mathematics

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$\mathcal{F}$-Harmonic Maps between Doubly Warped Product Manifolds

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Academic Editor: Lokenath Debnath
Received: 7 January 2017; Accepted: 21 March 2017; Published: 23 March 2017

Abstract: In this paper, some properties of $\mathcal{F}$-harmonic and conformal $\mathcal{F}$-harmonic maps between doubly warped product manifolds are studied and new examples of non-harmonic $\mathcal{F}$-harmonic maps are constructed.

Keywords: harmonic maps; $\mathcal{F}$-harmonic maps; doubly warped product manifolds

1. Introduction

Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. The map ϕ is called *harmonic* if it is a critical point of the energy functional:

$$E(\phi) = \int_K e(\phi) dv_g \tag{1}$$

for any compact sub-domain $K \subseteq M$, where $e(\phi) := \frac{1}{2} \mid d\phi \mid^2$ is the energy density of ϕ. Here, $\mid d\phi \mid$ denotes the Hilbert–Schmidt norm of the differential $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$ with respect to g and h.

The Euler–Lagrange equation corresponding to the energy functional is given by vanishing of the tension field $\tau(\phi) := trace_g \nabla d\phi$. Many researchers have been studying this topic extensively (see, for instance, [1–3]).

$\mathcal{F}$-harmonic maps as a generalization of harmonic maps, geodesics and minimal surfaces were first studied by Ara [4]. Let $\mathcal{F} : [0, \infty) \longrightarrow [0, \infty)$ be a C^2-function such that $F' > 0$ on $(0, \infty)$. The map ϕ is said to be $\mathcal{F}$-*harmonic* if ϕ is a critical point of the $\mathcal{F}$-energy functional:

$$E_{\mathcal{F}}(\phi) = \int_M \mathcal{F}(\frac{\mid d(\phi) \mid^2}{2}) dv_g \tag{2}$$

$\mathcal{F}$-energy functional can be considered as energy , p-energy or exponential energy when $\mathcal{F}(t)$ is equal to $t, (2t)^{\frac{p}{2}}/p \ \ (p \geq 4)$ or e^t, respectively. The Euler–Lagrange equation associated with the $\mathcal{F}$-energy functional is given by:

$$\tau_{\mathcal{F}}(\phi) := \mathcal{F}'(\frac{\mid d\phi \mid^2}{2})\tau(\phi) + d\phi(grad_g(\mathcal{F}'(\frac{\mid d\phi \mid^2}{2}))) = 0 \tag{3}$$

The operator $\tau_{\mathcal{F}}(\phi)$ is called the $\mathcal{F}$-*tension field* of the map ϕ.

In view of physics, harmonic maps have been studied in various fields of physics, such as super conductor, ferromagnetic material, liquid crystal, etc. [5–8]. Additionally, p-harmonic maps have been extensively applied in image processing for denoising color images [9,10]. Furthermore, exponential harmonic maps have been studied on gravity [11]. The concept of $\mathcal{F}$-harmonic maps, as an extension of harmonic, p-harmonic and exponential harmonic maps, have an important role in physics and physical cosmology. For instance, instead of the scalar field in the Lagrangian, some of the $\mathcal{F}$-harmonic maps, such as the trigonometric functions, are studied in order to reproduce the inflation. Moreover, there are other $\mathcal{F}$-harmonic maps, such as exponential harmonic maps, are investigated in order to depict the phenomenon of the quintessence [12,13].

Since 2000, many scholars have done research in this regard. In [14], Ara extended two stability theorems due to Howard and Okayasu to $\mathcal{F}$-harmonic maps. He also showed that every stable $\mathcal{F}$-harmonic map into sufficiently pinched simply-connected Riemannian manifolds is constant. In [15], several Liouville theorems for $\mathcal{F}$-harmonic maps between Riemannian manifold are proved. In [16], the researchers studied new geometric techniques to deal with the Dirichlet problem for p-harmonic maps and $\mathcal{F}$-harmonic maps. In [17], Li studied $\mathcal{F}$-harmonic maps from a complete Riemannian manifold M and $\mathcal{F}$-Yang–Mills fields on M. He showed that, under some conditions, every $\mathcal{F}$-Yang–Mills field with finite $\mathcal{F}$-Yang–Mills energy vanishes on M.

It is worth noting that $\mathcal{F}$-harmonic maps should not be confused with f-harmonic maps between Riemannian manifolds. The notion of f-harmonic maps was first studied by Lichnerowicz [18] in 1970. Let $f \in C^\infty(M)$ be a positive smooth function on M. The map ϕ is called f-harmonic if ϕ is a critical point of the f-energy functional:

$$E_f(\phi) := \frac{1}{2}\int_M f \mid d\phi \mid^2 dv_g$$

In view of physics, these maps can be considered as the stationary solutions of an inhomogeneous Heisenberg spin system. For more details [19].

On the other hand, the notion of warped product manifolds generalizes that of a surface of revolution. It was studied by Bishop and O'Neill for constructing negative curvature manifolds [20]. These products play an important role in many branches of physics, especially in general relativity, string and super gravity theories [21]. For instance, in gravity theories, the best model of space-time that describes the outer space near black holes with large gravitational force is given as a warped product manifold [22]. Due to the role of these products in construction of examples with negative curvature, warped product manifolds have an important role in geometry [20,23].

The notion of doubly warped product manifolds can be considered as an extension of warped product manifolds. These products are studied for Lorentzian manifolds by Beem and Powell in [24]. In [25,26], global hyperbolicity and null pseudocovexity of Lorentzian doubly warped product manifolds are investigated. In [27], the author investigates conformal properties of doubly warped products manifolds. In [28], globally hyperbolicity of generalized Robertson–Walker space-times with doubly warped product fiber is studied.

One of the important examples of doubly warped product manifolds is a doubly warped space-time. This is mainly because this space-time produces many exact solutions to Einstein's field equations. The Gibbons-Maeda-Garfinkle-Horowitz-Strominger solutions of the Einstein field equations represent the geometry exterior to a spherically symmetric static charged black hole and GMGHS metric in the doubly warped product space-time have the same form as the Kantowski–Sachs solution [29]. In [30], Choi studied the GMGHS space-time in order to analyze the anisotropic cosmological model, which represents homogeneous but anisotropically expanding or contracting cosmology. He also investigated the solution of GMGHS space-time in the form of doubly warped products and obtained the Ricci curvature associated with three phases in the evolution of the universe.

In this paper, following the ideas in [31], $\mathcal{F}$-harmonic and conformally $\mathcal{F}$-harmonic maps between doubly warped product manifolds are studied. More precisely, the methods provided in [3,31] are used to investigate the $\mathcal{F}$-harmonicity of these maps. Sections 3–5 contain our main results. Owing to the importance of $\mathcal{F}$-harmonic maps and doubly warped product space in physics and physical cosmology [5–13,24–30], the results of this paper can provide benefits for studying the $\mathcal{F}$-harmonicity of the maps from or into the doubly warped product manifolds in research on various fields of physical cosmology, such as the phenomena of the quintessence and the inflation.

This paper is organized as follows. In Section 2, some results on warped product manifolds are reviewed. In Section 3, the $\mathcal{F}$-harmonicity conditions of smooth maps from a Riemannian manifold P into a doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$ are analyzed. Especially, it is shown that both of the leaves $\{x_0\} \times N$ and $M \times \{y_0\}$ are never non-harmonic $\mathcal{F}$-harmonic maps as submanifolds of a doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$. In Section 4, some characterizations for $\mathcal{F}$-harmonicity of smooth maps from a doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$ into a Riemannian manifold P are given. Particularly, it is shown that the projection maps from a doubly warped product manifolds $M\,_{\mu}\!\times_{\lambda} N$, whose warping functions λ and μ are non-constant functions, are not $\mathcal{F}$-harmonic. In the last section, new classes of $\mathcal{F}$-harmonic maps between doubly warped product maps are constructed.

2. Some Results on Doubly Warped Product Manifolds

Let (M, g) and (N, h) be Riemannian manifolds and let $\lambda \in C^\infty(M)$ and $\mu \in C^\infty(N)$ be positive smooth functions. The doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$ is the product manifold $M \times N$ equipped with the Riemannian metric:

$$\bar{g}(X,Y) = (\mu \circ \pi_2)^2 g(d\pi_1(X), d\pi_1(Y)) + (\lambda \circ \pi_1)^2 h(d\pi_2(X), d\pi_2(Y)) \tag{4}$$

for any $X, Y \in \Gamma(T(M \times N))$, where $\pi_1 : M\,_{\mu}\!\times_{\lambda} N \longrightarrow M$ and $\pi_2 : M\,_{\mu}\!\times_{\lambda} N \longrightarrow N$ are canonical projection maps. A doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$ is called *direct product manifold* if λ and μ are constant. The relationship between the Levi–Civita connections on direct product manifold $M \times N$ and doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$ is given as follows:

Theorem 1. *Let (M, g) and (N, h) be Riemannian manifolds with the Levi–Civita connections ∇^M and ∇^N, respectively, and let ∇ and $\overline{\nabla}$ denote the Levi–Civita connections of direct product manifold $M \times N$ and doubly warped product manifold $M\,_{\mu}\!\times_{\lambda} N$, respectively. Then, the Levi–Civita connection of doubly warped product manifold is given as follows [19]:*

$$\begin{aligned} & \overline{\nabla}_{(X_1,Y_1)}(X_2,Y_2) \\ = \; & (\nabla^M_{X_1} X_2 + \frac{1}{2\mu^2} Y_1(\mu^2) X_2 + \frac{1}{2\mu^2} Y_2(\mu^2) X_1 - \frac{1}{2} h(Y_1,Y_2) grad_g \lambda^2, 0_2) \\ + \; & (0_1, \nabla^N_{Y_1} Y_2 + \frac{1}{2\lambda^2} X_1(\lambda^2) Y_2 + \frac{1}{2\lambda^2} X_2(\lambda^2) Y_1 - \frac{1}{2} g(X_1,X_2) grad_h \mu^2) \end{aligned} \tag{5}$$

for any $(X_1,Y_1),(X_2,Y_2) \in \Gamma(T(M \times N))$, where $X_1, X_2 \in \Gamma(TM)$ and $Y_1, Y_2 \in \Gamma(TN)$. Here, (X_i, Y_i) is identified with $(X_i, 0_2) + (0_1, Y_i)$, $i = 1, 2$, and $0_1 \in T_pM$ and $0_2 \in T_qN$.

Doubly warped product manifolds $M\,_{\mu}\!\times_{\lambda} N$, whose warping functions λ and μ are harmonic functions, have been interesting for researchers in mathematical physics. For instance, the following results were proved in [32].

Theorem 2. *Let $M\,_{\mu}\!\times_{\lambda} N$ be a doubly warped product manifold whose warping functions λ and μ are harmonic functions. Then, there exists no isometric minimal immersion of a doubly warped product $M\,_{\mu}\!\times_{\lambda} N$ into a Riemannian manifold of negative curvature [32].*

Theorem 3. *If M is a compact Riemannian manifold and μ is a harmonic function on N, then [32]:*

(1) *Every doubly warped product $M\,_{\mu}\!\times_{\lambda} N$ does not admit an isometric minimal immersion into any Riemannian manifold of negative curvature.*
(2) *Every doubly warped product $M\,_{\mu}\!\times_{\lambda} N$ does not admit an isometric minimal immersion into a Euclidean space.*

Next, the $\mathcal{F}$-harmonicity of some special maps from or into a doubly warped product manifold are studied.

3. $\mathcal{F}$-Harmonicity Conditions of $\Phi : (P,\rho) \longrightarrow (M\,{}_{\mu}\!\times_{\lambda} N, \bar{g})$

In this section, the $\mathcal{F}$-harmonicity conditions of smooth maps from a Riemannian manifold (P,ρ) into a doubly warped product manifold $M\,{}_{\mu}\!\times_{\lambda} N$ are studied. Particularly, it is shown that both of the leaves $\{x_0\} \times N$ and $M \times \{y_0\}$ are never non-harmonic $\mathcal{F}$-harmonic maps as submanifolds of a doubly warped product manifold $M\,{}_{\mu}\!\times_{\lambda} N$.

Let $(M^m, g), (N^n, h)$ and (P^p, ρ) be Riemannian manifolds of dimensions m, n and p, respectively. Let $\lambda \in C^\infty(M)$ and $\mu \in C^\infty(N)$ be two positive smooth functions and $(M\,{}_{\mu}\!\times_{\lambda} N, \bar{g})$ be the doubly warped product manifold. Denote the Levi–Civita connection on the doubly warped product manifold $M\,{}_{\mu}\!\times_{\lambda} N$ by $\overline{\nabla}$. In the sequel, the following convention of index ranges are used:

$$1 \leqslant i, j, k, \ldots \leqslant m, \qquad 1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n, \qquad 1 \leqslant a, b, c, \ldots \leqslant p.$$

According to the above notations, we have the following.

Theorem 4. *Let $(M^m, g), (N^n, h)$ and (P^p, ρ) be Riemannian manifolds. Let $\phi : (P,\rho) \longrightarrow (M,g)$ and $\psi : (P,\rho) \longrightarrow (N,h)$ be smooth maps. Then, the $\mathcal{F}$-tension field of:*

$$\begin{aligned} \Phi : (P^p, \rho) &\longrightarrow (M^m\,{}_{\mu}\!\times_{\lambda} N^n, \bar{g}) \\ x &\longrightarrow (\phi(x), \psi(x)) \end{aligned} \tag{6}$$

is given by:

$$\begin{aligned} \tau_{\mathcal{F}}(\Phi) = \mathcal{F}'(k)\Big\{&(\tau(\phi), \tau(\psi)) + \big(d\phi(grad_\rho \ln(\mu^2 \circ \psi)), d\psi(grad_\rho \ln(\lambda^2 \circ \phi))\big) \\ &- (e(\psi)(grad_g \lambda^2) \circ \phi, e(\phi)(grad_h \mu^2) \circ \psi)\Big\} + \mathcal{F}''(k)(d\phi(grad_\rho k), d\psi(grad_\rho k)) \end{aligned} \tag{7}$$

where $k := e(\phi)\mu^2 \circ \psi + e(\psi)\lambda^2 \circ \phi$.

Proof. Let $\{e_a\}$ be an orthogonal frame with respect to ρ on P. By means of Equation (5) and definition of tension field, we have:

$$\begin{aligned} \tau(\Phi) &= trace_\rho \nabla d\Phi \\ &= \sum_{a=1}^{p} \Big\{ \overline{\nabla}_{(d\phi(e_a), d\psi(e_a))}(d\phi(e_a), d\psi(e_a)) - (d\phi(\nabla^P_{e_a} e_a), d\psi(\nabla^P_{e_a} e_a)) \Big\} \\ &= \sum_{a=1}^{p} \Big\{ \Big(\nabla^M_{d\phi(e_a)} d\phi(e_a) - d\phi(\nabla^P_{e_a} e_a) + 2d\psi(e_a)(\ln\mu) d\phi(e_a) - \frac{1}{2} \mid d\psi \mid^2 (grad_g \lambda^2) \circ \phi, 0_2 \Big) \\ &+ \Big(0_1, \nabla^N_{d\psi(e_a)} d\psi(e_a) - d\psi(\nabla^P_{e_a} e_a) + 2d\phi(e_a)(\ln\lambda) d\psi(e_a) - \frac{1}{2} \mid d\phi \mid^2 (grad_h \mu^2) \circ \psi \Big) \Big\} \end{aligned} \tag{8}$$

Due to the fact that $d\psi(e_a)(\ln\mu) = e_a(\ln\mu \circ \psi)$ (resp. $d\phi(e_a)(\ln\lambda) = e_a(\ln\lambda \circ \phi)$) and considering Equation (8), we get:

$$\begin{aligned} \tau(\Phi) = {} &(\tau(\phi), \tau(\psi)) + (d\phi(grad_\rho \ln(\mu^2 \circ \psi)), d\psi(grad_\rho \ln(\lambda^2 \circ \phi))) \\ &- (e(\psi)(grad_g \lambda^2) \circ \phi, e(\phi)(grad_h \mu^2) \circ \psi) \end{aligned} \tag{9}$$

By means of Equations (3) and (9), the $\mathcal{F}$-tension field of Φ can be obtained as follows:

$$\begin{aligned}\tau_{\mathcal{F}}(\Phi) &= \mathcal{F}'(\frac{|\,d\Phi\,|^2}{2})\tau(\Phi) + d\Phi(grad_\rho(\mathcal{F}'(\frac{|\,d\Phi\,|^2}{2})))\\ &= \mathcal{F}''(\frac{|\,d\Phi\,|^2}{2})(d\phi(grad_\rho(\frac{|\,d\Phi\,|^2}{2})), d\psi(grad_\rho(\frac{|\,d\Phi\,|^2}{2})))\\ &+ \mathcal{F}'(\frac{|\,d\Phi\,|^2}{2})\Big\{(\tau(\phi),\tau(\psi)) + (d\phi(grad_\rho \ln(\mu^2\circ\psi)), d\psi(grad_\rho \ln(\lambda^2\circ\phi)))\\ &- (e(\psi)(grad_g\lambda^2)\circ\phi, e(\phi)(grad_h\mu^2)\circ\psi)\Big\}\end{aligned} \tag{10}$$

By calculating the energy density of Φ, we get:

$$\begin{aligned}e(\Phi) = \frac{1}{2}\,|\,d\Phi\,|^2 &= \frac{1}{2}\sum_{a=1}^{p}\bar{g}(d\Phi(e_a), d\Phi(e_a))\\ &= \frac{1}{2}\sum_{a=1}^{p}\Big\{\mu^2\circ\psi\, g(d\phi(e_a), d\phi(e_a)) + \lambda^2\circ\phi\, h(d\psi(e_a), d\psi(e_a))\Big\}\\ &= e(\phi)\mu^2\circ\psi + e(\psi)\lambda^2\circ\phi\end{aligned} \tag{11}$$

Substituting Equation (11) into (10) yields Equation (7) and hence completes the proof. □

$\mathcal{F}$-Harmonicity of the Inclusion Maps

Let $(M\,{}_\mu\!\times_\lambda N, \bar{g})$ be a doubly warped product manifold. Denote by:

$$\begin{aligned}i_{y_0} : (M, g) &\longrightarrow (M\,{}_\mu\!\times_\lambda N, \bar{g})\\ x &\longrightarrow (x, y_0)\end{aligned} \tag{12}$$

the inclusion map of M at the point $y_0 \in N$ level in $M\,{}_\mu\!\times_\lambda N$, and:

$$\begin{aligned}i_{x_0} : (N, h) &\longrightarrow (M\,{}_\mu\!\times_\lambda N, \bar{g})\\ y &\longrightarrow (x_0, y)\end{aligned} \tag{13}$$

the inclusion map of N at the point $x_0 \in M$ level in $M\,{}_\mu\!\times_\lambda N$. Now, the $\mathcal{F}$-harmonicity conditions for both of the leaves $\{x_0\} \times N$ and $M \times \{y_0\}$ as submanifolds of a non-trivial doubly warped product manifold $M\,{}_\mu\!\times_\lambda N$ are studied .

Proposition 1. *Let $M\,{}_\mu\!\times_\lambda N$ be a doubly warped product manifold whose warping functions λ and μ are non-constant functions. The inclusion map i_{y_0} defined by Equation (12) is never a non- harmonic $\mathcal{F}$-harmonic map.*

Proof. By means of Equations (7) and (9), the tension and $\mathcal{F}$-tension field of i_{y_0} are given by:

$$\tau(i_{y_0}) = -\frac{m}{2}(0_1, grad_h\mu^2)\circ i_{y_0} \tag{14}$$

and:

$$\tau_{\mathcal{F}}(i_{y_0}) = -\frac{m}{2}\mathcal{F}'(\frac{m}{2}\mu^2(y_0))(0_1, grad_h\mu^2)\circ i_{y_0} \tag{15}$$

By Equations (14) and (15), the inclusion map i_{y_0} is non-harmonic $\mathcal{F}$-harmonic map if and only if $\mathcal{F}'(\frac{m}{2}\mu^2(y_0)) = 0$. However, this is in contradiction with the fact that $F' > 0$ on $(0,\infty)$. This completes the proof. □

For the inclusion map $i_{x_0} : N \longrightarrow M\,{}_\mu\!\times_\lambda N$, defined by Equation (13), a similar proof gives:

Proposition 2. *Let $M\,_\mu\!\times_\lambda N$ be a doubly warped product manifold whose warping functions λ and μ are non-constant functions. The inclusion map i_{x_0}, defined by Equation (13), is never a non- harmonic $\mathcal{F}$-harmonic map.*

4. $\mathcal{F}$-Harmonicity Conditions of $\tilde{\Phi} : (M\,_\mu\!\times_\lambda N, \bar{g}) \longrightarrow (P, \rho)$

In this section, the $\mathcal{F}$-harmonicity conditions of some special maps from a doubly warped product manifold into a Riemannian manifold are studied and an example is given. Finally, it is obtained that the projection maps from a non-trivial doubly warped product manifolds $M\,_\mu\!\times_\lambda N$ can not be $\mathcal{F}$-harmonic.

First, we study the $\mathcal{F}$-harmonicity conditions of the map:

$$\begin{aligned} \widetilde{\Phi} : (M\,_\mu\!\times_\lambda N, \bar{g}) &\longrightarrow (P^p, \rho) \\ (x, y) &\longrightarrow \phi(x) \end{aligned} \tag{16}$$

where $\phi : M \longrightarrow P$ is a smooth map. We have the following.

Theorem 5. *Let $\phi : (M, g) \longrightarrow (P, \rho)$ be a smooth map. Then, the map $\widetilde{\Phi}$ defined by Equation (16) is $\mathcal{F}$-harmonic if and only if:*

$$\mathcal{F}'(\frac{e(\phi)}{\mu^2})d\phi(grad_g \ln \lambda^n) + \frac{1}{\mu^2}\{\mathcal{F}'(\frac{e(\phi)}{\mu^2})\tau(\phi) + \frac{1}{\mu^2}\mathcal{F}''(\frac{e(\phi)}{\mu^2})d\phi(grad_g(e(\phi)))\} = 0 \tag{17}$$

Proof. Let $\{e_i\}$ be an orthogonal frame on (M, g) and $\{f_\alpha\}$ be an orthogonal frame on (N, h). An orthogonal frame on a doubly warped product manifold $M\,_\mu\!\times_\lambda N$ is given by $\{\frac{1}{\mu}(e_i, 0_2), \frac{1}{\lambda}(0_1, f_\alpha)\}$. From the expression of tension field, we have:

$$\begin{aligned} \tau(\widetilde{\Phi}) &= trace_{\bar{g}} \nabla d\widetilde{\Phi} \\ &= \sum_{i=1}^{m} \left\{ \frac{1}{\mu} \nabla^P_{d\widetilde{\Phi}(e_i, 0_2)} d\widetilde{\Phi}(\frac{1}{\mu}e_i, 0_2) - d\widetilde{\Phi}(\frac{1}{\mu}\overline{\nabla}_{(e_i, 0_2)}(\frac{1}{\mu}e_i, 0_2)) \right\} \\ &+ \sum_{\alpha=1}^{n} \left\{ \frac{1}{\lambda} \nabla_{(0_1, f_\alpha)} d\widetilde{\Phi}(0_1, \frac{1}{\lambda}f_\alpha) - d\widetilde{\Phi}(\frac{1}{\lambda}\overline{\nabla}_{(0_1, f_\alpha)}(0_1, \frac{1}{\lambda}f_\alpha)) \right\} \end{aligned} \tag{18}$$

By Equation (5), we get:

$$\overline{\nabla}_{(e_i, 0_2)}(e_i, 0_2) = (\nabla^M_{e_i} e_i, 0_2) - \frac{1}{2} g(e_i, e_i)(0_1, grad_h\, \mu^2) \tag{19}$$

and:

$$\overline{\nabla}_{(0_1, f_\alpha)}(0_1, f_\alpha) = (0_1, \nabla^N_{f_\alpha} f_\alpha) - \frac{1}{2} h(f_\alpha, f_\alpha)(grad_g \lambda^2, 0_2) \tag{20}$$

By substituting Equations (19) and (20) into (18), we have:

$$\tau(\widetilde{\Phi}) = \frac{1}{\mu^2}\tau(\phi) + d\phi(grad_g \ln \lambda^n) \tag{21}$$

Let us now compute the $\mathcal{F}$-tension field of $\widetilde{\Phi}$. First, by Equation (4), we write down the energy density of $\widetilde{\Phi}$. We get:

$$
\begin{aligned}
e(\widetilde{\Phi}) &= \frac{1}{2} \mid d\widetilde{\Phi} \mid^2 \\
&= \frac{1}{2} \sum_{i=1}^{m} \rho\big(d\widetilde{\Phi}(\frac{1}{\mu}e_i, 0_2), d\widetilde{\Phi}(\frac{1}{\mu}e_i, 0_2)\big) + \sum_{\alpha=1}^{n} \rho\big(d\widetilde{\Phi}(0_1, \frac{1}{\lambda}f_\alpha), d\widetilde{\psi}(0_1, \frac{1}{\lambda}f_\alpha)\big) \\
&= \frac{1}{2\mu^2} \sum_{\alpha=1}^{n} \rho(d\phi(e_i), d\phi(e_i)) \\
&= \frac{e(\phi)}{\mu^2}
\end{aligned} \tag{22}
$$

By Equations (3), (21) and (22), the $\mathcal{F}$-tension field of $\widetilde{\Phi}$ can be obtained as follows:

$$
\begin{aligned}
\tau_{\mathcal{F}}(\widetilde{\Phi}) &= \mathcal{F}'(\frac{\mid d\widetilde{\Phi} \mid^2}{2})\tau(\widetilde{\Phi}) + d\widetilde{\Phi}(grad_{\bar{g}}\mathcal{F}'(\frac{\mid d\widetilde{\Phi} \mid^2}{2})) \\
&= \mathcal{F}'(\frac{e(\phi)}{\mu^2})\tau(\widetilde{\Phi}) + d\widetilde{\Phi}\Big(\sum_{i=1}^{m}(\frac{1}{\mu}e_i, 0_2)(\mathcal{F}'(\frac{e(\phi)}{\mu^2}))(\frac{1}{\mu}e_i, 0_2) \\
&+ \sum_{\alpha=1}^{n}(0_1, \frac{1}{\lambda}f_\alpha)(\mathcal{F}'(\frac{e(\phi)}{\mu^2}))(0_1, \frac{1}{\lambda}f_\alpha)\Big) \\
&= \mathcal{F}'(\frac{e(\phi)}{\mu^2})(\frac{1}{\mu^2}\tau(\phi) + d\phi(grad_g \ln \lambda^n)) + \frac{1}{\mu^4}\mathcal{F}''(\frac{e(\phi)}{\mu^2})d\phi(grad_g(e(\phi)))
\end{aligned} \tag{23}
$$

Thus, $\mathcal{F}$-harmonicity of $\widetilde{\Phi}$ implies that:

$$
\mathcal{F}'(\frac{e(\phi)}{\mu^2})d\phi(grad_g \ln \lambda^n) + \frac{1}{\mu^2}\Big\{\mathcal{F}'(\frac{e(\phi)}{\mu^2})\tau(\phi) + \frac{1}{\mu^2}\mathcal{F}''(\frac{e(\phi)}{\mu^2})d\phi(grad_g(e(\phi)))\Big\} = 0 \tag{24}
$$

This completes the proof. □

From Theorem 5, we obtain the following result.

Corollary 1. *Let $M^m\, {}_\mu\!\times_\lambda N^n$ be a doubly warped product manifold whose warping functions λ and μ are non-constant functions. The projection map $\pi_1 : (M^m\, {}_\mu\!\times_\lambda N^n, \bar{g}) \longrightarrow (M^m, g), \pi_1(x, y) = x$ is never an $\mathcal{F}$-harmonic map.*

Proof. By means of Equation (23), the $\mathcal{F}$-tension field of π_1 is given by:

$$
\tau_{\mathcal{F}}(\pi_1) = \mathcal{F}'(\frac{m}{2\mu^2})grad_g \ln \lambda^n \tag{25}
$$

Due to the fact that $F' > 0$ on $(0, \infty)$ and considering Equation (25), the $\mathcal{F}$-harmonicity of π_1 implies that λ is constant on M. However, this is in contradiction with $M\, {}_\mu\!\times_\lambda N$ being a non-trivial doubly warped product manifold. This completes the proof. □

Finally, we consider the map:

$$
\begin{aligned}
\widetilde{\Psi} : M\, {}_\mu\!\times_\lambda N &\longrightarrow N \\
(x, y) &\longrightarrow \psi(y)
\end{aligned} \tag{26}
$$

where $\psi : (N^n, h) \longrightarrow (N^n, h)$ is a smooth map. By calculating similarly to Equations (21) and (23), we get:

$$\tau_{\mathcal{F}}(\widetilde{\Psi}) = \frac{1}{\lambda^2}\left\{\mathcal{F}'(\frac{e(\psi)}{\lambda^2})\tau(\psi) + \frac{1}{\lambda^2}\mathcal{F}''(\frac{e(\psi)}{\lambda^2})d\psi(grad_h(e(\psi)))\right\}$$
$$+\mathcal{F}'(\frac{e(\psi)}{\lambda^2})d\psi(grad_h \ln \mu^m) \tag{27}$$

Thus, we have the following.

Theorem 6. *Let (M^m, g), (N^n, h) and (P^p, ρ) be Riemannian manifolds. Let $\psi : (N^n, h) \longrightarrow (P^p, \rho)$ be a smooth map. Then, the map $\widetilde{\Psi}$ defined by Equation (35) is $\mathcal{F}$-harmonic if and only if:*

$$\frac{1}{\lambda^2}\{\mathcal{F}'(\frac{e(\psi)}{\lambda^2})\tau(\psi) + \frac{1}{\lambda^2}\mathcal{F}''(\frac{e(\psi)}{\lambda^2})d\psi(grad_h(e(\psi))\} + \mathcal{F}'(\frac{e(\psi)}{\lambda^2})d\psi(grad_h \ln \mu^m) = 0 \tag{28}$$

As a similar proof of Corollary 1 and considering Equation (28), we get the following.

Corollary 2. *Let $M^m \,_{\mu}\times_{\lambda} N^n$ be a doubly warped product manifold whose warping functions λ and μ are non-constant functions. The projection map $\pi_2 : (M \,_{\mu}\times_{\lambda} N, \bar{g}) \longrightarrow (M^m, g), \pi_2(x, y) = y$ is never an $\mathcal{F}$-harmonic map.*

According to the proofs of Corollaries 1 and 2, we have the following.

Proposition 3. *Let $\overline{M} = M^m \,_{\mu}\times_{\lambda} N^n$ be a doubly warped product manifold. Then, the $\mathcal{F}$-harmonicity of the projection maps $\pi_1 : \overline{M} \longrightarrow M$ and $\pi_2 : \overline{M} \longrightarrow N$ implies that $\overline{M}$ is a direct product manifold.*

Proof. By Equations (17) and (28), it is obtained that the $\mathcal{F}$ -harmonicity of the projection maps π_1 and π_2 implies that λ and μ are constant functions on M and N, respectively. This completes the proof. □

Definition 1. *A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is said to be conformal if there exists a positive real function σ on M such that $\phi^* h = \sigma^2 g$. The function σ is called the dilation of ϕ [33].*

Proposition 4. *The tension field of a conformal map $\phi : (M^m, g) \to (N^n, h)$ with dilation σ, is given by [33]:*

$$\tau(\phi) = (2 - n)\; d\phi(grad\;\ln \sigma) - (m - n) d\phi(H) \tag{29}$$

where H is the mean curvature vector field of the fibers.

Note that any conformal map $\phi : (M^n, g) \to (N^n, h)$ between manifolds of the same dimension $(n > 2)$ is a local conformal diffeomorphism [3].

Now, an example of non-harmonic $\mathcal{F}$-harmonic maps is given.

Example 1. *Let $N = \mathbb{R}^4 - \{0\}$ be an Euclidean manifold and $\psi : N \longrightarrow N$ be a smooth map defined by:*

$$\psi(y) = \frac{y}{\mid y \mid^2} \tag{30}$$

and the map ψ is conformal with dilation:

$$\sigma = \frac{1}{\mid y \mid^2} = \frac{1}{r^2} \tag{31}$$

where $r := \mid y \mid$. Due to the fact that ψ is a conformal map between manifolds of the same dimension and using Equations (29) and (31), one can easily check that:

$$\tau(\psi) = 4 d\psi(grad \ln r) \tag{32}$$

and:

$$e(\psi) = \frac{2}{r^4} \tag{33}$$

Let $\mathbb{S}^3$ be the three-dimensional Euclidean unit sphere and:

$$\begin{aligned} \widetilde{\Psi} : \mathbb{S}^3 \,{}_{\mu}\!\times_{\lambda} N &\longrightarrow N \\ (x,y) &\longmapsto \psi(y) \end{aligned} \tag{34}$$

where $\mu = r^{\frac{4}{3}} e^{\frac{2}{3r^4}}$ and $\lambda = 1$. By setting $\mathcal{F}(t) = e^t$ and substituting Equations (32) and (33) in (28), it could be concluded that $\widetilde{\Psi}$ is $\mathcal{F}$-harmonic.

5. $\mathcal{F}$-Harmonicity Conditions of $\phi : (M\,{}_{\mu}\!\times_{\lambda} N, \bar{g}) \longrightarrow (P\,{}_{\beta}\!\times_{\alpha} Q, \tilde{g})$

In this section, we study $\mathcal{F}$-harmonic and conformal $\mathcal{F}$-harmonic maps between doubly warped product manifolds. Let $(M^m, g), (N^n, h), (P^p, \rho)$ and (Q^q, ρ) be Riemannian manifolds, the functions $\lambda \in C^\infty(M), \mu \in C^\infty(N), \alpha \in C^\infty(P)$ and $\beta \in C^\infty(Q)$ are positive and $(M\,{}_{\mu}\!\times_{\lambda} N, \bar{g})$ and $(P\,{}_{\beta}\!\times_{\alpha} Q, \tilde{g})$ are doubly warped product manifolds. We consider the map:

$$\begin{aligned} \widehat{\Phi} : (M\,{}_{\mu}\!\times_{\lambda} N, \bar{g}) &\longrightarrow (P\,{}_{\beta}\!\times_{\alpha} Q, \tilde{g}) \\ (x,y) &\longrightarrow (\phi(x), \psi(y)) \end{aligned} \tag{35}$$

where $\phi : (M, g) \longrightarrow (P, \rho)$ and $\psi : (N, h) \longrightarrow (Q, \rho)$ are smooth maps.

Theorem 7. *Let $\phi : (M, g) \longrightarrow (P, \rho)$ and $\psi : (N, h) \longrightarrow (Q, \rho)$ be smooth maps and let $(M\,{}_{\mu}\!\times_{\lambda} N, \bar{g})$ and $(P\,{}_{\beta}\!\times_{\alpha} Q, \tilde{g})$ be doubly warped product manifolds. Then, the $\mathcal{F}$-tension field of the map $\widehat{\Phi}$, defined by Equation (35), is given by:*

$$\begin{aligned} \tau_{\mathcal{F}}(\widehat{\Phi}) = {} & \mathcal{F}'(l)\{(\frac{1}{\mu^2}\tau(\phi) + d\phi(grad_g \ln \lambda^n) - \frac{1}{\lambda^2}e(\psi)(grad_\rho \alpha^2) \circ \phi, 0_2) \\ & + (0_1, \frac{1}{\lambda^2}\tau(\psi) + d\psi(grad_h \ln \mu^m) - \frac{1}{\mu^2}e(\phi)(grad_\rho \beta^2) \circ \psi)\} \\ & + \mathcal{F}''(l)\{(\frac{\beta^2 \circ \psi}{\mu^4} d\phi(grad_g e(\phi)) + \frac{e(\psi)}{\mu^2} d\phi(grad_g(\frac{\alpha^2 \circ \phi}{\lambda^2})), 0_2) \\ & + (0_1, \frac{\alpha^2 \circ \phi}{\lambda^4} d\psi(grad_h e(\psi)) + \frac{e(\phi)}{\lambda^2} d\psi(grad_h(\frac{\beta^2 \circ \psi}{\mu^2})))\} \end{aligned} \tag{36}$$

where $e(\phi)$ and $e(\psi)$ are energy densities of ϕ and ψ, respectively, and $l := \frac{\beta^2 \circ \psi}{\mu^2} e(\phi) + \frac{\alpha^2 \circ \phi}{\lambda^2} e(\psi)$.

Proof. Let $\{e_i\}$ be an orthogonal frame on (M, g) and $\{f_\alpha\}$ be an orthogonal frame on (N, h). An orthogonal frame on the doubly warped product manifold $M\,{}_{\mu}\!\times_{\lambda} N$ is given by $\{\frac{1}{\mu}(e_i, 0_2), \frac{1}{\lambda}(0_1, f_\alpha)\}$. Denote the Levi–Civita connections on $(M^m\,{}_{\mu}\!\times_{\lambda} N^n, \bar{g})$ and $(P^p\,{}_{\beta}\!\times_{\alpha} Q^q, \tilde{g})$ by $\overline{\nabla}$ and $\widetilde{\nabla}$, respectively. By using the expression of tension field, we get:

$$
\begin{aligned}
\tau(\widehat{\Phi}) &= trace_{\tilde{g}} \nabla d\widehat{\Phi} \\
&= \sum_{i=1}^{m} \left\{ \frac{1}{\mu} \widetilde{\nabla}_{d\widehat{\Phi}(e_i,0_2)} d\widehat{\Phi}(\frac{1}{\mu}e_i, 0_2) - d\widehat{\Phi}(\frac{1}{\mu}\overline{\nabla}_{(e_i,0_2)}(\frac{1}{\mu}e_i, 0_2)) \right\} \\
&+ \sum_{\gamma=1}^{n} \left\{ \frac{1}{\lambda} \widetilde{\nabla}_{d\widehat{\Phi}(0_1,f_\gamma)} d\widehat{\Phi}(0_1, \frac{1}{\lambda}f_\gamma) - d\widehat{\Phi}(\frac{1}{\lambda}\overline{\nabla}_{(0_1,f_\gamma)}(0_1, \frac{1}{\lambda}f_\gamma)) \right\} \\
&= \frac{1}{\mu^2} \sum_{i=1}^{m} \left\{ \left(\nabla^P_{d\phi(e_i)} d\phi(e_i) - d\phi(\nabla^M_{e_i} e_i), 0_2 \right) \right. \\
&- \left. \frac{1}{2} \left(0_1, \rho(d\phi(e_i), d\phi(e_i))(grad_\rho \beta^2) \circ \psi - g(e_i, e_i) d\psi(grad_h \mu^2) \right) \right\} \\
&+ \frac{1}{\lambda^2} \sum_{\gamma=1}^{n} \left\{ \left(0_1, \nabla^Q_{d\psi(f_\gamma)} d\psi(f_\gamma) - d\psi(\nabla^N_{f_\gamma} f_\gamma) \right) - \frac{1}{2} \left(\rho(d\psi(f_\gamma), d\psi(f_\gamma))(grad_\rho \alpha^2) \circ \phi \right. \right. \\
&- \left. \left. h(f_\gamma, f_\gamma) d\phi(grad_g \lambda^2), 0_2 \right) \right\} \\
&= (\frac{1}{\mu^2} \tau(\phi) + d\phi(grad_g \ln \lambda^n) - \frac{1}{\lambda^2} e(\psi)(grad_\rho \alpha^2) \circ \phi, 0_2) \\
&+ (0_1, \frac{1}{\lambda^2} \tau(\psi) + d\psi(grad_h \ln \mu^m) - \frac{1}{\mu^2} e(\phi)(grad_\rho \beta^2) \circ \psi) \qquad (37)
\end{aligned}
$$

Moreover, the energy density of $\widehat{\Phi}$ can be calculated as follows:

$$
\begin{aligned}
e(\widehat{\Phi}) &= \frac{\mid d\widehat{\Phi} \mid^2}{2} \\
&= \frac{1}{2\mu^2} \sum_{i=1}^{m} \tilde{g}(d\widehat{\Phi}(e_i, 0_2), d\widehat{\Phi}(e_i, 0_2)) + \frac{1}{2\lambda^2} \sum_{\gamma=1}^{n} \tilde{g}(d\widehat{\Phi}(0_1, f_\gamma), d\tilde{\Phi}(0_1, f_\gamma)) \\
&= \frac{\beta^2 \circ \psi}{\mu^2} e(\phi) + \frac{\alpha^2 \circ \phi}{\lambda^2} e(\psi) \qquad (38)
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
& d\widehat{\Phi}(grad_{\tilde{g}} \mathcal{F}'(\frac{\mid d\widehat{\Phi} \mid^2}{2})) \\
&= \mathcal{F}''(\frac{\mid d\widehat{\Phi} \mid^2}{2}) d\widehat{\Phi}(grad_{\tilde{g}}(\frac{\mid d\widehat{\Phi} \mid^2}{2})) \\
&= \mathcal{F}''(\frac{\beta^2 \circ \psi}{\mu^2} e(\phi) + \frac{\alpha^2 \circ \phi}{\lambda^2} e(\psi)) \left\{ \left(\frac{\beta^2 \circ \psi}{\mu^4} d\phi(grad_g e(\phi)) + \frac{e(\psi)}{\mu^2} d\phi(grad_g(\frac{\alpha^2 \circ \phi}{\lambda^2})), 0_2 \right) \right. \\
&+ \left. \left(0_1, \frac{\alpha^2 \circ \phi}{\lambda^4} d\psi(grad_h e(\psi)) + \frac{e(\phi)}{\lambda^2} d\psi(grad_h(\frac{\beta^2 \circ \psi}{\mu^2})) \right) \right\} \qquad (39)
\end{aligned}
$$

By Equations (3), (37), (38) and (39), we have:

$$
\begin{aligned}
\tau_{\mathcal{F}}(\widehat{\Phi}) &= \mathcal{F}'(\frac{|\, d\widehat{\Phi}\,|^2}{2})\tau(\widehat{\Phi}) + d\widehat{\Phi}(grad_{\bar{g}}\mathcal{F}'(\frac{|\, d\widehat{\Phi}\,|^2}{2})) \\
&= \mathcal{F}'(l)\{(\frac{1}{\mu^2}\tau(\phi) + d\phi(grad_g \ln\lambda^n) - \frac{1}{\lambda^2}e(\psi)(grad_\rho\alpha^2)\circ\phi, 0_2) \\
&+ (0_1, \frac{1}{\lambda^2}\tau(\psi) + d\psi(grad_h \ln\mu^m) - \frac{1}{\mu^2}e(\phi)(grad_\rho\beta^2)\circ\psi)\} \\
&+ \mathcal{F}''(l)\{(\frac{\beta^2\circ\psi}{\mu^4}d\phi(grad_g e(\phi)) + \frac{e(\psi)}{\mu^2}d\phi(grad_g(\frac{\alpha^2\circ\phi}{\lambda^2})), 0_2) \\
&+ (0_1, \frac{\alpha^2\circ\phi}{\lambda^4}d\psi(grad_h e(\psi)) + \frac{e(\phi)}{\lambda^2}d\psi(grad_h(\frac{\beta^2\circ\psi}{\mu^2})))\}
\end{aligned} \tag{40}
$$

This completes the proof. □

From Theorem 7, we get the following:

Remark 1. *Let $\psi : N \longrightarrow N$ be a harmonic map. The $\mathcal{F}$-tension fields of:*

$$
\begin{aligned}
\widehat{\Psi} : (M^m\, {}_\mu\!\times_\lambda N^n, \bar{g}) &\longrightarrow (M\times N, g\oplus h) \\
(x,y) &\longrightarrow (x, \psi(y))
\end{aligned} \tag{41}
$$

is given by:

$$
\begin{aligned}
\tau_{\mathcal{F}}(\widehat{\Psi}) &= \mathcal{F}'(s_1)(grad_g \ln\lambda^n, d\psi(grad_h \ln\mu^m)) \\
&+ \frac{\mathcal{F}''(s_1)}{\lambda^2}(0_1, d\psi(grad_h(\frac{e(\psi)}{\lambda^2} + \frac{m}{2\mu^2})))
\end{aligned} \tag{42}
$$

where $s_1 := \frac{m}{2\mu^2} + \frac{1}{\lambda^2}e(\psi)$.

According to the above Remark, we have the following.

Proposition 5. *Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimension m and n $(n > 2)$, respectively, and let $\lambda \in C^\infty(M)$ and $\mu \in C^\infty(N)$ be non-constant positive functions. Let $\psi : N \longrightarrow N$ be a conformal map with dilation σ. Then, the map $\widehat{\Psi}$, defined by Equation (41), is an $\mathcal{F}$-harmonic map if and only if λ, μ and σ are non constant solutions of the following equations:*

$$
\mathcal{F}'(s)\lambda^2\mu^2 - \mathcal{F}''(s)\sigma^2 = 0 \tag{43}
$$

and:

$$
\mathcal{F}'(s) grad_h \ln(\sigma^{\frac{2-n}{\lambda^2}}\mu^m) + \mathcal{F}''(s) grad_h(\frac{n\sigma^2}{2\lambda^4} + \frac{m}{2\lambda^2\mu^2}) = 0 \tag{44}
$$

where $s := \frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2}$.

Proof. By calculating similarly to Equation (37) and considering Equation (29), we have:

$$
\begin{aligned}
\tau(\widehat{\Psi}) &= \frac{1}{\mu}\sum_{i=1}^{m}\left\{\nabla_{Id_{TM}\times d\psi(\frac{1}{\mu}e_i,0_2)} Id_{TM}\times d\psi(\frac{1}{\mu}e_i, 0_2) - Id_{TM}\times d\psi(\overline{\nabla}_{(\frac{1}{\mu}e_i,0_2)}(\frac{1}{\mu}e_i, 0_2))\right\} \\
&+ \sum_{\gamma=1}^{n}\left\{\nabla_{Id_{TM}\times d\psi(0_1,\frac{1}{\lambda}f_\gamma)} Id_{TM}\times d\psi(0_1, \frac{1}{\lambda}f_\gamma) - Id_{TM}\times d\psi(\overline{\nabla}_{(0_1,\frac{1}{\lambda}f_\gamma)}(0_1, \frac{1}{\lambda}f_\gamma))\right\} \\
&= (grad_g \ln\lambda^n, d\psi(grad_h \ln(\sigma^{\frac{2-n}{\lambda^2}}\mu^m)))
\end{aligned} \tag{45}
$$

One can easily check that the energy density of $\widehat{\Psi}$ can be obtained as follows:

$$e(\widehat{\Psi}) = \frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2} \tag{46}$$

From Equations (3), (45) and (46), the $\mathcal{F}$-tension field of $\widehat{\Psi}$ can be calculated as follows:

$$\begin{aligned}\tau_{\mathcal{F}}(\widehat{\Psi}) &= \mathcal{F}'(\frac{\mid d\widehat{\Psi}\mid^2}{2})(grad_g \ln\lambda^n, d\psi(grad_h \ln(\sigma^{\frac{2-n}{\lambda^2}}\mu^m))) \\ &+ \mathcal{F}''(\frac{\mid d\widehat{\Psi}\mid^2}{2})d\widehat{\Psi}(grad_{\tilde{g}}(\frac{\mid d\widehat{\Psi}\mid^2}{2})) \\ &= \mathcal{F}'(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})(grad_g \ln\lambda^n, d\psi(grad_h \ln(\sigma^{\frac{2-n}{\lambda^2}}\mu^m))) \\ &+ \mathcal{F}''(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})(-\frac{\sigma^2}{\lambda^2\mu^2}grad_g \ln\lambda^n, d\psi(grad_h(\frac{n\sigma^2}{2\lambda^4} + \frac{m}{2\lambda^2\mu^2})))\end{aligned} \tag{47}$$

Therefore, the $\mathcal{F}$-harmonicity of $\widehat{\Psi}$ implies that:

$$(\mathcal{F}'(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})\lambda^2\mu^2 - \mathcal{F}''(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})\sigma^2)grad_g \ln\lambda^n = 0 \tag{48}$$

and:

$$\mathcal{F}'(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})grad_h \ln(\sigma^{\frac{2-n}{\lambda^2}}\mu^m) + \mathcal{F}''(\frac{m}{2\mu^2} + \frac{n\sigma^2}{2\lambda^2})grad_h(\frac{n\sigma^2}{2\lambda^4} + \frac{m}{2\lambda^2\mu^2}) \in Ker(d\psi) \tag{49}$$

Due to the fact that ψ is a conformal map between equidimensional manifolds and λ is non constant, the last two equations imply Equations (43) and (44). This completes the proof. □

Author Contributions: This work is a part of Seyed Mehdi Kazemi Torbaghan's Ph.D's Thesis which is done under supervision of Morteza Mirmohammad Rezaii.

Conflicts of Interest: The authors declare no conflict of interest.

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