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Article

Modular Forms and Weierstrass Mock Modular Forms

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Abstract: Alfes, Griffin, Ono, and Rolen have shown that the harmonic Maass forms arising from Weierstrass ζ-functions associated to modular elliptic curves "encode" the vanishing and nonvanishing for central values and derivatives of twisted Hasse-Weil L-functions for elliptic curves. Previously, Martin and Ono proved that there are exactly five weight 2 newforms with complex multiplication that are eta-quotients. In this paper, we construct a canonical harmonic Maass form for these five curves with complex multiplication. The holomorphic part of this harmonic Maass form arises from the Weierstrass ζ-function and is referred to as the Weierstrass mock modular form. We prove that the Weierstrass mock modular form for these five curves is itself an eta-quotient or a twist of one. Using this construction, we also obtain p-adic formulas for the corresponding weight 2 newform using Atkin's U-operator.

Keywords: modular forms; weierstrass mock modular forms; eta-quotients

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1. Introduction

In a recent paper, Alfes, Griffin, Ono, and Rolen [1] obtain canonical weight 0 harmonic Maass forms that arise from Eisenstein's corrected Weierstrass zeta-function for elliptic curves over $\mathbf{Q}$. The holomorphic part of this harmonic Maass form is a mock modular form, referred to as the *Weierstrass mock modular form*. The harmonic Maass form for a specific elliptic curve E encodes the central L-values and L-derivatives that occur in the Birch and Swinnerton-Dyer Conjecture for elliptic curves in a family of quadratic twists [1,2]. Guerzhoy [3] has studied the construction of harmonic Maass forms using the Weierstrass ζ function in his work on the Kaneko-Zagier hypergeometric differential equation.

In [4], Martin and Ono prove that there are exactly twelve weight 2 newforms $F_E(\tau)$ that are products and quotients of the Dedekind eta-function

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q := e^{2\pi i \tau}$. By the modularity of elliptic curves, there is an isogeny class of $E/\mathbf{Q}$ for each of these eta-quotients. Martin and Ono present a table of elliptic curves E corresponding to these cusp forms and describe the Grössencharacters for the five curves with complex multiplication.

In this paper, we prove that the derivative of the Weierstrass mock modular form of each such elliptic curve E is a weight 2 weakly holomorphic modular form which also turns out to be an eta-quotient or a twist of one. We also obtain p-adic formulas for the corresponding weight 2 newforms using Atkin's U-operator.

Let E be one of the five elliptic curves with complex multiplication whose associated newform, $F_E(\tau)$, is an eta-quotient. Let N_E denote the conductor of this curve and label its coefficients a_i such that they belong to the Weierstrass model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

The following Table 1 contains a strong Weil curve for each of the weight 2 newforms with complex multiplication that are eta-quotients.

Table 1. Table of five elliptic curves.

N_E	$F_E(\tau)$	a_1	a_2	a_3	a_4	a_6
27	$\eta^2(3\tau)\eta^2(9\tau)$	0	0	1	0	-7
32	$\eta^2(4\tau)\eta^2(8\tau)$	0	0	0	4	0
36	$\eta^4(6\tau)$	0	0	0	0	1
64	$\frac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}$	0	0	0	-4	0
144	$\frac{\eta^{12}(12\tau)}{\eta^4(6\tau)\eta^4(24\tau)}$	0	0	0	0	-1

Let $\widehat{\mathfrak{Z}}^+_E(\tau)$ denote the Weierstrass mock modular form of E, and let $Z_{N_E}(\tau) := q \cdot \frac{d}{dq}\widehat{\mathfrak{Z}}^+_E(\tau)$ denote the derivative of the Weierstrass mock modular form (see Section 2.1 for details). Let $\chi_D := \left(\frac{D}{\cdot}\right)$ denote the usual Kronecker symbol so that $\left(\sum a(n)q^n\right)|_{\chi_D} = \sum \chi_D(n)a(n)q^n$.

Theorem 1. *The derivative of the Weierstrass mock modular form for each of the five elliptic curves E given in Table 1 is an eta-quotient or a twist of one, as described below.*

$$\begin{aligned}
Z_{27}(\tau) &= -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)\\
Z_{32}(\tau) &= -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)\\
Z_{36}(\tau) &= -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)\\
Z_{64}(\tau) &= -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)|_{\chi_8}\\
Z_{144}(\tau) &= -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)|_{\chi_{12}}
\end{aligned}$$

We also obtain p-adic formulas for the corresponding weight 2 newform using Atkin's U-operator,

$$\sum a(n)q^n|U(m) := \sum a(mn)q^n$$

By taking a p-adic limit, we can retrieve the coefficients of the original cusp form, $F_E(\tau)$, of the elliptic curve. Let $Z_{N_E}(\tau) = \sum_{n=-1}^{\infty} d(n)q^n$ be the derivative of the Weierstrass mock modular form as before.

Theorem 2. *For each of the five elliptic curves listed in Table 1, if p is inert in the field of complex multiplication, then as a p-adic limit we have*

$$F_E(\tau) = \lim_{\omega\to\infty} \frac{Z_{N_E}(\tau)|U(p^{2\omega+1})}{d(p^{2\omega+1})}$$

Example 1. *Here we illustrate Theorem 2 for the prime $p = 5$ and the newform with conductor 27. Let*

$$Z_{E,\omega}(p,\tau) = \frac{Z_{N_E}(\tau)|U(p^{2\omega+1})}{d(p^{2\omega+1})}$$

If $p = 5$, then we have

$$\begin{aligned} Z_{E,0}(5,\tau) &= q + 8q^4 + 49q^7 + 75q^{10} + \dots & \equiv F_E(z) \pmod 5 \\ Z_{E,1}(5,\tau) &= q + \frac{195040}{480}q^4 + \frac{6821395}{480}q^7 - \frac{114840625}{480}q^{10} + \dots & \equiv F_E(z) \pmod{5^2} \end{aligned}$$

We prove this theorem using techniques outlined in [5]. Similar results can be found in both [1,6]. In [6], El-Guindy and Ono study a modular function that arises from Gauss's hypergeometric function that gives a modular parameterization of period integrals of E_{32}, the elliptic curve with conductor 32. In [1], Theorem 1.3 builds p-adic formulas for the corresponding weight 2 newforms using the action of the Hecke algebra on the Weierstrass mock modular forms.

2. Background

2.1. Weierstrass Mock Modular Forms

Let E be an elliptic curve over $\mathbf{Q}$ such that $E \simeq \mathbf{C}/\Lambda_E$, where Λ_E is a two-dimensional lattice in $\mathbf{C}$. By the modularity of elliptic curves over $\mathbf{Q}$, we have the modular parameterization

$$\phi_E : X_0(N_E) \to \mathbf{C}/\Lambda_E \simeq E$$

where N_E is the conductor of E. Suppose E is a strong Weil curve and let

$$F_E(z) = \sum_{n=1}^{\infty} a_E(n)q^n \in S_2(\Gamma_0(N_E))$$

be the associated newform where $q = e^{2\pi i z}$.

Let $\wp(\Lambda_E;\mathfrak{z})$ be the usual Weierstrass $\wp$-function given by

$$\wp(\Lambda_E;\mathfrak{z}) := \frac{1}{\mathfrak{z}^2} + \sum_{\omega\in\Lambda_E\setminus\{0\}} \left(\frac{1}{(\mathfrak{z}-\omega)^2} - \frac{1}{\omega^2}\right)$$

All elliptic functions with respect to Λ_E are naturally generated from the Weierstrass $\wp$-functions. While there can never be a single-order elliptic function, Eisenstein constructed a simple function with a single pole that can be modified, at the expense of holomorphicity, to become lattice-invariant (see [7]). Eisenstein began with the Weierstrass zeta-function denoted $\zeta(\Lambda_E,\mathfrak{z})$ for Λ_E, the function whose derivative is $-\wp(\Lambda_E;\mathfrak{z})$. The Weierstrass zeta-function is defined for $\mathfrak{z} \notin \Lambda_E$ by

$$\zeta(\Lambda_E;\mathfrak{z}) := \frac{1}{\mathfrak{z}} + \sum_{\omega\in\Lambda_E\setminus\{0\}} \left(\frac{1}{\mathfrak{z}-\omega} + \frac{1}{\omega} + \frac{\mathfrak{z}}{\omega^2}\right) = \frac{1}{\mathfrak{z}} - \sum_{n\geq 1} G_{2n+2}(\Lambda_E)\mathfrak{z}^{2n+1}$$

Eisenstein's corrected ζ-function is given by

$$\mathfrak{Z}_E(\mathfrak{z}) := \zeta(\Lambda_E;\mathfrak{z}) - S(\Lambda_E)\mathfrak{z} - \frac{\deg(\text{Œ}_E)}{4\pi||F_E||^2}\bar{\mathfrak{z}},$$

where $S(\Lambda_E) := \lim_{s\to 0^+} \sum_{0\neq\omega\in\Lambda_E} \frac{1}{\omega^2|\omega|^{2s}}$, $\deg(\text{Œ}_E)$ is the degree of the modular parameterization and $||F_E||$ is the Petersson norm of F_E. In [8], Rolen provides a new, direct proof of the lattice-invariance of

$\mathfrak{Z}_E(\mathfrak{z})$ using the standard theory of differential operators for Jacobi forms.

The canonical harmonic Maass form arises from the corrected Weierstrass zeta-function. Define $\mathfrak{Z}_E^+(\mathfrak{z}) := \zeta(\Lambda_E; \mathfrak{z}) - S(\Lambda_E)\mathfrak{z}$. Let $\mathcal{E}_E(z)$ be the Eichler integral of F_E defined by

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} F_E(\tau) d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

The nonholomorphic function $\widehat{\mathfrak{Z}}_E(z)$ is given by

$$\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z) = \mathfrak{Z}_E(\mathcal{E}(z))$$

Alfes, Griffin, Ono, and Rolen proved the following.

Theorem 3 (Theorem 1.1 of [1])**.** *Assume the notation and hypotheses above. Then the following are true:*

1. *The poles of $\widehat{\mathfrak{Z}}_E^+(z)$ are precisely those points z for which $\mathcal{E}_E(z) \in \Lambda_E$.*
2. *If $\widehat{\mathfrak{Z}}_E^+(z)$ has poles in $\mathcal{H}$, then there is a canonical modular function $M_E(z)$ with algebraic coefficients on $\Gamma_0(N_E)$ for which $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$ is holomorphic on $\mathcal{H}$.*
3. *We have that $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$ is a weight 0 harmonic Maass form on $\Gamma_0(N_E)$.*

In particular, the holomorphic part of $\widehat{\mathfrak{Z}}_E(z)$ is $\widehat{\mathfrak{Z}}_E^+(z) = \mathfrak{Z}_E^+(\mathcal{E}_E(z))$, where $\widehat{\mathfrak{Z}}_E^+(z)$ is a weight 0 mock modular form known as the *Weierstrass mock modular form* for E.

We are interested in computing the Weierstrass mock modular form for the elliptic curves with conductors 27, 32, 36, 64, and 144 given by Table 1. The value of $S(\Lambda_E)$ is 0 for each of these curves and so the Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ is $\zeta(\Lambda_E; \mathcal{E}_E(z))$. Bruinier, Rhoades, and Ono [2], and Candelori [9] proved that if a normalized newform has complex multiplication then the holomorphic part of a certain harmonic Maass form has algebraic coefficients; in particular, the coefficients of $\widehat{\mathfrak{Z}}_E^+(z)$ are algebraic.

Relabeling z as τ so that $q = e^{2\pi i\tau}$, we can now define the derivative of the Weierstrass mock modular form as $Z_{N_E}(\tau) = q \cdot \frac{d}{dq}\widehat{\mathfrak{Z}}_E^+(\tau)$. The list below (Table 2) gives the first few terms of the q-expansion for the derivative of the Weierstrass mock modular form for each of the five curves.

Table 2. Table of Z_{N_E}.

N_E	q-Expansion for $Z_{N_E}(\tau)$
27	$-q^{-1} + q^2 + q^5 + 6q^8 - 6q^{11} - 7q^{14} - 9q^{17} + 8q^{20} + 15q^{23} - 13q^{26} + 19q^{29} + \ldots$
32	$-q^{-1} + 2q^3 + q^7 - 2q^{11} + 5q^{15} - 14q^{19} - 4q^{23} + 12q^{27} - 5q^{31} + \ldots$
36	$-q^{-1} + 3q^5 + q^{11} - 5q^{17} - 8q^{23} - q^{29} + 28q^{35} + \ldots$
64	$-q^{-1} - 2q^3 + q^7 + 2q^{11} + 5q^{15} + 14q^{19} - 4q^{23} - 12q^{27} - 5q^{31} + \ldots$
144	$-q^{-1} - 3q^5 + q^{11} + 5q^{17} - 8q^{23} + q^{29} + 28q^{35} + \ldots$

2.2. Eta-Quotient

After the proof of Fermat's Last Theorem and the subsequent expository articles describing the modularity theorem, Martin and Ono wrote an article compiling the complete list of all weight 2 newforms that are eta-quotients. Five of these curves have complex multiplication, and using q-series infinite product identities, they described the Grössencharacters for these curves. The curves with conductors 27, 36, and 144 have complex multiplication by $\mathbf{Q}(\sqrt{-3})$ and the curves with conductors 32 and 64 have complex multiplication by $\mathbf{Q}(i)$. In addition, Martin and Ono in [4] proved that the curves with $N_E = 36$ and $N_E = 144$ are quadratic twists of each other.

If the derivative of the Weierstrass mock modular form, $Z_{N_E}(\tau)$, is an eta-quotient, certain properties must hold. In [10], Ono described the following result of Gordon, Hughes, and Newman on eta-quotients.

Theorem 4 (Theorem 1.64 of [10]). *If $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbf{Z}$, with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$$

then $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

Here the character χ is defined for $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s = \prod_{\delta|N} \delta^{r_\delta}$.

In Section 3.6, we will prove that the derivative of the Weierstrass mock modular form $Z_{N_E}(\tau)$ is an eta-quotient or a twist of one. In order to help us identify plausible eta-quotients to describe $Z_{N_E}(\tau)$, note that any such eta-quotient $\prod_{\delta|N_E} \eta(\delta\tau)^{r_\delta}$ must satisfy the following:

$$\begin{aligned} &\sum_{\delta|N_E} r_\delta = 4 \\ &\sum_{\delta|N_E} \delta r_\delta = -24 \\ &\sum_{\delta|N_E} \frac{N_E}{\delta} r_\delta \equiv 0 \pmod{24} \\ &\prod_{\delta|N_E} \delta^{r_\delta} = a^2 \text{ for some integer } a \end{aligned} \tag{1}$$

This description follows from Theorem 4, together with the fact that $Z_{N_E}(\tau)$ has weight 2, level N_E and leading term q^{-1}.

3. Examples and Proof

3.1. $N_E = 27$

Consider the curve $E : y^2 + y = x^3 - 7$, which has conductor $N_E = 27$. The eta-quotient $\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$ satisfies the four properties described in Equation (1) for $N_E = 27$ and its initial terms match with those of $Z_{27}(\tau)$, as shown below:

$$\begin{aligned} \eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau) &= q^{-1} - q^2 - q^5 - 6q^8 + 6q^{11} + 7q^{14} + 9q^{17} - 8q^{20} - 15q^{23} + O(q^{26}) \\ Z_{27}(\tau) &= -q^{-1} + q^2 + q^5 + 6q^8 - 6q^{11} - 7q^{14} - 9q^{17} + 8q^{20} + 15q^{23} - O(q^{26}) \end{aligned}$$

Thus we define $\eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$ and guess that $Z_{27} = \eta_{27}$. This will be proven in Section 3.6 in order to establish Theorem 1.

3.2. $N_E = 32$

Consider the curve, $E : y^2 = x^3 + 4x$. The eta-quotient $\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$ satisfies the four properties described in Equation (1) for $N_E = 32$ and its initial terms match with those of $Z_{32}(\tau)$, as shown below:

$$\begin{aligned}\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) &= q^{-1} - 2q^3 - q^7 + 2q^{11} - 5q^{15} + 14q^{19} + 4q^{23} - 12q^{27} + 5q^{31} - O(q^{35}) \\ Z_{32}(\tau) &= -q^{-1} + 2q^3 + q^7 - 2q^{11} + 5q^{15} - 14q^{19} - 4q^{23} + 12q^{27} - 5q^{31} + O(q^{35})\end{aligned}$$

Letting $\eta_{32} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$, we will later prove $Z_{32} = \eta_{32}$ in Section 3.6 to establish Theorem 1.

3.3. $N_E = 36$

Consider the curve with level 36, $E : y^2 = x^3 + 1$. The eta-quotient $\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$ satisfies the four properties described in Equation (1) for $N_E = 36$ and its initial terms match with those of $Z_{36}(\tau)$, as shown below:

$$\begin{aligned}\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) &= q^{-1} - 3q^5 - q^{11} + 5q^{17} + 8q^{23} + q^{29} - 28q^{35} \\ &\quad - 11q^{41} + 10q^{47} + O(q^{53}) Z_{36}(\tau) \\ &= -q^{-1} + 3q^5 + q^{11} - 5q^{17} - 8q^{23} - q^{29} + 28q^{35} \\ &\quad + 11q^{41} - 10q^{47} - O(q^{53})\end{aligned} \tag{2}$$

Letting $\eta_{36} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$, we will later prove $Z_{36} = \eta_{36}$.

3.4. $N_E = 64$

Consider the curve with level 64, $E : y^2 = x^3 - 4x$. The eta-quotient $\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$ satisfies the four properties described in Equation (1) for $N_E = 64$. Note $-\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) = \eta_{32}$. The initial terms of this eta-quotient match with those of $Z_{64}(\tau)$, as shown below:

$$\begin{aligned}\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) &= q^{-1} - 2q^3 - q^7 + 2q^{11} - 5q^{15} + 14q^{19} + 4q^{23} - 12q^{27} + 5q^{31} - O(q^{35}) \\ Z_{64}(\tau) &= -q^{-1} - 2q^3 + q^7 + 2q^{11} + 5q^{15} + 14q^{19} - 4q^{23} - 12q^{27} - 5q^{31} - O(q^{35})\end{aligned}$$

Letting $\eta_{64} = \eta_{32}|_{\chi_8}$, we will later prove $Z_{64} = \eta_{64}$.

3.5. $N_E = 144$

Consider the curve with level 144, $E : y^2 = x^3 - 1$. The eta-quotient $\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$ satisfies the four properties described in Equation (1) for $N_E = 144$. Note $-\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) = \eta_{36}$. The initial terms of this eta-quotient match with those of $Z_{144}(\tau)$, as shown below:

$$\begin{aligned}\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) &= q^{-1} - 3q^5 - q^{11} + 5q^{17} + 8q^{23} + q^{29} - 28q^{35} - 11q^{41} + 10q^{47} + O(q^{53}) \\ Z_{144}(\tau) &= -q^{-1} - 3q^5 + q^{11} + 5q^{17} - 8q^{23} + q^{29} + 28q^{35} - 11q^{41} - 10q^{47} + O(q^{53})\end{aligned}$$

Letting $\eta_{144} = \eta_{36}|_{\chi_{12}}$, we will later prove $Z_{144} = \eta_{144}$.

3.6. Proof of Theorems 1 and 2

Proof of Theorem 1. When the conductor of E is 27, 32, and 36, the modular parameterization of these 3 curves has degree 1 (as computed in Sage [11]) and each Weierstrass mock modular form has only a single pole at infinity. Let S_{N_E} denote Sturm's bound for the space of modular forms on $\Gamma_0(N_E)$ of weight 2, and let η_{N_E} denote the eta-quotient described in Section 3. For example, recall $\eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$. Consider the difference of the eta-quotients, η_{N_E}, and the derivatives of the Weierstrass mock modular form, $Z_{N_E}(\tau)$. Both q-expansions have a simple pole at infinity. The principal part of $Z_{N_E}(\tau)$ for $N_E = 27, 32, 36$ is constant at every cusp except infinity because the degree of modular parameterization for E_{27}, E_{32} and E_{36} is 1. Using the following formula, one can verify with a few Sage computations that the order of vanishing of η_{N_E} is nonnegative at each cusp c/d (except at infinity, where there is a simple pole) [11].

Theorem 5 (Theorem 1.65 of [10])**.** *Let c, d and N be positive integers with $d|N$ and $gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 1.64 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{gcd(d,\delta)^2 r_\delta}{gcd(d, \frac{N}{d}) d\delta}$$

Since the difference $Z_{N_E}(\tau) - \eta_{N_E}$ is holomorphic, as shown above, if $Z_{N_E}(\tau) - \eta_{N_E}$ is 0 for S_{N_E} coefficients, the identities claimed for $N_E = 27, 32, 36$ are correct. The following table (Table 3) gives Sturm's bound for the space of modular forms on $\Gamma_0(N_E)$ of weight 2.

Table 3. Table of S_{N_E}.

N_E	S_{N_E}
27	13
32	17
36	25

After checking the coefficients of the expansions up to the corresponding bound, we see $Z_{27}(\tau) = \eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$, $Z_{32}(\tau) = \eta_{32} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$, and $Z_{36}(\tau) = \eta_{36} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$, as claimed.

The modular parametrization for E_{64} has degree 2, and the modular parametrization for E_{144} has degree 4 (as computed in Sage [11]); therefore we cannot apply Sturm's bound to the difference of the associated Weierstrass mock modular forms and eta-quotients. Instead we prove Z_{64} is a twist of Z_{32} by χ_8, and Z_{144} is a twist of Z_{36} by χ_{12}. Consider first Z_{64}, Z_{32}, and χ_8, where χ_8 denotes the Kronecker symbol as before. We have already shown $(Z_{32} - \eta_{32})|_{\chi_8} = 0$. Therefore, $Z_{32}|_{\chi_8} - \eta_{32}|_{\chi_8} = 0$. Since $Z_{32}|_{\chi_8} - \eta_{32}|_{\chi_8}$ is a twist of a holomorphic difference, we can use Sturm's bound to check up to S_{32} coefficients and confirm $Z_{32}|_{\chi_8} = \eta_{32}|_{\chi_8} = \eta_{64}$. To prove $Z_{32}|_{\chi_8} = Z_{64}$, note the q-expansions are equal up to 17 coefficients and their difference is holomorphic (as the principal part of each is constant at every cusp except infinity as shown before). Therefore, $Z_{32}|_{\chi_8} = Z_{64}$ so $Z_{64} = \eta_{32}|_{\chi_8} = \eta_{64} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)|_{\chi_8}$. The proof for $Z_{36}|_{\chi_{12}} = Z_{144}$ is similar, giving us the equality $Z_{144} = \eta_{36}|_{\chi_{12}} = \eta_{144} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)|_{\chi_{12}}$. □

Proof of Theorem 2. Theorem 2 is a consequence of Theorem 6 of Guerzhoy, Kent, and Ono. Let $g(\tau) = \sum_{n=1}^{\infty} b(n)q^n \in S_2(\Gamma_0(N_E))$ denote the normalized newform and $\mathcal{E}_E(\tau)$ its Eichler integral. Recall, g has rational coefficients. Let $f = f^+ + f^-$ denote a weight-0 harmonic Maass form where f^+ is the holomorphic part. If $\xi = \xi_2 := 2iy^2 \frac{d}{d\bar{\tau}}$, then we say that g is a shadow of f^+ if $\xi(f) = g$. We say $f \in H_0(\Gamma_0(N_E))$ is *good* for $g(\tau)$ if the following hold;

1. The principal part of f at the cusp ∞ belongs to $\mathbf{Q}[q^{-1}]$.
2. The principal part of f at other cusps is constant.
3. $\xi(f) = \frac{g}{<g,g>}$ where $< \cdot, \cdot >$ denotes the usual Petersson inner product.

Let D denote the operator $D := \frac{1}{2\pi i}\frac{d}{d\tau}$ so that $D(f^+) = \sum_{n=1}^{\infty} d(n)q^n$ is the derivative of the holomorphic part of the harmonic Maass form, i.e. the mock modular form. Guerzhoy, Kent, and Ono relate the coefficients of g and f using the following theorem.

Theorem 6 (Theorem 1.2 (2) of [5]). *Suppose $g(\tau) \in S_2(\Gamma_0(N))$ has CM and g is good for f. If p is inert in the field of complex multiplication, then we have that*

$$g = \lim_{\omega \to \infty} \frac{D(f^+)|U(p^{2\omega+1})}{d(p^{2\omega+1})}.$$

Consider $F_E(\tau) \in S_2(\Gamma_0(N_E))$, the normalized newform equal to an eta-quotient for one of the elliptic curves E with complex multiplication listed in Table 1, $\widehat{\mathfrak{Z}}_E(z)$ the canonical harmonic Maass form and $Z_{N_E}(\tau)$ the derivative of the Weierstrass mock modular form. The harmonic Maass form $\widehat{\mathfrak{Z}}_E(z)$ is good for F_E as follows:

1. The principal part of $\widehat{\mathfrak{Z}}_E(z)$ at ∞ belongs to $\mathbf{Q}[q^{-1}]$.
2. There are no poles at other cusps for $N_E = 27, 32, 36$. Since Z_{64} is a twist of Z_{32} and Z_{144} is a twist of Z_{36}, the principal parts of $\widehat{\mathfrak{Z}}_E(z)$ for E_{64} and E_{144} must have constant principal parts at other cusps.
3. By definition of $\widehat{\mathfrak{Z}}_E(z)$, we have $\xi(f) = \frac{g}{<g,g>}$.

Therefore, $\widehat{\mathfrak{Z}}_E(z)$ is good for F_E and we can apply Theorem 6 to show the p-adic limit holds for the derivative of the Weierstrass mock modular form. □

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