we deduce that

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\gamma\right),d\phi\right)d\phi = {} & R^N\left(d\phi\left(grad\gamma\right),d\phi\left(e_i\right)\right)d\phi\left(e_i\right) \\ = {} & d\phi\left(Ricci^M\left(grad\gamma\right)\right)+(n-2)\,d\ln\lambda\left(grad\gamma\right)d\phi\left(grad\ln\lambda\right) \\ & -(n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\gamma\right)-(n-2)\,\nabla d\ln\lambda\left(grad\gamma,e_i\right)d\phi\left(e_i\right) \\ & -\left(\Delta\ln\lambda\right)d\phi\left(grad\gamma\right). \end{aligned}$$

To complete the proof, we will simplify the term $\nabla d\ln\lambda\left(grad\gamma,e_i\right)d\phi\left(e_i\right)$, we obtain

$$\begin{aligned} \nabla d\ln\lambda\left(grad\gamma,e_i\right)d\phi\left(e_i\right) &= \left\{e_i\left(g\left(grad\ln\lambda,grad\gamma\right)\right)-d\ln\lambda\left(\nabla_{e_i}grad\gamma\right)\right\}d\phi\left(e_i\right) \\ &= g\left(\nabla_{e_i}grad\ln\lambda,grad\gamma\right)d\phi\left(e_i\right) \\ &= g\left(\nabla_{grad\gamma}grad\ln\lambda,e_i\right)d\phi\left(e_i\right) \\ &= d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right), \end{aligned}$$

which finally gives

$$\begin{aligned} Tr_g R^N\left(d\phi\left(grad\gamma\right),d\phi\right)d\phi = {} & d\phi\left(Ricci^M\left(grad\gamma\right)\right)-(n-2)\,d\phi\left(\nabla_{grad\gamma}grad\ln\lambda\right) \\ & -\left(\Delta\ln\lambda+(n-2)\left|grad\ln\lambda\right|^2\right)d\phi\left(grad\gamma\right) \\ & +(n-2)\,d\ln\lambda\left(grad\gamma\right)d\phi\left(grad\ln\lambda\right). \end{aligned}$$

This completes the proof of Lemma 3. We are now able to prove Theorem 2.
Proof of Theorem 2. By definition, the bi-tension field is given by

$$\tau_2\left(\phi\right) = -Tr_g\left(\nabla^{\phi}\right)^2\tau\left(\phi\right)-Tr_g R^N\left(\tau\left(\phi\right),d\phi\right)d\phi.$$

The tension field of the conformal map ϕ is given by

$$\tau\left(\phi\right) = (2-n)\,d\phi\left(grad\ln\lambda\right),$$

it follows that

$$\tau_2\left(\phi\right) = (n-2)\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right)+Tr_g R^N\left(d\phi\left(grad\ln\lambda\right),d\phi\right)d\phi\right). \tag{31}$$

By Lemma 2, we have

$$\begin{aligned} Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right) = {} & d\phi\left(grad\Delta\ln\lambda\right)+2d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right) \\ & -\left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)+d\phi\left(Ricci^M\left(grad\ln\lambda\right)\right) \\ & -(n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right). \end{aligned} \tag{32}$$

By using lemma 3 and the fact that $\nabla_{grad \ln \lambda} grad \ln \lambda = \frac{1}{2} grad \left(|grad \ln \lambda|^2 \right)$

$$\begin{aligned} Tr_g R^N \left(d\phi \left(grad \ln \lambda \right), d\phi \right) d\phi = d\phi \left(Ricci^M \left(grad \ln \lambda \right) \right) - \left(\Delta \ln \lambda \right) d\phi \left(grad \ln \lambda \right) \\ - \frac{(n-2)}{2} d\phi \left(grad \left(|grad \ln \lambda|^2 \right) \right). \end{aligned} \tag{33}$$

If we replace (32) and (33) in (31), we deduce that

$$\begin{aligned} \tau_2 (\phi) = {} & (n-2)\, d\phi \left(grad \Delta \ln \lambda \right) - \frac{(n-2)(n-6)}{2} d\phi \left(grad \left(|grad \ln \lambda|^2 \right) \right) \\ & - (n-2) \left(2 \left(\Delta \ln \lambda \right) + (n-2) |grad \ln \lambda|^2 \right) d\phi \left(grad \ln \lambda \right) \\ & + 2 (n-2)\, d\phi \left(Ricci^M \left(grad \ln \lambda \right) \right). \end{aligned}$$

Then the bi-tension field of ϕ is given by :

$$\tau_2(\phi) = (n-2)\, d\phi\, (H)$$

where

$$\begin{aligned} H = {} & grad \Delta \ln \lambda - \frac{(n-6)}{2} grad \left(|grad \ln \lambda|^2 \right) + 2 Ricci^M \left(grad \ln \lambda \right) \\ & - \left(2 \left(\Delta \ln \lambda \right) + (n-2) |grad \ln \lambda|^2 \right) grad \ln \lambda. \end{aligned}$$

The proof of Theorem 2 is complete. By application of Theorem 2, we get the following result (see [15]).

Theorem 3. *([12]) Let $\phi : (M^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ, then ϕ is biharmonic if and only if the dilation λ satisfies*

$$\begin{aligned} & grad \left(\Delta \ln \lambda \right) - \left(2 \left(\Delta \ln \lambda \right) + (n-2) |grad \ln \lambda|^2 \right) grad \ln \lambda \\ & + \frac{6-n}{2} grad \left(|grad \ln \lambda|^2 \right) + 2 Ricci^M (grad \ln \lambda) = 0. \end{aligned}$$

In particular, we prove that the biharmonicity of the conformal map $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ where the dilation λ is radial $\left(\ln \lambda = \alpha (r), r = |x| \text{ and } \alpha \in C^\infty (\mathbb{R}, \mathbb{R}) \right)$ is equivalent to an ordinary differential equation of the second order. More precisely, we have

Corollary 2. *Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ when we suppose that $\ln \lambda$ is radial $\left(\ln \lambda = \alpha (r), r = |x| \text{ and } \alpha \in C^\infty (\mathbb{R}, \mathbb{R}) \right)$. Then ϕ is biharmonic if and only if $\beta = \alpha'$ satisfies the following ordinary differential equation :*

$$\beta'' - (n-4) \beta \beta' + \frac{n-1}{r} \beta' - \frac{n-1}{r^2} \beta - \frac{2(n-1)}{r} \beta^2 - (n-2) \beta^3 = 0. \tag{34}$$

Proof of Corollary 2 Let $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ $(n \geq 3)$ to be a conformal map of dilation λ such that $\ln \lambda = \alpha (r)$. By Theorem 3, ϕ is biharmonic if and only if the dilation λ satisfies

$$\begin{aligned} & grad \left(\Delta \ln \lambda \right) - \left(2 \left(\Delta \ln \lambda \right) + (n-2) |grad \ln \lambda|^2 \right) grad \ln \lambda \\ & + \frac{6-n}{2} grad \left(|grad \ln \lambda|^2 \right) + 2 Ricci^M (grad \ln \lambda) = 0. \end{aligned}$$

A direct calculation gives

$$grad \ln \lambda = \alpha' \frac{\partial}{\partial r},$$

$$|grad \ln \lambda|^2 = (\alpha')^2,$$

$$grad\left(|grad \ln \lambda|^2\right) = 2\alpha'\alpha''\frac{\partial}{\partial r},$$

$$\Delta \ln \lambda = \alpha'' + \frac{n-1}{r}\alpha'$$

and

$$grad\,(\Delta \ln \lambda) = \left(\alpha''' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha'\right)\frac{\partial}{\partial r}.$$

Therefore ϕ is biharmonic if and only if the function α satisfies the following differential equation

$$\alpha''' - (n-4)\,\alpha'\alpha'' + \frac{n-1}{r}\alpha'' - \frac{n-1}{r^2}\alpha' - \frac{2\,(n-1)}{r}\,(\alpha')^2 - (n-2)\,(\alpha')^3 = 0.$$

If we denote $\beta = \alpha'$, the biharmonicity of ϕ is equivalent to the differential equation

$$\beta'' - (n-4)\,\beta\beta' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta - \frac{2\,(n-1)}{r}\beta^2 - (n-2)\,\beta^3 = 0.$$

As a consequence of the Corollary 2, We will present some remarks which we give a particular solutions of the equation (34) that allows us to construct a biharmonic non-harmonic maps.

Remark 3. *. Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$). By (34), we deduce that $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ ($n \geq 3$) is biharmonic if and only if a is a solution of the algebraic equation*

$$(n-2)\,a^2 + (n+2)\,a + 2n - 2 = 0.$$

This equation has real solutions if and only if $n \in \{3, 4\}$.

1. *If $n = 3$, we find $a = \frac{-5+\sqrt{17}}{2}$ or $a = \frac{-5-\sqrt{17}}{2}$, so $\lambda = Cr^{-\left(\frac{5-\sqrt{17}}{2}\right)}$ or $\lambda = Cr^{-\left(\frac{5+\sqrt{17}}{2}\right)}$ ($C \in \mathbb{R}_+^*$). It follows that any conformal map $\phi : (\mathbb{R}^3, g) \to (N^3, h)$ of dilation $\lambda = Cr^{-\left(\frac{5-\sqrt{17}}{2}\right)}$ or $\lambda = Cr^{-\left(\frac{5+\sqrt{17}}{2}\right)}$ is biharmonic non-harmonic.*
2. *If $n = 4$, we find $a = -1$ or $a = -2$, so $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ ($C \in \mathbb{R}_+^*$). Then, in this case any conformal map $\phi : (\mathbb{R}^4, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^2}$ or $\lambda = \frac{C}{r}$ is biharmonic non-harmonic. For example, the inversion $\phi : (\mathbb{R}^n \backslash \{0\}, g_{\mathbb{R}^n}) \longrightarrow (\mathbb{R}^n \backslash \{0\}, g_{\mathbb{R}^n})$ definded by $\phi(x) = \frac{x}{|x|^2}$ is a conformal map with dilation $\lambda = \frac{1}{r^2}$. By (34), the inversion is biharmonic non-harmonic if and only if $n = 4$.*

Remark 4. *. Looking for particular solutions of type $\beta = \frac{ar}{1+r^2}$ ($a \in \mathbb{R}^*$). By (34), $\phi : (\mathbb{R}^n, g) \to (N^n, h)$ ($n \geq 3$) is biharmonic if and only we have*

$$(n-2)\,a^2 + (n+2)\,a + 2n - 2 = 0$$

and

$$3\,(n-2)\,a + 2n + 4 = 0.$$

These two equations gives $a = -2$ and $n = 4$, it follows that the dilation is equal to $\lambda = \frac{C}{r^2+1}$ ($C \in \mathbb{R}_+^*$). Then, all conformal maps $\phi : (\mathbb{R}^4, g) \to (N^4, h)$ of dilation $\lambda = \frac{C}{r^2+1}$ are biharmonic non-harmonic. For example, the inverse of the stereographic projection of the sphere $\phi : \mathbb{R}^n \longrightarrow S^n$ definded by $\phi(x) = \frac{1}{|x|^2+1}\left(|x|^2 - 1, 2x\right)$ is a conformal map with dilation $\lambda = \frac{2}{r^2+1}$. By (34), the inverse of the stereographic projection is biharmonic non-harmonic if and only if $n = 4$.

The last part of this paper is devoted to the study of biharmonic maps between warped product manifolds, these maps were also studied in [17]. We will give some results of the biharmonicity in other particular cases.

3. Biharmonic maps and the warped product

Let (M^m, g) and (N^n, h) two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The warped product $M \times_f N$ is the product manifolds $M \times N$ endowed with the Riemannian metric G_f defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_f(X, Y) = g(d\pi(X), d\pi(Y)) + (f \circ \pi)^2 h(d\eta(X), d\eta(Y)),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The function f is called the warping function of the warped product. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$. The Levi-Civita connection $\widetilde{\nabla}$ of the warped product $M \times_f N$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + X_1(\ln f)(0, Y_2) + Y_1(\ln f)(0, X_2) - f^2 h(X_2, Y_2)(grad \ln f, 0). \tag{35}$$

In the first, we consider a smooth map $\phi : (M^m, g) \longrightarrow (P^p, k)$ and we defined the map $\widetilde{\phi} : \left(M^m \times_f N^n, G_f\right) \longrightarrow (P^p, k)$ by $\widetilde{\phi}(x, y) = \phi(x)$. We will study the biharmonicity of $\widetilde{\phi}$. By calculating the tension field of $\widetilde{\phi}$, we obtain the following result :

Proposition 2. *Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ be a smooth map. The tension field of the map $\widetilde{\phi} : \left(M^m \times_f N^n, G_f\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x, y) = \phi(x)$ is given by*

$$\tau\left(\widetilde{\phi}\right) = \tau(\phi) + n d\phi(grad \ln f) \tag{36}$$

Proof of Proposition 2. Let us choose $\{e_i\}_{1 \le i \le m}$ to be an orthonormal frame on M and $\{f_j\}_{1 \le j \le n}$ to be an orthonormal frame on N. An orthonormal frame on $M \times_f N$ is given by $\left\{(e_i, 0), \frac{1}{f}(0, f_j)\right\}$. Note that in this case we have $d\widetilde{\phi}(X, Y) = (d\phi(X), 0)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. By definition to the tension field, we have

$$\begin{aligned}
\tau\left(\widetilde{\phi}\right) &= Tr_{G_f} \nabla d\widetilde{\phi} \\
&= \nabla^{\widetilde{\phi}}_{(e_i,0)} d\widetilde{\phi}(e_i, 0) + \frac{1}{f^2} \nabla^{\widetilde{\phi}}_{(0,f_j)} d\widetilde{\phi}(0, f_j) \\
&\quad - d\widetilde{\phi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i, 0)\right) - \frac{1}{f^2} d\widetilde{\phi}\left(\widetilde{\nabla}_{(0,f_j)}(0, f_j)\right).
\end{aligned}$$

A simple calculation gives

$$\nabla^{\widetilde{\phi}}_{(e_i,0)} d\widetilde{\phi}(e_i, 0) = \nabla^{\phi}_{e_i} d\phi(e_i)$$

and

$$\nabla^{\widetilde{\phi}}_{(0,f_j)} d\widetilde{\phi}(0, f_j) = 0,$$

By using the equation (35), we deduce that

$$\widetilde{\nabla}_{(e_i,0)}(e_i, 0) = (\nabla_{e_i} e_i, 0)$$

and

$$\widetilde{\nabla}_{(0,f_j)}(0, f_j) = \left(0, \nabla_{f_j} f_j\right) - n f^2 (grad \ln f, 0).$$

It follows that

$$\tau\left(\widetilde{\phi}\right)=\nabla^{\phi}_{e_i}d\phi\left(e_i\right)-d\phi\left(\nabla^{M}_{e_i}e_i\right)+nd\phi\left(grad\ln f\right),$$

then, we obtain

$$\tau\left(\widetilde{\phi}\right)=\tau\left(\phi\right)+nd\phi\left(grad\ln f\right).$$

Remark 5. *If $\phi:(M^m,g)\longrightarrow(P^m,k)$ ($m\geq 3$) is a conformal map with dilation λ, the tension field of $\widetilde{\phi}$ is given by*

$$\tau\left(\widetilde{\phi}\right)=(2-m)\,d\phi\left(grad\ln\lambda\right)+nd\phi\left(grad\ln f\right)=d\phi\left(grad\ln\left(\lambda^{2-m}f^n\right)\right).$$

Then $\widetilde{\phi}$ is harmonic if and only if the function $\lambda^{2-m}f^n$ is constant.

We will now calculate the bitension field of the map $\widetilde{\phi}:\left(M^m\times_f N^n,G_f\right)\longrightarrow(P^p,k)$.

Theorem 4. *Let $\phi:(M^m,g)\longrightarrow(P^p,k)$ be a smooth map. The bitension field of the map $\widetilde{\phi}:\left(M^m\times_f N^n,G_f\right)\longrightarrow(P^p,k)$ defined by $\widetilde{\phi}(x,y)=\phi(x)$ is given by*

$$\begin{aligned}\tau_2\left(\widetilde{\phi}\right)=&\,\tau_2\left(\phi\right)-n\left(Tr_g\nabla^2d\phi\left(grad\ln f\right)+Tr_gR^p\left(d\phi\left(grad\ln f\right),d\phi\right)d\phi\right)\\&-n\nabla_{grad\ln f}\tau\left(\phi\right)-n^2\nabla_{grad\ln f}d\phi\left(grad\ln f\right).\end{aligned}\tag{37}$$

Proof of Theorem 4. By definition of the bi-tension field, we have

$$\tau_2\left(\widetilde{\phi}\right)=-Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)-Tr_{G_f}R^P\left(\tau\left(\widetilde{\phi}\right),d\widetilde{\phi}\right)d\widetilde{\phi}\tag{38}$$

For the first term $Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)$, we have

$$\begin{aligned}Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)=&\,\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\widetilde{\phi}\right)+\frac{1}{f^2}\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\tau\left(\widetilde{\phi}\right)\\&-\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\phi}\right)-\frac{1}{f^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\phi}\right).\end{aligned}$$

We will study term by term the right-hand of this expression. A simple calculation gives

$$\begin{aligned}\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\widetilde{\phi}\right)&=\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\tau\left(\phi\right)+n\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}d\phi\left(grad\ln f\right)\\&=\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}\tau\left(\phi\right)+n\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i}d\phi\left(grad\ln f\right)\end{aligned}$$

and

$$\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\tau\left(\widetilde{\phi}\right)=0.$$

By using the equation (35), we obtain

$$\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\phi}\right)=\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau\left(\phi\right)+n\nabla^{\phi}_{\nabla^M_{e_i}e_i}d\phi\left(grad\ln f\right),$$

and

$$\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\phi}\right)=-nf^2\nabla^{\phi}_{grad\ln f}\tau\left(\phi\right)-n^2f^2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln f\right).$$

Then, we deduce that

$$\begin{aligned}Tr_{G_f}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)=&\,Tr_g\left(\nabla^{\phi}\right)^2\tau\left(\phi\right)+nTr_g\left(\nabla^{\phi}\right)^2d\phi\left(grad\ln f\right)\\&+n\nabla^{\phi}_{grad\ln f}\tau\left(\phi\right)+n^2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln f\right).\end{aligned}\tag{39}$$

To complete the proof, we will simplify the term $Tr_{G_f} R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right) d\widetilde{\phi}$, we have

$$\begin{aligned} Tr_{G_f} R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right) d\widetilde{\phi} = & R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(e_i, 0\right)\right) d\widetilde{\phi}\left(e_i, 0\right) \\ & + \frac{1}{f^2} R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(0, f_j\right)\right) d\widetilde{\phi}\left(0, f_j\right) \\ = & R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\left(e_i, 0\right)\right) d\widetilde{\phi}\left(e_i, 0\right) \\ = & R^P\left(\tau\left(\phi\right), d\phi\left(e_i\right)\right) d\phi\left(e_i\right) \\ & + nR^P\left(d\phi\left(grad \ln f\right), d\phi\left(e_i\right)\right) d\phi\left(e_i\right). \end{aligned}$$

It follows that

$$Tr_{G_f} R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right) d\widetilde{\phi} = Tr_g R^P\left(\tau\left(\phi\right), d\phi\right) d\phi + nTr_g R^P\left(d\phi\left(grad \ln f\right), d\phi\right) d\phi. \tag{40}$$

If we replace (39) and (40) in (38), we obtain

$$\begin{aligned} \tau_2\left(\widetilde{\phi}\right) = & \tau_2\left(\phi\right) - n\left(Tr_g \nabla^2 d\phi\left(grad \ln f\right) + Tr_g R^p\left(d\phi\left(grad \ln f\right), d\phi\right) d\phi\right) \\ & - n\nabla_{grad \ln f} \tau\left(\phi\right) - n^2 \nabla_{grad \ln f} d\phi\left(grad \ln f\right). \end{aligned}$$

The proof of Theorem 4 is complete.

Remark 6. *Theorem 4 is a particular result of generalized warped product manifolds (see [18]).*

As a consequence, if ϕ is harmonic, we have

Corollary 3. *Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ a harmonic map. The map $\widetilde{\phi} : \left(M^m \times_f N^n, G_f\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}\left(x, y\right) = \phi\left(x\right)$ is biharmonic if and only if*

$$Tr_g \nabla^2 d\phi\left(grad \ln f\right) + Tr_g R^P\left(d\phi\left(grad \ln f\right), d\phi\right) d\phi + n\nabla_{grad \ln f} d\phi\left(grad \ln f\right) = 0.$$

In particular if $\phi = Id_M$, the first projection $P_1 : \left(M^m \times_f N^n, G_f\right) \longrightarrow (M^m, g)$ defined by $P_1\left(x, y\right) = x$ is biharmonic if and only if (see [17])

$$grad \Delta \ln f + \frac{n}{2} grad\left(\left|grad \ln f\right|^2\right) + 2Ricci^M\left(grad \ln f\right) = 0.$$

In the following we shall present an example of biharmonic non-harmonic maps.

Example 1. *Let $\widetilde{\varphi} : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$ defined by $\widetilde{\varphi}\left(x, y\right) = \frac{x}{|x|^2}$ when we suppose that $\ln f$ is radial ($\ln f = \alpha\left(r\right)$). Then by Theorem 4, we deduce that the map $\widetilde{\varphi} : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$ is biharmonic if and only if the function α satisfies the following differential equation*

$$n\alpha''' + \frac{n\left(m-5\right)}{r}\alpha'' - \frac{3n\left(3m-7\right)}{r^2}\alpha' + n^2\alpha'\alpha'' - \frac{2n^2}{r}\left(\alpha'\right)^2 - \frac{8\left(m-2\right)\left(m-4\right)}{r^3} = 0.$$

Let $\beta = \alpha'$, this equation becomes

$$n\beta'' + \frac{n\left(m-5\right)}{r}\beta' - \frac{3n\left(3m-7\right)}{r^2}\beta + n^2\beta\beta' - \frac{2n^2}{r}\beta^2 - \frac{8\left(m-2\right)\left(m-4\right)}{r^3} = 0.$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ ($a \in \mathbb{R}^*$), then $\widetilde{\varphi} : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$ is biharmonic if and only if

$$3n^2a^2 + 2n\left(5m-14\right)a + 8\left(m-2\right)\left(m-4\right) = 0.$$

This equation has two solutions $a = \frac{4-2m}{n}$ and $a = \frac{4(4-m)}{3n}$.

1. *For* $a = \frac{4-2m}{n}$, *we obtain* $f(r) = Cr^{\frac{4-2m}{n}}$ *and in this case* $\widetilde{\varphi} : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$ *is harmonic so biharmonic.*
2. *For* $a = \frac{4(4-m)}{3n}$, *we obtain* $f(r) = Cr^{\frac{4(4-m)}{3n}}$ *and in this case* $\widetilde{\varphi} : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$ *is biharmonic non-harmonic.*

Now, we consider a smooth map $\psi : (N^n, g) \longrightarrow (P^p, k)$ and we define the map $\widetilde{\psi} : \left(M^m \times_f N^n, G_f\right) \longrightarrow (P^p, k)$ by $\widetilde{\psi}(x, y) = \psi(y)$. We will study the biharmonicity of $\widetilde{\psi}$, we obtain the following result :

Theorem 5. *Let* $\psi : (N^n, h) \to (P^p, k)$ *be a smooth map, we define* $\widetilde{\psi} : \left(M^m \times_{f^2} N^n, G_{f^2}\right) \to (P^p, k)$ *by* $\widetilde{\psi}(x, y) = \psi(y)$ *. The tension field and the bi-tension field of* $\widetilde{\psi}$ *are given by*

$$\tau(\widetilde{\psi}) = \frac{1}{f^2 \circ \pi} \tau(\psi) \tag{41}$$

and

$$\tau_2(\widetilde{\psi}) = \frac{1}{f^4 \circ \pi} \tau_2(\psi) - \frac{2}{f^2 \circ \pi} \left(\left(\Delta \ln f + (n-2) |grad \ln f|^2 \right) \circ \pi \right) \tau(\psi). \tag{42}$$

Proof of Theorem 5. In the first, we calculate the tension field of of $\widetilde{\psi}$. By definition to the tension field, we have

$$\begin{aligned} \tau(\widetilde{\psi}) &= Tr_{G_f} \nabla d\widetilde{\psi} \\ &= \nabla^{\widetilde{\psi}}_{(e_i,0)} d\widetilde{\psi}(e_i, 0) + \frac{1}{f^2 \circ \pi} \nabla^{\widetilde{\psi}}_{(0,f_j)} d\widetilde{\psi}(0, f_j) \\ &\quad - d\widetilde{\psi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i, 0)\right) - \frac{1}{f^2 \circ \pi} d\widetilde{\psi}\left(\widetilde{\nabla}_{(0,f_j)}(0, f_j)\right). \end{aligned}$$

By using the equation (35), we obtain

$$\tau(\widetilde{\psi}) = \frac{1}{f^2 \circ \pi} \nabla^{\psi}_{f_j} d\psi(f_j) - \frac{1}{f^2 \circ \pi} d\psi\left(\nabla_{f_j} f_j\right) = \frac{1}{f^2 \circ \pi} \tau(\psi),$$

then

$$\tau(\widetilde{\psi}) = \frac{1}{f^2 \circ \pi} \tau(\psi).$$

By this expression, we deduce that $\widetilde{\psi}$ is harmonic if and only if ψ is harmonic. Now, we will calculate the bi-tension field of $\widetilde{\psi}$. By definition, we have

$$\tau_2(\widetilde{\psi}) = -Tr_{G_f} \left(\nabla^{\widetilde{\psi}}\right)^2 \tau(\widetilde{\psi}) - Tr_{G_f} R^P \left(\tau(\widetilde{\psi}), d\widetilde{\psi}\right) d\widetilde{\psi}. \tag{43}$$

For the first term $Tr_{G_f} \left(\nabla^{\widetilde{\psi}}\right)^2 \tau(\widetilde{\psi})$, we have

$$\begin{aligned} Tr_{G_f} \left(\nabla^{\widetilde{\psi}}\right)^2 \tau(\widetilde{\psi}) &= \nabla^{\widetilde{\psi}}_{(e_i,0)} \nabla^{\widetilde{\psi}}_{(e_i,0)} \tau(\widetilde{\psi}) + \frac{1}{f^2 \circ \pi} \nabla^{\widetilde{\psi}}_{(0,f_j)} \nabla^{\widetilde{\psi}}_{(0,f_j)} \tau(\widetilde{\psi}) \\ &\quad - \nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)} \tau(\widetilde{\psi}) - \frac{1}{f^2 \circ \pi} \nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)} \tau(\widetilde{\psi}). \end{aligned}$$

A long calculation gives

$$\nabla^{\widetilde{\psi}}_{(e_i,0)} \nabla^{\widetilde{\psi}}_{(e_i,0)} \tau(\widetilde{\psi}) = \frac{2}{f^2 \circ \pi} \left(\left(2 |grad \ln f|^2 - e_i(e_i(\ln f)) \right) \circ \pi \right) \tau(\psi)$$

and

$$\frac{1}{f^2 \circ \pi} \nabla^{\widetilde{\psi}}_{(0,f_j)} \nabla^{\widetilde{\psi}}_{(0,f_j)} \tau(\widetilde{\psi}) = \frac{1}{f^4 \circ \pi} \nabla^{\psi}_{f_j} \nabla^{\psi}_{f_j} \tau(\psi).$$

Finally, by (35), we obtain

$$\nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\tau\left(\widetilde{\psi}\right) = \frac{2}{f^2\circ\pi}\left(\nabla_{e_i}e_i\left((\ln f)\right)\circ\pi\right)\tau\left(\psi\right)$$

and

$$\frac{1}{f^2\circ\pi}\nabla^{\widetilde{\psi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\tau\left(\widetilde{\psi}\right) = \frac{1}{f^4\circ\pi}\nabla^{\psi}_{\nabla_{f_j}f_j}\tau\left(\psi\right) + \frac{2n}{f^2\circ\pi}\left(\left(|grad\ln f|^2\right)\circ\pi\right)\tau\left(\psi\right).$$

Which gives us

$$Tr_{G_f}\left(\nabla^{\widetilde{\psi}}\right)^2\tau\left(\widetilde{\psi}\right) = \frac{1}{f^4\circ\pi}Tr_h\nabla^2\tau\left(\psi\right) - \frac{2}{f^2\circ\pi}\left(\left(\Delta\ln f + (n-2)\,|grad\ln f|^2\right)\circ\pi\right)\tau\left(\psi\right) \quad (44)$$

Finally for the first term $Tr_{G_f}R^P\left(\tau\left(\widetilde{\psi}\right), d\widetilde{\psi}\right)d\widetilde{\psi}$, it is easy to verify that

$$Tr_{G_f}R^P\left(\tau\left(\widetilde{\psi}\right), d\widetilde{\psi}\right)d\widetilde{\psi} = \frac{1}{f^4\circ\pi}Tr_hR^P\left(\tau\left(\psi\right), d\psi\right)d\psi. \quad (45)$$

If we substitute (44) and (45) in (43), we obtain

$$\tau_2\left(\widetilde{\psi}\right) = \frac{1}{f^4\circ\pi}\tau_2\left(\psi\right) - \frac{2}{f^2\circ\pi}\left(\left(\Delta\ln f + (n-2)\,|grad\ln f|^2\right)\circ\pi\right)\tau\left(\psi\right).$$

This completes the proof of Theorem 5. An immediate consequence of Theorem 5 is given by the following corollary :

Corollary 4. *Let* $\psi : (N^n, h) \longrightarrow (P^p, k)$ *a biharmonic non-harmonic map. The map* $\widetilde{\phi} : \left(M^m \times_f N^n, G_{f^2}\right) \longrightarrow (P^p, k)$ *defined by* $\widetilde{\psi}(x, y) = \psi(y)$ *is biharmonic if and only if the function* f^{n-2} *is harmonic.*

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