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Existence of Mild Solutions for Impulsive Fractional Integro-Differential Inclusions with State-Dependent Delay

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Abstract: In this manuscript, we implement Bohnenblust–Karlin's fixed point theorem to demonstrate the existence of mild solutions for a class of impulsive fractional integro-differential inclusions (IFIDI) with state-dependent delay (SDD) in Banach spaces. An example is provided to illustrate the obtained abstract results.

Keywords: Fractional order differential equations; impulsive conditions; state-dependent delay (SDD); multivalued map; fixed point theorem; Banach space; semigroup theory

JEL Classification: 26A33; 34A08; 35R12; 34A60; 34G20; 34K05; 45J05

1. Introduction

The notion of fractional derivatives, as is long familiar, has its commencement in an inquiry postured amid a correspondence in the middle of Leibnitz and L'hospital. The five millennium extremely ancient inquiry has turned into a significant zone of exploration. As of late, it has been demonstrated that the differential designs including derivatives of fractional order emerge in numerous technological innovations and scientific disciplines as the statistical modeling of frameworks and procedures in numerous fields—case in point: physical science, chemical industry, aerodynamics, electrodynamics of complex medium, etc. For information, such as some uses and latest outcomes, think about the treatise of Abbas et al. [1], Baleanu et al. [2], Podlubny [3], Diethelm [4], Kilbas et al. [5], and Tarasov [6], and the papers [7–21], and the references cited therein.

Fractional differential inclusions (FDI) are speculation of fractional differential equations (FDE). Along these lines, all models viewed regarding FDE that may be existence of solutions, continuous dependence and parameters are also available in the concept of FDI—considering the fact that FDI occur in the mathematical modelling of specific models in financial aspects, optimal control, etc. and are usually investigated by numerous writers (see, for instance, [22–24] and the references therein). Fractional equation with delay properties arise in several fields such as biological and physical with state-dependent delay (SDD) or non-constant delay. Nowadays, existence results of mild solutions for such problems became very attractive and several researchers are working on it. Recently, several papers have been written on the fractional order problems with SDD [23,25–36] and the sources therein.

On the flip side, the concept of impulsive differential framework has been a target consideration due to the fact of its extensive uses in physics, biology, engineering, medicinal fields, industry and technology. The purpose behind this pertinence emerges from the way that impulsive differential frameworks are a proper model for portraying procedures that, at specific moments, change their state quickly and which cannot depict utilization of the traditional differential models. For additional purposes of enthusiasm on this concept and on its uses (see, for example, the treatise by Lakshmikantham et al. [37], Ivanka M. Stamova [38], Graef et al. [39], Bainov et al. [40], Benchohra et al. [41], the papers [22,42–50], and the references cited therein).

The existence, controllability, and other qualitative and quantitative attributes of differential and FDEs are the most advancing area of interest (for instance, see [20,26,31–33,35,51–54]). In particular, in [20,53,54], the authors investigate the different types of impulsive fractional differential systems in Banach spaces under different fixed point theorems with weak conditions. In particular, in [20], the authors define more suitable PC-mild solutions for the impulsive FDI with non-local conditions. As of late, Carvalho dos Santos et al. [32] have analyzed the existence of solutions for FIDE with SDD in Banach spaces. Kavitha et al. [35] acknowledged the existence of mild solutions for FIDE with SDD by using an appropriate fixed point theorem. In [31,33], the authors offer adequate circumstances for the existence of solutions of FDE with SDD. Lately, Benchohra et al. [26] researched the existence of mild solutions on a compact interval for FIDE with SDD in Banach spaces. However, existence results for IFIDI with SDD in $\mathscr{B}_h$ phase space adages have not yet been completely examined.

To think seriously about fractional frameworks in the infinite dimensional space, the essential imperative move is to focus on a latest technique of the mild solution. As of late, in Wang et al. [20], a proper thought of mild solutions was exhibited. Furthermore, they profoundly examined the current PC-mild solution characterized by a few scientists.

Motivated by the effort of the aforementioned papers [20,22,26,30], the primary inspiration driving this manuscript is to research the existence of mild solutions for an IFIDI with SDD of the model:

$$ {}^{C}D_t^{\alpha}x(t) - \mathscr{A}x(t) \in \mathscr{F}\left(t, x_{\varrho(t,x_t)}\right) + \int_0^t e(t,s,x_s)\,ds, \quad \text{a.e. on} \quad \mathscr{J} - \{t_1, t_2, \cdots, t_m\}, \tag{1} $$

$$ \Delta x(t_k) = \mathcal{I}_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \tag{2} $$

$$ x(t) = \varsigma(t), \quad \varsigma(t) \in \mathscr{B}_h, \tag{3} $$

where $\mathscr{J} = [0, b]$ with $b > 0$ is settled, ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of the order $\alpha \in (0, 1)$ with the lower limit zero, $\mathscr{A}$ is a fractional sectorial operator similar to [55] described on a Banach space $\mathbb{X}$, having its norm recognized as $\|\cdot\|_{\mathbb{X}}$, $\mathscr{F} : \mathscr{J} \times \mathscr{B}_h \to \mathcal{P}(\mathbb{X})$ is a multivalued map, where ($\mathcal{P}(\mathbb{X})$ is the family of all non-empty subsets of $\mathbb{X}$), $e : \mathscr{D} \times \mathscr{B}_h \to \mathbb{X}, \varrho : \mathscr{J} \times \mathscr{B}_h \to (-\infty, b]$ are apposite functions, and $\mathscr{B}_h$ is a theoretical phase space adages outlined in Preliminaries. Here, $\mathscr{D} = \{(t, s) \in \mathscr{J} \times \mathscr{J} : 0 \leq s \leq t \leq b\}$. Here, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b, \mathcal{I}_k : \mathbb{X} \to \mathbb{X} (k = 1, 2, \cdots, m)$ are impulsive functions which portray the jump of the solutions at impulse points t_k, and $x(t_k^+) = \lim_{h \to 0} x(t_k + h), x(t_k^-) = \lim_{h \to 0} x(t_k - h)$ are the right and left limits of x at the points t_k separately.

For almost any continuous function x characterized on $(-\infty, b]$ and any $t \geq 0$, we designate by x_t the part of $\mathscr{B}_h$ characterized by $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$. Now, $x_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta \in (-\infty, 0]$, likely the current time t.

This manuscript is composed as the following. In Section 2, we show a few preliminaries and lemmas that are to be utilized subsequently to demonstrate our primary outcomes. In Section 3, the existence of mild solutions for the model (1)–(3) is discussed under a suitable fixed point theorem. Section 4 is saved for a case to delineate the conceptual results.

To the best of our insight, there is no work giving an account of the existence results for an IFIDI with SDD, which is communicated in the structure (1)–(3). To fill this gap, in this manuscript, we contemplate this fascinating model.

2. Preliminaries

In this part, we display a few documentations, definitions and preparatory facts from functional analysis, solution operator and fractional calculus theory that will be utilized throughout this manuscript.

Let $L(\mathbb{X})$ symbolize the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$, having its norm recognized as $\|\cdot\|_{L(\mathbb{X})}$.

Let $C(\mathscr{J},\mathbb{X})$ symbolize the space of all continuous functions from $\mathscr{J}$ into $\mathbb{X}$, having its norm recognized as $\|\cdot\|_{C(\mathscr{J},\mathbb{X})}$.

Let $L^1(\mathscr{J},\mathbb{X})$ be the space of $\mathbb{X}$-valued Bochner integrable functions on $\mathscr{J}$ with the norm:

$$\|y\|_{L^1} = \int_0^b \|y(t)\|dt.$$

It needs to be outlined that, once the delay is infinite, we need to talk about the theoretical phase space $\mathscr{B}_h$ in a beneficial way. In this manuscript, we deliberate phase spaces $\mathscr{B}_h, \mathscr{B}'_h$ that are the same as described in [30]. Therefore, we bypass the details.

If $x : (-\infty, b] \to \mathbb{X}, b > 0$ is continuous on $\mathscr{J}$ and $x_0 \in \mathscr{B}_h$, then, for every $t \in \mathscr{J}$, the accompanying conditions hold:

- (P_1) x_t is in $\mathscr{B}_h$;
- (P_2) $\|x(t)\|_{\mathbb{X}} \le H\|x_t\|_{\mathscr{B}_h}$;
- (P_3) $\|x_t\|_{\mathscr{B}_h} \le \mathscr{D}_1(t)\sup\{\|x(s)\| : 0 \le s \le t\} + \mathscr{D}_2(t)\|x_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0,+\infty) \to [0,+\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0,+\infty) \to [0,+\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(\cdot)$.
- (P_4) The function $t \to \varsigma_t$ is well described and continuous from the set:

$$\mathcal{R}(\varrho^-) = \{\varrho(s,\psi) : (s,\psi) \in [0,b] \times \mathscr{B}_h\}$$

into $\mathscr{B}_h$ and there is a continuous and bounded function $J^{\varsigma} : \mathcal{R}(\varrho^-) \to (0,\infty)$ to ensure that $\|\varsigma_t\|_{\mathscr{B}_h} \le J^{\varsigma}(t)\|\varsigma\|_{\mathscr{B}_h}$ for every $t \in \mathcal{R}(\varrho^-)$.

Lemma 1. *Let $x : (-\infty, b] \to \mathbb{X}$ be a function in a way that $x_0 = \varsigma, x|_{\mathscr{J}_k} \in C(\mathscr{J}_k, \mathbb{X})$ ([56] Lemma 2.1), and, if (P_4) hold, then:*

$$\|x_s\|_{\mathscr{B}_h} \le (\mathscr{D}_2^* + J^{\varsigma})\|\varsigma\|_{\mathscr{B}_h} + \mathscr{D}_1^* \sup\{\|x(\theta)\|_{\mathbb{X}} : \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{J},$$

where $J^{\varsigma} = \sup\limits_{t\in\mathcal{R}(\varrho^-)} J^{\varsigma}(t), \quad \mathscr{D}_1^* = \sup\limits_{s\in[0,b]} \mathscr{D}_1(s), \quad \mathscr{D}_2^* = \sup\limits_{s\in[0,b]} \mathscr{D}_2(s).$

Now, we show some known results from multivalued analysis that we will apply in the spin-off. Denote:

$$\mathcal{P}_{cl}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ closed}\}, \quad \mathcal{P}_b(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ bounded }\},$$
$$\mathcal{P}_{cp}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ compact}\},$$
$$\mathcal{P}_{cp,c}(\mathbb{X}) = \{Y \in \mathcal{P}(\mathbb{X}) : Y \text{ compact and convex}\}.$$

Remark 1. *In multi-valued analysis, the definitions of convex, upper semi-continuous, completely continuous, closed graph and measurable are classical. Hence, we omit it. For extra points of interest on this, we suggest the reader to [13,22].*

Definition 1. *The multivalued map $\mathscr{F} : \mathscr{J} \times \mathscr{B}_h \to \mathcal{P}(\mathbb{X})$ is said to be Carathéodory if:*

(i) $t \mapsto \mathscr{F}(t,u)$ *is measurable for each* $u \in \mathscr{B}_h$;
(ii) $u \mapsto \mathscr{F}(t,u)$ *is upper semicontinuous for almost all* $t \in \mathscr{J}$.

Let $S_{\mathscr{F},x}$ be a set characterized by:

$$S_{\mathscr{F},x} = \{v \in L^1(\mathscr{J},\mathbb{X}) : v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)}) \text{ a.e. } t \in \mathscr{J}\}.$$

Presently, we speak about the subsequent lemmas which are essential to set up our primary outcome [57,58].

Lemma 2. *Let $\mathbb{X}$ be a Banach space. Let $\mathscr{F} : \mathscr{J} \times \mathscr{B}_h \to \mathcal{P}_{cp,c}(\mathbb{X})$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(\mathscr{J}, \mathbb{X})$ to $C(\mathscr{J}, \mathbb{X})$. Then, the operator:*

$$\begin{aligned} \Psi \circ S_{\mathscr{F}} &: C(\mathscr{J}, \mathbb{X}) \to \mathcal{P}_{cp,c}(C(\mathscr{J}, \mathbb{X})), \\ x &\mapsto (\Psi \circ S_{\mathscr{F}})(x) := \Psi(S_{\mathscr{F},x}), \end{aligned}$$

is a closed graph operator in $C(\mathscr{J}, \mathbb{X}) \times C(\mathscr{J}, \mathbb{X})$.

Lemma 3 (Bohnenblust–Karlin's fixed point theorem)**.** *Let $\mathbb{X}$ be a Banach space and $D \in \mathcal{P}_{cl,c}(\mathbb{X})$. Suppose that the operator $G : D \to \mathcal{P}_{cl,c}(D)$ is upper semicontinuous and the set $G(D)$ is relatively compact in $\mathbb{X}$. Then, G has a fixed point in D.*

For surplus points of benefit on multivalued maps, think about the monographs of Graef et al. [39] and Górniewicz et al. [59].

Currently, we offer some fundamental definitions and results of the fractional calculus [3,5] concept that are used further as an aspect of this manuscript.

Definition 2. *The fractional integral of order γ with the lower limit zero for a function f is determined as:*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \quad \gamma > 0,$$

the right part offered is point-wise described on $[0, +\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 3. *The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f \in L^1(\mathscr{J}, \mathbb{X})$ is characterized as:*

$$D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds, \quad t > 0, \quad n-1 < \gamma < n.$$

Definition 4. *The Caputo derivative of order γ for a function $f \in L^1(\mathscr{J}, \mathbb{X})$ could be consisting as:*

$${}^{C}D_t^{\gamma} f(t) = D_t^{\gamma}(f(t) - f(0)), \quad t > 0, \quad 0 < \gamma < 1.$$

Remark 2.

(i) *Generally, the definition of solution operator and its outcomes are too standard. Hence, we will not discuss it. For extra points of interest on this, we suggest the reader to [18,20,55].*

(ii) *To be able to determine a mild solution of the model (1)–(3), we require providing the mild solution of the subsequent Cauchy problem:*

$$\begin{cases} {}^{C}D_t^{\alpha} x(t) = \mathscr{A} x(t) + f(t), & t \in \mathscr{J}, \\ x(0) = x_0 \in \mathbb{X}. \end{cases}$$

The mild solution [18,55] of the above Cauchy problem can be described by:

$$x(t) = S_\alpha(t) x_0 + \int_0^t T_\alpha(t-s) f(s) ds,$$

where:

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda, \quad T_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda$$

for a suitable path Γ and $f : \mathscr{J} \to \mathbb{X}$ is continuous.

Lemma 4. *If $\mathscr{A} \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$, then:*

$$\|S_\alpha(t)\|_{L(\mathbb{X})} \leq Me^{\omega t} \quad \text{and} \quad \|T_\alpha(t)\|_{L(\mathbb{X})} \leq Ce^{\omega t}(1 + t^{\alpha-1})$$

for every $t > 0$, $\omega > \omega_0$. Thus, using:

$$\widetilde{M}_S = \sup_{0 \leq t \leq b} \|S_\alpha(t)\|_{L(\mathbb{X})}, \quad \widetilde{M}_T = \sup_{0 \leq t \leq b} Ce^{\omega t}(1 + t^{1-\alpha}),$$

we get:

$$\|S_\alpha(t)\|_{L(\mathbb{X})} \leq \widetilde{M}_S, \quad \|T_\alpha(t)\|_{L(\mathbb{X})} \leq t^{\alpha-1}\widetilde{M}_T.$$

In accordance with the above discussion, we determine the mild solution of the model (1)–(3).

Definition 5. *A function $x : (-\infty, b] \to \mathbb{X}$ is known as a mild solution of the model (1)–(3) if the accompanying retains: $x_0 = \varsigma \in \mathscr{B}_h$ on $(-\infty, 0]; \Delta x|_{t=t_k} = \mathcal{I}_k(x(t_k^-))$, $k = 1, 2, \ldots, m$, the constraint of $x(\cdot)$ to the interval $\mathscr{J}_k$, $k = 0, 1, 2, \ldots, m$, is continuous and there exists $v(\cdot) \in L^1(\mathscr{J}_k, \mathbb{X})$, such that $v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)})$ a.e. $t \in \mathscr{J}$, and x fulfills the subsequent integral equation:*

$$x(t) = \begin{cases} \varsigma(t), \quad t \in (-\infty, 0], \\ S_\alpha(t)\varsigma(0) + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s) \int_0^s e\big(s, \tau, x_\tau\big)d\tau ds, & t \in [0, t_1], \\ S_\alpha(t)\varsigma(0) + S_\alpha(t - t_1)\mathcal{I}_1(x(t_1^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s) \int_0^s e\big(s, \tau, x_\tau\big)d\tau ds, & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)\varsigma(0) + \sum_{k=1}^{m} S_\alpha(t - t_k)\mathcal{I}_k(x(t_k^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s) \int_0^s e\big(s, \tau, x_\tau\big)d\tau ds, & t \in (t_m, b]. \end{cases} \tag{4}$$

Now, we list the subsequent hypotheses:

Hypothesis 1. *The semigroup $T_\alpha(t)$ is compact for $t > 0$.*

Hypothesis 2. *The multivalued map $\mathscr{F} : \mathscr{J} \times \mathscr{B}_h \to \mathbb{X}$ is Carathéodory, with compact convex values.*

Hypothesis 3. *There exists a function $p \in L^1(\mathscr{J}, \mathbb{R}^+)$ and a continuous non-decreasing function $\Omega_1 : \mathbb{R}^+ \to (0, \infty)$ such that:*

$$\|\mathscr{F}(t, u)\|_{\mathbb{X}} \leq p(t)\Omega_1(\|u\|_{\mathscr{B}_h}), \quad (t, u) \in \mathscr{J} \times \mathscr{B}_h.$$

Hypothesis 4. *For every $(t, s) \in \mathscr{D}$, the function $e(t, s, \cdot) : \mathscr{B}_h \to \mathbb{X}$ is continuous and for every $u \in \mathscr{B}_h$, the function $e(\cdot, \cdot, u) : \mathscr{D} \to \mathbb{X}$ is strongly measurable. We can find an integrable function $m_1 : \mathscr{J} \to [0, \infty)$ and a constant $a > 0$ to ensure that:*

$$\| \int_0^t e(t, s, u)ds\|_{\mathbb{X}} \leq am_1(t)\Omega_2(\|u\|_{\mathscr{B}_h}),$$

where $\Omega_2 : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function.

Hypothesis 5. *$\mathcal{I}_k \in C(\mathbb{X},\mathbb{X})$ and we can find $L_k \in C[\mathscr{J},\mathbb{R}^+]$ such that:*

$$\|\mathcal{I}_k(x)\|_{\mathbb{X}} \le L_k(t)\|x\|_{\mathbb{X}}, \quad x \in \mathbb{X}, \quad t \in \mathscr{J}.$$

3. Existence Results

In this part, we show and demonstrate the existence results for the model (1)–(3).

Theorem 1. *Assume that the hypotheses (1)–(5) hold. Then, model (1)–(3) has a mild solution on $(-\infty, b]$.*

Proof. We will transmute the structure (1)–(3) into a fixed point problem. Recognize the multivalued operator $\Upsilon : \mathscr{B}_h \to \mathcal{P}(\mathscr{B}_h)$ specified by $\Upsilon(h) = \{h \in \mathscr{B}_h\}$ with:

$$h(t) = \begin{cases} \varsigma(t), \quad t \in (-\infty,0], \\ S_\alpha(t)\varsigma(0) + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,x_\tau\big)d\tau ds, \qquad t \in [0,t_1], \\ S_\alpha(t)\varsigma(0) + S_\alpha(t-t_1)\mathcal{I}_1(x(t_1^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,x_\tau\big)d\tau ds, \qquad t \in (t_1,t_2], \\ \vdots \\ S_\alpha(t)\varsigma(0) + \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,x_\tau\big)d\tau ds, \quad v \in S_{F,x}, \qquad t \in (t_m,b]. \end{cases}$$

It is evident that the fixed points of the operator Υ are mild solutions of the model (1)–(3). We express the function $y(\cdot) : (-\infty,b] \to \mathbb{X}$ as:

$$y(t) = \begin{cases} \varsigma(t), \quad t \in (-\infty,0], \\ S_\alpha(t)\varsigma(0), \quad t \in \mathscr{J}; \end{cases}$$

then, $y_0 = \varsigma$. For every function $z \in C(\mathscr{J},\mathbb{R})$ with $z(0) = 0$, we allocate that $\bar{z}$ is characterized by:

$$\bar{z}(t) = \begin{cases} 0, \qquad t \in (-\infty,0], \\ z(t), \quad t \in \mathscr{J}. \end{cases}$$

If $x(\cdot)$ fulfilled Equation (4), we are able to decompose it as $x(\cdot)$ as $x(t) = y(t) + \bar{z}(t)$ for $t \in \mathscr{J}$, which suggests that $x_t = y_t + \bar{z}_t$ for $t \in \mathscr{J}$, and the function $z(\cdot)$ meets:

$$z(t) = \begin{cases} \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, \qquad t \in [0,t_1], \\ S_\alpha(t-t_1)\mathcal{I}_1(y(t_1^-) + \bar{z}(t_1^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, \qquad t \in (t_1,t_2], \end{cases}$$

$$z(t) = \begin{cases} \vdots \\ \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \bar{z}(t_k^-)) \\ \qquad + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, \qquad t \in (t_m,b], \end{cases}$$

where $v(s) \in S_{F,y+\bar{z}}$.

Let $\mathscr{B}''_h = \{z \in \mathscr{B}'_h \colon z_0 = 0\}$. Let $\|\cdot\|_{\mathscr{B}''_h}$ be the seminorm in $\mathscr{B}''_h$ described by:

$$\|z\|_{\mathscr{B}''_h} = \sup_{t\in\mathscr{J}} \|z(t)\|_{\mathbb{X}} + \|z_0\|_{\mathscr{B}_h} = \sup_{t\in\mathscr{J}} \|z(t)\|_{\mathbb{X}}, \quad z \in \mathscr{B}''_h.$$

As a result, $(\mathscr{B}''_h, \|\cdot\|_{\mathscr{B}''_h})$ is a Banach space. We delimit the operator $\overline{\Upsilon} : \mathscr{B}''_h \to \mathscr{B}''_h$ by $\overline{\Upsilon}(z) = \{h \in \mathscr{B}''_h\}$ with:

$$h(t) = \begin{cases} \displaystyle\int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\overline{z}_\tau + y_\tau\big)d\tau ds, & t \in [0,t_1], \\ S_\alpha(t-t_1)\mathcal{I}_1(y(t_1^-) + \overline{z}(t_1^-)) & \\ \quad \displaystyle + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\overline{z}_\tau + y_\tau\big)d\tau ds, & t \in (t_1,t_2], \\ \vdots & \\ \displaystyle\sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \overline{z}(t_k^-)) & \\ \quad \displaystyle + \int_0^t T_\alpha(t-s)v(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\overline{z}_\tau + y_\tau\big)d\tau ds, & t \in (t_m,b], \end{cases}$$

where $v(s) \in S_{\mathscr{F},y+\overline{z}}$. It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point.

Remark 3. *From Lemma 1 and above assumptions, we have the following estimates:*

(i)

$$\begin{aligned} &\|\overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \\ &\le \|\overline{z}_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} + \|y_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_2^* + J^\varsigma)\|z_0\|_{\mathscr{B}_h} + \mathscr{D}_1^*|y(s)| + (\mathscr{D}_2^* + J^\varsigma)\|y_0\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\|S_\alpha(t)\|_{L(\mathbb{X})}|\varsigma(0)| + (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_1^*\widetilde{M}_S H\|\varsigma\|_{\mathscr{B}_h} + (\mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h} \\ &\le \mathscr{D}_1^* \sup_{0\le\tau\le s} \|z(\tau)\|_{\mathbb{X}} + (\mathscr{D}_1^*\widetilde{M}_S H + \mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h}. \end{aligned}$$

If $\|z\|_{\mathbb{X}} < r$, $r > 0$, then:

$$\|\overline{z}_{\varrho(s,\overline{z}_s+y_s)} + y_{\varrho(s,\overline{z}_s+y_s)}\|_{\mathscr{B}_h} \le \mathscr{D}_1^* r + c_n,$$

where $c_n = (\mathscr{D}_1^*\widetilde{M}_S H + \mathscr{D}_2^* + J^\varsigma)\|\varsigma\|_{\mathscr{B}_h}$.

(ii)

$$\|\sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \overline{z}(t_k^-))\|_{\mathbb{X}} \le m\widetilde{M}_S\|\mathcal{I}_k(y(t_k^-) + \overline{z}(t_k^-))\|_{\mathbb{X}}. \tag{5}$$

Since:

$$\begin{aligned} |\mathcal{I}_k(y(t_k^-) + \overline{z}(t_k^-))| &\le L_k(t)\big(|y(t_k^-) + \overline{z}(t_k^-)|\big) \\ &\le L_k(t)\big(\sup_{t\in\mathscr{J}} |y(t) + \overline{z}(t)|\big) \\ &\le L_0 H\|y_t + \overline{z}_t\|_{\mathscr{B}_h}, \end{aligned}$$

where $L_0 = \max\{L_k(t) | t \in \mathscr{J}, \quad k = 1,2,3,\ldots,m\}$.

Now:

$$\begin{aligned}
\|y_t + \bar{z}_t\|_{\mathscr{B}_h} &\leq \|y_t\|_{\mathscr{B}_h} + \|\bar{z}_t\|_{\mathscr{B}_h} \\
&\leq \mathscr{D}_1(t) \sup_{0\leq\tau\leq t} \|y(\tau)\|_{\mathbb{X}} + \mathscr{D}_2(t)\|y_0\|_{\mathscr{B}_h} + \mathscr{D}_1(t) \sup_{0\leq\tau\leq t} \|z(\tau)\|_{\mathbb{X}} + \mathscr{D}_2(t)\|z_0\|_{\mathscr{B}_h} \\
&\leq \mathscr{D}_1(t)\Big[\|S_\alpha(t)\|_{L(\mathbb{X})}|\varsigma(0)|\Big] + \mathscr{D}_2(t)\|\varsigma\|_{\mathscr{B}_h} + \mathscr{D}_1(t) \sup_{0\leq\tau\leq t} \|z(\tau)\|_{\mathbb{X}} \\
&\leq \mathscr{D}_1^* r + (\mathscr{D}_1^* \widetilde{M}_S H + \mathscr{D}_2^*)\|\varsigma\|_{\mathscr{B}_h} \\
&\leq \mathscr{D}_1^* r + \widetilde{c_n},
\end{aligned}$$

where $\widetilde{c_n} = (\mathscr{D}_1^* \widetilde{M}_S H + \mathscr{D}_2^*)\|\varsigma\|_{\mathscr{B}_h}$. *Hence, Equation (5) becomes:*

$$\|\sum_{k=1}^{m} S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \bar{z}(t_k^-))\|_{\mathbb{X}} \leq m\widetilde{M}_S L_0 H\big[\mathscr{D}_1^* r + \widetilde{c_n}\big].$$

Let:

$$B_r = \{z \in \mathscr{B}_h'' : z(0) = 0; \quad \|z\|_{\mathscr{B}_h''} \leq r\},$$

where r is any fixed finite real number that fulfills the inequality:

$$\begin{aligned}
r > m\widetilde{M}_S L_0 H\big[\mathscr{D}_1^* r + \widetilde{c_n}\big] + \frac{\widetilde{M}_T b^\alpha}{\alpha}\Omega_1(\mathscr{D}_1^* r + c_n)\int_0^b p(s)ds \\
+ \frac{a\widetilde{M}_T b^\alpha}{\alpha}\Omega_2(\mathscr{D}_1^* r + \widetilde{c_n})\int_0^b m_1(s)ds.
\end{aligned} \tag{6}$$

It is obvious that B_r is a closed, convex, bounded set in $\mathscr{B}_h''$. Now, we shall display that $\overline{\Upsilon}$ meets all the presumptions of Lemma 3. Now, we split the proof into grouping of subsequent steps:

Step 1:

$\overline{\Upsilon}(z)$ is convex for every $z \in \mathscr{B}_h''$. In fact, if h_1 and h_2 belong to $\overline{\Upsilon}(z)$, then we can find $v_1, v_2 \in S_{\mathscr{F},y+\bar{z}}$ in a way that, for $t \in \mathscr{J}$ and $i = 1,2$, we sustain:

$$h_i(t) = \begin{cases}
\int_0^t T_\alpha(t-s)v_i(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, & t \in [0,t_1], \\
S_\alpha(t-t_1)\mathcal{I}_1(y(t_1^-) + \bar{z}(t_1^-)) & \\
\quad + \int_0^t T_\alpha(t-s)v_i(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, & t \in (t_1,t_2], \\
\vdots & \\
\sum_{k=1}^{m} S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \bar{z}(t_k^-)) & \\
\quad + \int_0^t T_\alpha(t-s)v_i(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds, & t \in (t_m,b].
\end{cases}$$

Let $\lambda \in [0,1]$. Then, for every $t \in [0,t_1]$, we get:

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \int_0^t T_\alpha(t-s)\big[\lambda v_1(s) + (1-\lambda)v_2(s)\big]ds + \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds.$$

In the same way, for any $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we receive:

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \sum_{k=1}^{m} S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \bar{z}(t_k^-)) + \int_0^t T_\alpha(t-s)\big[\lambda v_1(s) + (1-\lambda)v_2(s)\big]ds$$
$$+ \int_0^t T_\alpha(t-s)\int_0^s e\big(s,\tau,\bar{z}_\tau + y_\tau\big)d\tau ds.$$

Since $\mathscr{F}$ has convex values, $S_{\mathscr{F},y+\bar{z}}$ is convex, and we realize that $\lambda h_1(t) + (1-\lambda)h_2(t) \in \overline{\Upsilon}(z)$.

Step 2:
$\overline{\Upsilon}(B_r) \subset B_r$. Let $h \in \overline{\Upsilon}(z)$ and $z \in B_r$, for $t \in [0, t_1]$. Then, by Remark 3, we sustain:

$$\begin{aligned}
\|h(t)\|_{\mathbb{X}} &\leq \int_0^t \|T_\alpha(t-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds + \int_0^t \|T_\alpha(t-s)\|_{L(\mathbb{X})}\int_0^s \|e\big(s,\tau,\bar{z}_\tau + y_\tau\big)\|_{\mathbb{X}}d\tau ds \\
&\leq \widetilde{M}_T \int_0^t (t-s)^{\alpha-1}p(s)\Omega_1(\|\bar{z}_{\varrho(s,\bar{z}_s+y_s)} + y_{\varrho(s,\bar{z}_s+y_s)}\|_{\mathscr{B}_h})ds \\
&\quad + \widetilde{M}_T a \int_0^t (t-s)^{\alpha-1}m_1(s)\Omega_2(\|\bar{z}_s + y_s\|_{\mathscr{B}_h})ds \\
&\leq \frac{\widetilde{M}_T b^\alpha}{\alpha}\Omega_1(\mathscr{D}_1^* r + c_n)\int_0^b p(s)ds + \frac{a\widetilde{M}_T b^\alpha}{\alpha}\Omega_2(\mathscr{D}_1^* r + \widetilde{c_n})\int_0^b m_1(s)ds < r.
\end{aligned}$$

Moreover, when $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, from Remark 3, we have the estimate:

$$\begin{aligned}
\|h(t)\|_{\mathbb{X}} &\leq \|\sum_{k=1}^{m} S_\alpha(t-t_k)\mathcal{I}_k(y(t_k^-) + \bar{z}(t_k^-))\|_{\mathbb{X}} + \int_0^t \|T_\alpha(t-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds \\
&\quad + \int_0^t \|T_\alpha(t-s)\|_{L(\mathbb{X})}\int_0^s \|e\big(s,\tau,\bar{z}_\tau + y_\tau\big)\|_{\mathbb{X}}d\tau ds \\
&\leq m\widetilde{M}_S L_0 H\big[\mathscr{D}_1^* r + \widetilde{c_n}\big] + \frac{\widetilde{M}_T b^\alpha}{\alpha}\Omega_1(\mathscr{D}_1^* r + c_n)\int_0^b p(s)ds \\
&\quad + \frac{a\widetilde{M}_T b^\alpha}{\alpha}\Omega_2(\mathscr{D}_1^* r + \widetilde{c_n})\int_0^b m_1(s)ds < r,
\end{aligned}$$

which proves that $\overline{\Upsilon}(B_r) \subset B_r$.

Step 3:
We will confirm that $\overline{\Upsilon}(B_r)$ is equicontinuous. Let $u, v \in [0, t_1]$, with $u < v$, we get:

$$\begin{aligned}
\|h(v) - h(u)\|_{\mathbb{X}} &\leq \int_0^u \|T_\alpha(v-s) - T_\alpha(u-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds + \int_u^v \|T_\alpha(v-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds \\
&\quad + \int_0^u \|T_\alpha(v-s) - T_\alpha(u-s)\|_{L(\mathbb{X})}\int_0^s \|e(s,\tau,\bar{z}_\tau + y_\tau)\|_{\mathbb{X}}d\tau ds \\
&\quad + \int_u^v \|T_\alpha(v-s)\|_{L(\mathbb{X})}\int_0^s \|e(s,\tau,\bar{z}_\tau + y_\tau)\|_{\mathbb{X}}d\tau ds \\
&\leq Q_1 + Q_2 + Q_3 + Q_4,
\end{aligned}$$

where:

$$Q_1 = \int_0^u \|T_\alpha(v-s) - T_\alpha(u-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds$$

$$\leq \Omega_1(\mathscr{D}_1^* r + c_n)\int_0^u \|T_\alpha(v-s) - T_\alpha(u-s)\|_{L(\mathbb{X})}p(s)ds.$$

Since $\|T_\alpha(v-s)-T_\alpha(u-s)\|_{L(\mathbb{X})} \leq 2\widetilde{M}_T(t_1-s)^{\alpha-1} \in L^1(\mathscr{J},\mathbb{R}^+)$ for $s \in [0,t_1]$, and $T_\alpha(v-s)-T_\alpha(u-s) \to 0$ as $u \to v$, T_α is strongly continuous. This infers that $\lim_{u\to v} Q_1 = 0$,

$$\begin{aligned} Q_2 &= \int_u^v \|T_\alpha(v-s)\|_{L(\mathbb{X})}\|v(s)\|_{\mathbb{X}}ds \\ &\leq \frac{\Omega_1(\mathscr{D}_1^* r + c_n)\widetilde{M}_T(v-u)^\alpha}{\alpha} \int_u^v p(s)ds. \end{aligned}$$

As a result, we deduce that $\lim_{u\to v} Q_2 = 0$,

$$\begin{aligned} Q_3 &= \int_0^u \|T_\alpha(v-s)-T_\alpha(u-s)\|_{L(\mathbb{X})} \int_0^s \|e(s,\tau,\bar{z}_\tau+y_\tau)\|_{\mathbb{X}} d\tau ds \\ &\leq a\Omega_2(\mathscr{D}_1^* r + \widetilde{c_n}) \int_0^u \|T_\alpha(v-s)-T_\alpha(u-s)\|_{L(\mathbb{X})} m_1(s)ds. \end{aligned}$$

This suggests that $\lim_{u\to v} Q_3 = 0$,

$$\begin{aligned} Q_4 &= \int_u^v \|T_\alpha(v-s)\|_{L(\mathbb{X})} \int_0^s \|e(s,\tau,\bar{z}_\tau+y_\tau)\|_{\mathbb{X}} d\tau ds \\ &\leq \frac{a\Omega_2(\mathscr{D}_1^* r + \widetilde{c_n})\widetilde{M}_T(v-u)^\alpha}{\alpha} \int_u^v m_1(s)ds. \end{aligned}$$

Therefore, we deduce that $\lim_{u\to v} Q_4 = 0$.

In the same way, for $u, v \in (t_k, t_{k+1}]$, with $u < v$, $k = 1, 2, \ldots, m$, we sustain:

$$\begin{aligned} \|h(v)-h(u)\|_{\mathbb{X}} &\leq \sum_{k=1}^m \|S_\alpha(v-t_k)-S_\alpha(u-t_k)\|_{\mathbb{X}} \|\mathcal{I}_k(y(t_k^-)+\bar{z}(t_k^-))\|_{\mathbb{X}} + Q_1+Q_2+Q_3+Q_4 \\ &\leq L_0 H\left[\mathscr{D}_1^* r + \widetilde{c_n}\right] \sum_{k=1}^m \|S_\alpha(v-t_k)-S_\alpha(u-t_k)\|_{\mathbb{X}} + Q_1+Q_2+Q_3+Q_4. \end{aligned}$$

Since S_α is also strongly continuous, $S_\alpha(v-t_k)-S_\alpha(u-t_k) \to 0$ as $u \to v$. Hence, from the aforementioned inequalities, we receive:

$$\lim_{u\to v} \|h(v)-h(u)\|_{\mathbb{X}} = 0.$$

Thus, $\overline{\Upsilon}(B_r)$ is equicontinuous.

As an impact of actions 1, 2 and 3 with Arzela–Ascoli's theorem ([60] (Chapter 2)), we understand that the operator $\overline{\Upsilon} : \mathscr{B}_h'' \to \mathcal{P}(\mathscr{B}_h'')$ is completely continuous.

Step 4:

$\overline{\Upsilon}$ has a closed graph. Expect that $z_n \to z_*$, $h_n \in \overline{\Upsilon}(z_n)$ with $h_n \to h_*$. We claim that $h_* \in \overline{\Upsilon}(z_*)$. In fact, the assumption $h_n \in \overline{\Upsilon}(z_n)$ suggests that we can find $v_n \in S_{\mathscr{F},y_n+\bar{z}_n}$ in a way that, for each $t \in [0,t_1]$:

$$h_n(t) = \int_0^t T_\alpha(t-s)v_n(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\bar{z}_{n_\tau}+y_\tau)d\tau ds.$$

We need to demonstrate that there exists $v_* \in S_{\mathscr{F},\bar{z}_*+y_*}$ such that, for each $t \in [0,t_1]$:

$$h_*(t) = \int_0^t T_\alpha(t-s)v_*(s)ds + \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\bar{z}_{*_\tau}+y_\tau)d\tau ds.$$

Set:

$$\Theta_n(t) = h_n(t) - \int_0^t T_\alpha(t-s)v_n(s)ds - \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{n_\tau}+y_\tau)d\tau ds,$$
$$\Theta_*(t) = h_*(t) - \int_0^t T_\alpha(t-s)v_*(s)ds - \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{*_\tau}+y_\tau)d\tau ds.$$

We now have, for every $t \in [0,t_1]$:

$$\|\Theta_n(t) - \Theta_*(t)\|_{\mathbb{X}} \to 0, \quad \text{as} \quad n \to \infty.$$

Recognize the linear continuous operator $\Psi : L^1([0,t_1],\mathbb{X}) \to C([0,t_1],\mathbb{X})$, specified by:

$$\Psi(v)(t) = \int_0^t T_\alpha(t-s)v(s)ds.$$

From Lemma 2 and the definition of Ψ, it follows that $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, and for every $t \in [0,t_1]$, $\Theta_n(t) \in \Psi(S_{\mathscr{F},y_n+\overline{z}_n})$.

Since $z_n \to z_*$ and $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, then there exists $v_* \in S_{\mathscr{F},y_*+\overline{z}_*}$ such that, for every $t \in [0,t_1]$:

$$h_*(t) - \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{*_\tau}+y_\tau)d\tau ds = \int_0^t T_\alpha(t-s)v_*(s)ds.$$

In the same way, for any $t \in (t_k,t_{k+1}]$, $k = 1,2,\ldots,m$, we get:

$$h_n(t) = \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y_n(t_k^-)+\overline{z}_n(t_k^-)) + \int_0^t T_\alpha(t-s)v_n(s)ds$$
$$+ \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{n_\tau}+y_\tau)d\tau ds.$$

We need to demonstrate that there exists $v_* \in S_{F,y_*+\overline{z}_*}$ such that, for every $t \in (t_k,t_{k+1}]$:

$$h_*(t) = \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y_*(t_k^-)+\overline{z}_*(t_k^-)) + \int_0^t T_\alpha(t-s)v_*(s)ds$$
$$+ \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{*_\tau}+y_\tau)d\tau ds.$$

For every $t \in (t_k,t_{k+1}]$, $k = 1,2,\ldots,m$, we sustain:

$$\|\Theta_n(t) - \Theta_*(t)\|_{\mathbb{X}} \to 0, \quad \text{as} \quad n \to \infty,$$

where:

$$\Theta_n(t) = h_n(t) - \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y_n(t_k^-)+\overline{z}_n(t_k^-)) - \int_0^t T_\alpha(t-s)v_n(s)ds$$
$$- \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{n_\tau}+y_\tau)d\tau ds,$$

$$\Theta_*(t) = h_*(t) - \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y_*(t_k^-)+\overline{z}_*(t_k^-)) - \int_0^t T_\alpha(t-s)v_*(s)ds$$
$$- \int_0^t T_\alpha(t-s)\int_0^s e(s,\tau,\overline{z}_{*_\tau}+y_\tau)d\tau ds.$$

Now, for every $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we recognize the linear operator $\Psi : L^1((t_k, t_{k+1}], \mathbb{X}) \to C((t_k, t_{k+1}], \mathbb{X})$, characterized by:

$$\Psi(v)(t) = \int_0^t T_\alpha(t-s)v(s)ds.$$

From Lemma 2, it follows that $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, and, for every $t \in (t_k, t_{k+1}]$, $\Theta_n(t) \in \Psi(S_{\mathscr{F}, y_n + \bar{z}_n})$.

Since $z_n \to z_*$, and $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, then there exists $v_* \in S_{\mathscr{F}, y_* + \bar{z}_*}$ such that, for every $t \in (t_k, t_{k+1}]$:

$$h_*(t) - \sum_{k=1}^m S_\alpha(t-t_k)\mathcal{I}_k(y_*(t_k^-) + \bar{z}_*(t_k^-)) - \int_0^t T_\alpha(t-s)\int_0^s e(s, \tau, \bar{z}_{*_\tau} + y_\tau)d\tau ds = \int_0^t T_\alpha(t-s)v_*(s)ds.$$

Hence, $\overline{\Upsilon}$ has a closed graph. It follows that Lemma 3 that $\overline{\Upsilon}$ has a fixed point $z \in \mathscr{B}_h''$. Then, the operator Υ has a fixed point that offers ascent to a mild solution to the model (1)–(3). The proof is now completed. □

4. Applications

To exemplify our theoretical results, we treat IFIDI with SDD of the model:

$$\begin{aligned} {}^C D_t^q u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) &\in \int_{-\infty}^t \mu_1(t, x, s-t)u(s - \varrho_1(t)\varrho_2(\|u(t)\|), x)ds \\ &+ \int_0^t \int_{-\infty}^s k(s-\tau)P(u(\tau, x))d\tau ds,\ x \in [0, \pi],\ 0 \le t \le b,\ t \ne t_k, \end{aligned} \tag{7}$$

$$u(t, 0) = 0 = u(t, \pi), \quad t \ge 0, \tag{8}$$

$$u(t, x) = \varsigma(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi], \tag{9}$$

$$\Delta u(t_k)(x) = \int_{-\infty}^{t_k} q_k(t_k - s)u(s, x)ds, \quad x \in [0, \pi],\ k = 1, 2, \ldots, m, \tag{10}$$

where ${}^C D_t^q$ is Caputo's fractional derivative of order $0 < q < 1$, $0 < t_1 < t_2 < \cdots < t_n < b$ are pre-fixed numbers and $\varsigma \in \mathscr{B}_h$. We consider $\mathbb{X} = L^2[0, \pi]$ with the norm $|\cdot|_{L^2}$ and delineate the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ by $\mathscr{A}w = w''$ with the domain:

$$D(\mathscr{A}) = \{w \in \mathbb{X} : w, w' \quad \text{are absolutely continuous}, \quad w'' \in \mathbb{X},\ w(0) = w(\pi) = 0\}.$$

Then:

$$\mathscr{A}w = \sum_{n=1}^\infty n^2 \langle w, w_n \rangle w_n, \quad w \in D(\mathscr{A}),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, \ldots,$. is the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \ge 0}$ in $\mathbb{X}$ and is given by:

$$T(t)w = \sum_{n=1}^\infty e^{-n^2 t}\langle w, w_n \rangle w_n, \quad \text{for all} \quad w \in \mathbb{X}, \quad \text{and every} \quad t > 0.$$

From these outflows, it follows that $(T(t))_{t \ge 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, \mathscr{A}) = (\lambda - \mathscr{A})^{-1}$ is a compact operator for all $\lambda \in \rho(\mathscr{A})$; that is, $\mathscr{A} \in \mathbb{A}^\alpha(\theta_0, \omega_0)$. In addition, the subordination principle of solution operator $(S_\alpha(t))_{t \ge 0}$ such that $\|S_\alpha(t)\|_{L(\mathbb{X})} \le \widetilde{M}_S$ for

$t \in [0, b]$. For phase space, we choose $h = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$, and determine:

$$\|\varsigma\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\varsigma(\theta)\|_{L^2} ds.$$

Therefore, $(t, \varsigma) \in [0, b] \times \mathscr{B}_h$, where $\varsigma(\theta)(x) = \varsigma(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set:

$$u(t)(x) = u(t, x), \quad \varrho(t, \varsigma) = \varrho_1(t)\varrho_2(\|\varsigma(0)\|),$$

and we receive:

$$f(t, \varsigma)(x) = \int_{-\infty}^{0} \mu_1(t, x, \theta)(\varsigma(\theta)(x))d\theta,$$

and:

$$(B\varsigma)(x) = \int_0^t e(t, s, \varsigma)(x)ds = \int_0^t \int_{-\infty}^{0} k(s - \theta)P_1(\varsigma(\theta)(x))d\theta ds.$$

Along these adjustments, the aforementioned model (7)–(10) can be written in the theoretical form as model (1)–(3).

Suppose further that:

(i) the functions $\varrho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$ are continuous;
(ii) the function $\mu_1(t, x, \theta)$ is continuous in $[0, b] \times [0, \pi] \times (-\infty, 0]$; and $\mu_1(t, x, \theta) \geq 0$; $\int_{-\infty}^{0} \mu_1(t, x, \theta)d\theta = p_1(t, x) < \infty$.
(iii) the function $k(t - s)$ is continuous in $[0, b]$ and $k(t - s) \geq 0$, $\int_0^t \int_{-\infty}^{0} k(s - \theta)d\theta ds = m_1(t) < \infty$.
(iv) the functions $q_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$ are continuous and $d_i = \int_{-\infty}^{0} h(s)q_i^2(s)ds < \infty$ for $i = 1, 2, \ldots, n$.
(v) The function $P(\cdot)$ is continuous and for each $(\theta, x) \in (-\infty, 0] \times [0, \pi]$; $0 \leq P(u(\theta)(x)) \leq \Phi\Big(\int_{-\infty}^{0} e^{2s}\|u(s, \cdot)\|_{L^2}ds\Big)$, where $\Phi : [0, +\infty) \to (0, +\infty)$ is a continuous non-decreasing function.

Now, consider:

$$\begin{aligned}
\|B\varsigma\|_{L^2} &= \left[\int_0^{\pi} \left(\int_0^t \int_{-\infty}^{0} k(s - \theta)P(\varsigma(\theta)(x))d\theta ds\right)^2 dx\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^{\pi} \left(\int_0^t \int_{-\infty}^{0} k(\tau - \theta)\Phi\left(\int_{-\infty}^{0} e^{2s}\|\varsigma(s)(\cdot)\|_{L^2}ds\right)d\theta d\tau\right)^2 dx\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^{\pi} \left(\int_0^t \int_{-\infty}^{0} k(s - \theta)d\theta ds\right)^2 dx\right]^{\frac{1}{2}} \Phi(\|\varsigma\|_{\mathscr{B}_h}) \\
&\leq \left[\int_0^{\pi} (m_1^*(t))^2 dx\right]^{\frac{1}{2}} \Phi(\|\varsigma\|_{\mathscr{B}_h}) \\
&\leq \sqrt{\pi}\,\overline{m}_1(t)\Phi(\|\varsigma\|_{\mathscr{B}_h}).
\end{aligned}$$

Since $\Phi : [0, +\infty) \to (0, +\infty)$ is a continuous non-decreasing function, we will take $m_1(t) = \overline{m}_1(t)$ with $a = \sqrt{\pi}$ and $\Omega_2(r) = \Phi(r)$ in hypothesis (4). Observe that $\mathscr{F}$ meets the hypothesis (3) with $\Omega_1(r) = r$, $p(t) = \overline{p}_1(t)$, and, if the bounds in Equation (6) are fulfilled, then model (7)–(10) has a mild solution on $(-\infty, b]$.

5. Conclusions

In this paper, we have studied the existence results for impulsive fractional integro-differential systems with SDD conditions in a Banach space. More precisely, by utilizing the fractional calculus, semigroup theory and the Bohnenblust–Karlin's fixed point theorem, we investigate the IFIDI with SDD in a Banach space. To validate the obtained theoretical results, we analyze one example. The FDEs are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability.

There are two direct issues that require further study. First, we will investigate the approximate controllability of fractional neutral integro-differential systems with SDD in the cases of a noncompact operator and a normal topological space. Second, we will study the approximate controllability of a new class of impulsive fractional integro-differential equations with SDD and non-instantaneous impulses, as discussed in [48].

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