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Lie Symmetries, Optimal System and Invariant Reductions to a Nonlinear Timoshenko System

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Abstract: Lie symmetries and their Lie group transformations for a class of Timoshenko systems are presented. The class considered is the class of nonlinear Timoshenko systems of partial differential equations (PDEs). An optimal system of one-dimensional sub-algebras of the corresponding Lie algebra is derived. All possible invariant variables of the optimal system are obtained. The corresponding reduced systems of ordinary differential equations (ODEs) are also provided. All possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations are given. As an application, explicit solutions of the system are given where the beam is hinged at one end and free at the other end.

Keywords: Timoshenko beam system; similarity reduction; optimal system; invariant solutions

1. Introduction

The problem of transverse vibration of beams is of importance in many engineering problems. Some of the early studies were based on the Euler–Bernoulli model, which takes into account the bending moment and the lateral displacement. Later models were based on adding shear or rotary inertia effect. In [1,2], Timoshenko proposed taking into consideration the shear, as well as the rotation effects, which proved to be suitable for non-slender beams and high frequency vibrations. Dimplekumar et al. [3] discuss the mathematical modeling for the mechanics of a solid using the distribution theory of Schwartz to the beam bending differential equations; the governing differential equations of a Timoshenko beam with jump discontinuities in slope, deflection, flexural stiffness and shear stiffness were obtained in this paper in the space of generalized functions. Mustafa and Messaoudi [4] and Fatiha [5] studied stability of the following Timoshenko system with the nonlinear frictional damping term in one equation:

$$\begin{aligned}\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\ \rho_2 \psi_{tt} - EI\psi_{xx} + k(\varphi_x + \psi) + \chi(\psi_t) &= 0,\end{aligned}\tag{1}$$

where the functions φ, ψ depend on $(t, x) \in (0, \infty) \times (0, L)$ and model the vertical displacement of a beam and the rotation angle of a filament, respectively. The shear angle is $\psi - \varphi_x$, and L denotes the length of the beam. The physical parameters appearing in the system are ρ_1 , the mass density per unit length, ρ_2 , the polar moment of inertia of a cross-section, E , Young's modulus of elasticity, I , the moment of inertia of the cross-section, and k , the shear modulus. In this paper the beam is assumed to be uniform, that is the physical parameters ρ_1, ρ_2, EI and k are all positive constants, and χ is an arbitrary nonlinear function of the damping term ψ_t .

We have solved this system by determining its symmetries and performing a Lie theoretic analysis as detailed below.

The classification of group invariant solutions of differential equations by means of the optimal systems is one of the main applications of Lie group analysis to differential equations. The main idea behind the method is discussed in detail in Ovsiannikov [6], Ibragimov [7], Olver [8] and Hydon [9]. We can always construct a family of group invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there is an infinite number of such subgroups, it is not possible to list all of the group invariant solutions. An effective and systematic way of classifying these group invariant solutions is to obtain optimal systems of subalgebras of the Lie symmetry algebra. This leads to non-similar invariant solutions under symmetry transformations.

Recall that one calls a list $\{\tilde{v}_\alpha\}_{\alpha \in \mathcal{A}}$ a one-dimensional optimal system, if it satisfies the conditions: (1) completeness, i.e., any one-dimensional subalgebra is equivalent to some \tilde{v}_α ; (2) inequivalence, i.e., \tilde{v}_α and \tilde{v}_β are inequivalent for distinct α and β . In this paper, we have used all of the basic invariants to determine the conjugacy classes of one-dimensional subalgebras and at the same time shown that these representatives are comprehensive and mutually inequivalent. This is done by using the formula given in Section 3 of [10]. This formula is more or less a direct consequence of definitions. For the convenience of the reader, a detailed explanation is given in Section 2. The idea of using invariant functions to determine conjugacy classes is also discussed in [11,12].

Here is a brief outline of the paper. In Section 2, a formula for computing invariants in the adjoint representation is given. In Section 3, Lie symmetries and their Lie group transformations for the Timoshenko system are presented. In Section 4, an optimal system of one-dimensional subalgebras of the corresponding Lie algebra is derived. In Section 5, all possible invariant variables of the optimal system are presented; moreover, the corresponding reduced systems of ordinary differential equations (ODEs) are also given. As an illustration, some invariant solutions are given explicitly by solving the reduced systems of ODEs. Furthermore, all possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations are given. A hinged-free beam has been considered in Example 1, with a constant torque control at the hinged end and a linear force control at the free end.

2. Invariants in the Adjoint Representations

The following formula is stated without a proof in [10]. Let X_1, \dots, X_N be a basis of the Lie algebra \mathcal{L} and $\omega_1, \dots, \omega_N$ be the corresponding dual basis of \mathcal{L}^* . Let X be an element of \mathcal{L} and $Y = \sum_{i=1}^N x_i X_i$ be a general element of \mathcal{L} . Then, $X_{\mathcal{L}}$, the fundamental vector field corresponding to X in the adjoint representation, is given by:

$$X_{\mathcal{L}}(Y) = \sum_{1 \leq i, j \leq N} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j}.$$

To see this, recall that a vector field V on an open set Ω of \mathbb{R}^N assigns to each point p of Ω a vector $V(p) = (V_1(p), \dots, V_N(p))$. Therefore, if e_1, \dots, e_N is a basis of \mathbb{R}^N and $\omega_1, \dots, \omega_N$ is the dual basis, then:

$$V(p) = \sum_{i=1}^N \omega_i(V(p)) e_i.$$

The vector field V gives rise to the differential operator of taking the directional derivative of any function f in the direction of $V(p)$. We identify V with the operator $\sum_{j=1}^N \omega_j(V(p)) \frac{\partial}{\partial x_j} |_p$.

Specializing to the adjoint representation, the fundamental vector field $X_{\mathcal{L}}$ is given by:

$$X_{\mathcal{L}}(Y) = \frac{\partial}{\partial t} \Big|_{t=0} e^{t \text{ad} X}(Y) = [X, Y].$$

Therefore, if $Y = \sum_{i=1}^N x_i X_i$ then:

$$X_{\mathcal{L}}(Y) = \sum_{1 \leq i, j \leq N} x_i \omega_j([X, X_i]) X_j = \sum_{1 \leq i, j \leq N} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j}.$$

Application to Computations of Invariants

Let f be a function on \mathcal{L} , which is invariant in the adjoint representation of \mathcal{L} . This means that $f(\exp ad tX)(Y) = f(Y)$ for all X in the Lie algebra \mathcal{L} and all reals t . Therefore, if we write $f(x_1 X_1 + \dots + x_n X_n) = f(x_1, \dots, x_n)$, then $\sum_{1 \leq i, j \leq N} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j} f = 0$. Letting X run through the basis $\{X_1, \dots, X_N\}$, we get the system:

$$\sum_{1 \leq i, j \leq N} x_i \omega_j([X_k, X_i]) \frac{\partial f}{\partial x_j} = 0, \quad k = 1, \dots, N. \quad (2)$$

System (2) needs to be solved only for a set of generators of the Lie algebra \mathcal{L} . The solution of this linear system of partial differential equations (PDEs) gives the invariants in the adjoint representations.

For a solvable Lie algebra \mathcal{L} , it is best to work with the central series for the commutator \mathcal{L}' of \mathcal{L} , which is a nilpotent subalgebra. Therefore, one can find a chain of subalgebras, each an ideal in its successor and each of codimension one. Using this chain, one is essentially dealing with invariants of a single vector field, namely if we have found the basic invariants for a subalgebra S , which is an ideal of codimension one in \tilde{S} , then \tilde{S}/S operates on invariants of S , and this gives the invariants of \tilde{S} . Having obtained the invariants of \mathcal{L}' , one can again reach \mathcal{L} by a series of ideals, each of codimension one, to determine the basic invariants of \mathcal{L} . Such a series also gives a factorization of the group $G = \langle \exp(Ad X) : X \in \mathcal{L} \rangle$ in terms of one-dimensional subgroups.

The advantage of working with invariant functions is that if f is an invariant of \mathcal{L} in the adjoint representation, then the adjoint group operates on the level sets, as well as sub-level sets of f , and one can systematically use invariants to work on level and sub-level sets of low dimensions to determine all of the conjugacy classes of one-dimensional subalgebras. This procedure will be used in Section 4 to determine one-dimensional optimal systems of the nonlinear damped Timoshenko system. This in turn will be used in Section 5 to determine optimal reduction.

3. Lie's Infinitesimal Method

Lie symmetry method [13,14], described extensively in the literature, is invoked in the sequel to apply Lie symmetry analysis for Timoshenko System (1).

Using the invariance condition of the system of PDEs (1):

$$\begin{aligned} X^{[2]}(\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x)|_{(1)} &= 0, \\ X^{[2]}(\rho_2 \psi_{tt} - EI\psi_{xx} + k(\varphi_x + \psi) + \chi(\psi_t))|_{(1)} &= 0, \end{aligned} \quad (3)$$

where $X^{[2]}$ is the second prolongation of the vector field differential operator:

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial \varphi} + \eta^2 \frac{\partial}{\partial \psi}.$$

Comparing coefficients of the various derivatives of the dependent variables φ and ψ yields an over-determined linear PDE system. Carrying out the Janet basis of this over-determined system in the degree reverse lexicographical ordering as $\psi > \varphi > x > t$ and $\eta_2 > \eta_1 > \xi_2 > \xi_1$ by using the command "JanetBasis" involved in the Maple package "Janet" [15] leads to the following determining equations:

$$\begin{aligned} &[\eta_\psi^1, \xi_\psi^2, \xi_\psi^1, \eta_\varphi^2, \eta_\varphi^1 - \eta_\psi^2, \xi_\varphi^2, \xi_\varphi^1, \xi_x^2, \xi_x^1, \xi_t^2, \xi_t^1, \eta_{\psi,\psi}^2, \eta_{\psi,x}^2, \eta_{\psi,t}^2, \eta_{\psi,t}^1, \eta_{\varphi,x}^2, \eta_{\varphi,t}^2, \eta_{\varphi,t}^1] \\ &k\eta_x^1 + k\eta^2 + (\psi_t \chi' - k\psi - \chi)\eta_\psi^2 + \eta_t^2 \chi' - EI\eta_{x,x}^2 + \rho_2 \eta_{t,t}^2, \quad \eta_{x,x}^1 - \frac{\rho_1}{k} \eta_{t,t}^1 + \eta_x^2]. \end{aligned} \quad (4)$$

The command “Denominators” involved in the Maple package “Janet” returns the functions by which the Janet basis algorithm had to divide. These functions may give rise to new cases. The command “Denominators” tells us that Janet basis of the determining Equations (4) is produced when $k \neq \frac{\rho_1 EI}{\rho_2}$. However, Janet basis of the determining equations for this case is equivalent to (4). The solution to the determining Equation (4) is:

$$\zeta^1 = c_1, \quad \zeta^2 = c_2, \quad \eta^1 = c_0 \varphi + f(t, x), \quad \eta^2 = c_0 \psi + g(t, x) \quad (5)$$

where $f(t, x)$ and $g(t, x)$ satisfy the following system:

$$\begin{aligned} \rho_1 f_{tt} - k f_{xx} - k g_x &= 0, \\ k f_x + k g + g_t \chi'(\psi_t) + \rho_2 g_{tt} - EI g_{xx} &= c_0 (\chi(\psi_t) - \psi_t \chi'(\psi_t)). \end{aligned} \quad (6)$$

Differentiating the second equation of System (6) with respect to ψ_t gives:

$$(g_t + c_0 \psi_t) \chi''(\psi_t) = 0. \quad (7)$$

Since we are concerned with nonlinear damping term, $\chi''(\psi_t) \neq 0$, which implies $g = g(x)$ and $c_0 = 0$. Solving System (6) yields:

$$\begin{aligned} f(t, x) &= c_3 + c_4 t - c_5 x - \frac{1}{2} c_6 x^2 + \left(-\frac{1}{6} x^3 + \frac{EI}{k} x\right) c_7 + \left(-\frac{1}{24} x^4 + \frac{EI}{2k} x^2 + \frac{EI}{2\rho_1} t^2\right) c_8, \\ g(x) &= c_5 + c_6 x + \frac{1}{2} c_7 x^2 + \frac{1}{6} c_8 x^3. \end{aligned} \quad (8)$$

Thus, the Lie point symmetry generators admitted by System (1) are given by:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_5 &= -x \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \psi}, \\ X_2 &= \frac{\partial}{\partial x}, & X_6 &= -\frac{1}{2} x^2 \frac{\partial}{\partial \varphi} + x \frac{\partial}{\partial \psi}, \\ X_3 &= \frac{\partial}{\partial \varphi}, & X_7 &= \left(-\frac{1}{6} x^3 + \frac{EI}{k} x\right) \frac{\partial}{\partial \varphi} + \frac{1}{2} x^2 \frac{\partial}{\partial \psi}, \\ X_4 &= t \frac{\partial}{\partial \varphi}, & X_8 &= \left(-\frac{1}{24} x^4 + \frac{EI}{2k} x^2 + \frac{EI}{2\rho_1} t^2\right) \frac{\partial}{\partial \varphi} + \frac{1}{6} x^3 \frac{\partial}{\partial \psi}. \end{aligned} \quad (9)$$

In order to obtain the group transformations, which are generated by the resulting infinitesimal symmetry generators (9), we need to solve the following initial value problem:

$$\begin{aligned} \frac{d\tilde{t}(\epsilon)}{d\epsilon} &= \zeta^1(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), & \tilde{t}(0) &= t, & \left| \frac{d\tilde{\varphi}(\epsilon)}{d\epsilon} &= \eta^1(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), & \tilde{\varphi}(0) &= \varphi, \\ \frac{d\tilde{x}(\epsilon)}{d\epsilon} &= \zeta^2(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), & \tilde{x}(0) &= x, & \left| \frac{d\tilde{\psi}(\epsilon)}{d\epsilon} &= \eta^2(\tilde{t}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), & \tilde{\psi}(0) &= \psi. \end{aligned} \quad (10)$$

The one parameter group $G_i(\epsilon) = e^{\epsilon X_i}$ generated by X_i for $i = 1, \dots, 8$ is as follows:

$$\begin{aligned} G_1(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t + \epsilon, x, \varphi, \psi), & G_5(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi - \epsilon x, \psi + \epsilon), \\ G_2(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x + \epsilon, \varphi, \psi), & G_6(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi - \frac{1}{2} \epsilon x^2, \psi + \epsilon x), \\ G_3(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi + \epsilon, \psi), & G_7(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi - \frac{1}{6} \epsilon x^3 + \frac{EI}{k} \epsilon x, \psi + \frac{1}{2} \epsilon x^2), \\ G_4(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi + \epsilon t, \psi), & G_8(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi - \frac{1}{24} \epsilon x^4 + \frac{EI}{2k} \epsilon x^2 + \frac{EI}{2\rho_1} \epsilon t^2, \psi + \frac{1}{6} \epsilon x^3). \end{aligned} \quad (11)$$

Theorem 1. If $\varphi = f(t, x)$ and $\psi = g(t, x)$ is a solution of Timoshenko System (1), then so is:

$$\begin{aligned} \varphi &= f(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4(t + \epsilon_1) - \epsilon_5(x + \epsilon_2) - \epsilon_6 \left(\epsilon_2 x + \frac{1}{2} \epsilon_2^2 + \frac{1}{2} x^2 \right) - \epsilon_7 \left(\frac{1}{2} x \epsilon_2^2 \right. \\ &\quad \left. - \left(\frac{EI}{k} - \frac{1}{2} x^2 \right) \epsilon_2 + \frac{1}{6} x^3 - \frac{EI}{k} x + \frac{1}{6} \epsilon_2^3 \right) - \epsilon_8 \left(\frac{1}{6} x \epsilon_2^3 - \frac{EI}{2k} x^2 + \epsilon_2^2 \left(\frac{1}{4} x^2 - \frac{EI}{2k} \right) \right. \\ &\quad \left. + \frac{1}{24} x^4 - \epsilon_2 \left(\frac{EI}{k} x - \frac{1}{6} x^3 \right) - \frac{EI}{\rho_1} \epsilon_1 t + \frac{1}{24} \epsilon_2^4 - \frac{EI}{2\rho_1} \epsilon_1^2 - \frac{EI}{2\rho_1} t^2 \right), \\ \psi &= g(t + \epsilon_1, x + \epsilon_2) + \epsilon_5 + \epsilon_6(x + \epsilon_2) + \frac{1}{2} \epsilon_7(x + \epsilon_2)^2 + \frac{1}{6} \epsilon_8(x + \epsilon_2)^3, \end{aligned} \quad (12)$$

where $\{\epsilon_i\}_{i=1}^8$ are arbitrary real numbers.

Proof. The eight-parameter group:

$$G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ G_6(\epsilon_6) \circ G_5(\epsilon_5) \circ G_4(\epsilon_4) \circ G_3(\epsilon_3) \circ G_2(\epsilon_2) \circ G_1(\epsilon_1)$$

generated by X_i for $i = 1, \dots, 8$, can be given by the composition of the transformations (11) as follows:

$$\begin{aligned} G : (t, x, \varphi, \psi) \mapsto & \left(t + \epsilon_1, x + \epsilon_2, \phi + \epsilon_3 + \epsilon_4(t + \epsilon_1) - \epsilon_5(x + \epsilon_2) - \epsilon_6 \left(\epsilon_2 x + \frac{1}{2} \epsilon_2^2 + \frac{1}{2} x^2 \right) \right. \\ & - \epsilon_7 \left(\frac{1}{2} x \epsilon_2^2 - \left(\frac{EI}{k} - \frac{1}{2} x^2 \right) \epsilon_2 + \frac{1}{6} x^3 - \frac{EI}{k} x + \frac{1}{6} \epsilon_2^3 \right) - \epsilon_8 \left(\frac{1}{6} x \epsilon_2^3 - \frac{EI}{2k} x^2 \right. \\ & + \left(\frac{1}{4} x^2 - \frac{EI}{2k} \right) \epsilon_2^2 + \frac{1}{24} x^4 - \left(\frac{EI}{k} x - \frac{1}{6} x^3 \right) \epsilon_2 - \frac{EI}{\rho_1} \epsilon_1 t + \frac{1}{24} \epsilon_2^4 - \frac{EI}{2\rho_1} \epsilon_1^2 \\ & \left. - \frac{EI}{2\rho_1} t^2 \right), \psi + \epsilon_5 + \epsilon_6(x + \epsilon_2) + \frac{1}{2} \epsilon_7(x + \epsilon_2)^2 + \frac{1}{6} \epsilon_8(x + \epsilon_2)^3 \Big). \end{aligned} \quad (13)$$

□

4. Optimal System of One-Dimensional Sub-Algebras of the Nonlinear Damped Timoshenko System

In this section, we give the one-dimensional optimal system for the algebra with basis (9). In order to find the optimal system, one needs to classify the one-dimensional sub-algebras under the action of the adjoint representation. We follow the algorithm explained by Ibragimov [7] and Olver [8].

The non-zero commutators of the Lie algebra \mathcal{L}^8 with basis (9) are given by:

$$\begin{aligned} [X_1, X_4] &= X_3, & [X_1, X_8] &= \frac{EI}{\rho_1} X_4, & [X_2, X_5] &= -X_3, \\ [X_2, X_6] &= X_5, & [X_2, X_7] &= X_6 + \frac{EI}{k} X_3, & [X_2, X_8] &= X_7. \end{aligned} \quad (14)$$

Recall that the adjoint representation is given by:

$$Ad(\exp(\epsilon X_i)).X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \frac{\epsilon^3}{3!}[X_i, [X_i, [X_i, X_j]]] + \dots$$

The Lie algebra \mathcal{L}^8 is solvable, and the adjoint table is given in Table 1 below:

Table 1. The adjoint table.

$Ad(e^\epsilon)$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_3$	X_5	X_6	X_7	Y_1
X_2	X_1	X_2	X_3	X_4	$X_5 + \epsilon X_3$	Y_2	Y_3	Y_4
X_3	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_4	$X_1 + \epsilon X_3$	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_5	X_1	$X_2 - \epsilon X_3$	X_3	X_4	X_5	X_6	X_7	X_8
X_6	X_1	$X_2 + \epsilon X_5$	X_3	X_4	X_5	X_6	X_7	X_8
X_7	X_1	$X_2 + \epsilon(X_6 + \frac{EI}{k} X_3)$	X_3	X_4	X_5	X_6	X_7	X_8
X_8	$X_1 + \frac{EI}{k} \epsilon X_4$	$X_2 + \epsilon X_7$	X_3	X_4	X_5	X_6	X_7	X_8

$$\begin{aligned} Y_1 &= X_8 - \frac{EI}{\rho_1} \epsilon X_4 + \frac{EI}{2\rho_1} \epsilon^2 X_3, & Y_2 &= X_6 - \epsilon X_5 - \frac{1}{2} \epsilon^2 X_3, \\ Y_3 &= X_7 - \epsilon(X_6 + \frac{EI}{k} X_3) + \frac{\epsilon^2}{2} X_5 + \frac{\epsilon^3}{6} X_3, & Y_4 &= X_8 - \epsilon X_7 + \frac{\epsilon^2}{2} (X_6 + \frac{EI}{k} X_3) - \frac{\epsilon^3}{6} X_5 - \frac{\epsilon^4}{24} X_3. \end{aligned}$$

The adjoint group is generated by $\langle \exp(adX) : X \in \mathcal{L} \rangle$. Using the solvability of \mathcal{L} , this group consists of the elements:

$$A = Ad(e^{\epsilon_1 X_1}).Ad(e^{\epsilon_2 X_2}).Ad(e^{\epsilon_3 X_3}).Ad(e^{\epsilon_4 X_4}).Ad(e^{\epsilon_5 X_5}).Ad(e^{\epsilon_6 X_6}).Ad(e^{\epsilon_7 X_7}).Ad(e^{\epsilon_8 X_8}).$$

Therefore, A is given by:

$$A = \begin{pmatrix} 1 & 0 & \epsilon_4 & \frac{EI}{\rho_1} \epsilon_8 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{EI}{k} \epsilon_7 - \epsilon_5 & 0 & \epsilon_6 & \epsilon_7 & \epsilon_8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \epsilon_2^2 & 0 & -\epsilon_2 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{6k} \epsilon_2 (6EI - k\epsilon_2^2) & 0 & \frac{1}{2} \epsilon_2^2 & -\epsilon_2 & 1 & 0 \\ 0 & 0 & \frac{EI}{2\rho_1} \epsilon_1^2 + \frac{1}{24k} \epsilon_2^2 (12EI - k\epsilon_2^2) & -\frac{EI}{\rho_1} \epsilon_1 & -\frac{1}{6} \epsilon_2^3 & \frac{1}{2} \epsilon_2^2 & -\epsilon_2 & 1 \end{pmatrix}. \quad (15)$$

Theorem 2. An optimal system of one-dimensional sub-algebras of \mathcal{L}^8 with basis (9) is provided by the following generators:

$$\begin{array}{ll|ll} X^1 = X_1 + \alpha X_2 + \beta X_8, & \alpha, \beta \neq 0, & X^9 = \alpha X_3 + \beta X_5 + \gamma X_6 + X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\ X^2 = X_1 + \alpha X_2 + \beta X_4, & \alpha \neq 0, \beta \in \mathbb{R}, & X^{10} = X_4 + \alpha X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, \\ X^3 = X_1 + \alpha X_5 + \beta X_6 + \gamma X_8, & \alpha, \beta \in \mathbb{R}, \gamma \neq 0, & X^{11} = X_4 + \alpha X_6, & \alpha \neq 0, \\ X^4 = X_1 + \alpha X_5 + \beta X_7, & \alpha \in \mathbb{R}, \beta \neq 0, & X^{12} = X_4 + \alpha X_5, & \alpha \in \mathbb{R}, \\ X^5 = X_1 + \alpha X_6, & \alpha \neq 0, & X^{13} = \alpha X_3 + \beta X_5 + X_7, & \alpha, \beta \in \mathbb{R}, \\ X^6 = X_1 + \alpha X_5, & \alpha \in \mathbb{R}, & X^{14} = \alpha X_3 + X_6, & \alpha \in \mathbb{R}, \\ X^7 = X_2 + \alpha X_8, & \alpha \neq 0, & X^{15} = X_5, & \\ X^8 = X_2 + \alpha X_4, & \alpha \in \mathbb{R}, & X^{16} = X_3. & \end{array} \quad (16)$$

Proof. Let X and \tilde{X} be two elements in the Lie algebra \mathcal{L}^8 with basis (9) given by $X = \sum_{i=1}^8 a_i X_i$ and $\tilde{X} = \sum_{i=1}^8 \tilde{a}_i X_i$. For simplicity, we will write X and \tilde{X} as row vectors of the coefficients on the form $X = (a_1 \ a_2 \ \dots \ a_8)$ and $\tilde{X} = (\tilde{a}_1 \ \tilde{a}_2 \ \dots \ \tilde{a}_8)$. Then, in order that X and \tilde{X} are in the same conjugacy class, we must have $\tilde{X} = XA$, where A is given by (15). Therefore, the theorem is proven by solving the system:

$$\begin{aligned} \tilde{a}_1 &= a_1, \\ \tilde{a}_2 &= a_2, \\ \tilde{a}_3 &= a_1 \epsilon_4 + \frac{EI}{k} a_2 \epsilon_7 - a_2 \epsilon_5 + a_3 - a_4 \epsilon_1 + a_5 \epsilon_2 - \frac{1}{2} a_6 \epsilon_2^2 + \frac{1}{6} a_7 \epsilon_2^3 - \frac{EI}{k} a_7 \epsilon_2 - \frac{1}{24} a_8 \epsilon_2^4 + \frac{1}{2} \frac{EI}{\rho_1} a_8 \epsilon_1^2 + \frac{1}{2} \frac{EI}{k} a_8 \epsilon_2^2, \\ \tilde{a}_4 &= \frac{EI}{\rho_1} a_1 \epsilon_8 - \frac{EI}{\rho_1} a_8 \epsilon_1 + a_4, \\ \tilde{a}_5 &= a_2 \epsilon_6 + a_5 - a_6 \epsilon_2 + \frac{1}{2} a_7 \epsilon_2^2 - \frac{1}{6} a_8 \epsilon_2^3, \\ \tilde{a}_6 &= a_2 \epsilon_7 + a_6 - a_7 \epsilon_2 + \frac{1}{2} a_8 \epsilon_2^2, \\ \tilde{a}_7 &= a_2 \epsilon_8 - a_8 \epsilon_2 + a_7, \\ \tilde{a}_8 &= a_8, \end{aligned} \quad (17)$$

for $\{\epsilon_i\}_{i=1}^8$ in terms of $\{a_i\}_{i=1}^8$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^8$.

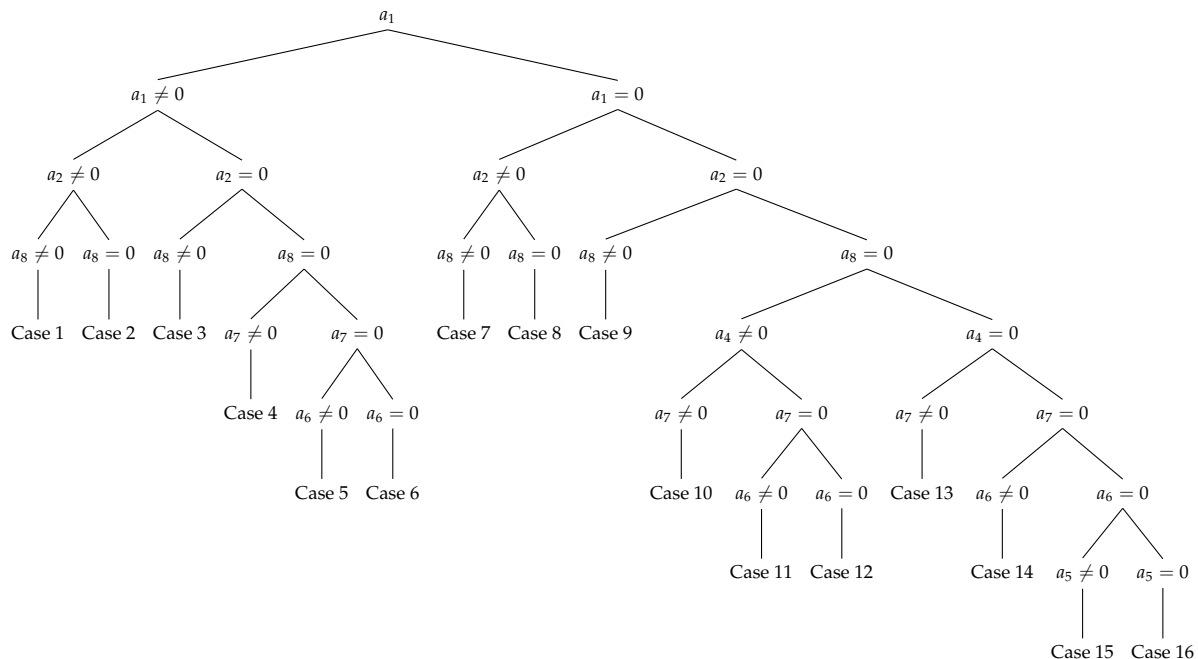
The results are presented for different cases in a tree diagram, where each of its vertices is an invariant function, and its leaves are given completely. Moreover, one can verify that all of the parameters α, β and γ appearing in each case are invariants. Therefore, the inequivalence and completeness conditions are satisfied.

All joint invariants are obtained by solving the following system of PDEs, which is given by using Formula (2):

$$\begin{cases} a_4 \frac{\partial \Phi}{\partial a_3} + \frac{EI}{\rho_1} a_8 \frac{\partial \Phi}{\partial a_4} = 0, \\ a_5 \frac{\partial \Phi}{\partial a_3} - \frac{EI}{k} a_7 \frac{\partial \Phi}{\partial a_3} - a_6 \frac{\partial \Phi}{\partial a_5} - a_7 \frac{\partial \Phi}{\partial a_6} - a_8 \frac{\partial \Phi}{\partial a_7} = 0, \\ a_1 \frac{\partial \Phi}{\partial a_3} = 0, \\ a_2 \frac{\partial \Phi}{\partial a_3} = 0, \\ a_2 \frac{\partial \Phi}{\partial a_5} = 0, \\ a_2 \frac{\partial \Phi}{\partial a_6} + \frac{EI}{k} a_2 \frac{\partial \Phi}{\partial a_3} = 0, \\ a_2 \frac{\partial \Phi}{\partial a_7} + \frac{EI}{\rho_1} a_1 \frac{\partial \Phi}{\partial a_4} = 0. \end{cases} \quad (18)$$

By solving this system, we obtain $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_2, a_8)$, where F is an arbitrary function of a_1, a_2, a_8 . Hence, the basic invariants of Timoshenko System (1) are a_1, a_2

and a_8 ; this means that the first three vertices of the tree will be these invariants in any order. In our case, we will consider the order a_1, a_2, a_8 .



The full details for each leaf are given as follows:

Case 1 $a_1 \neq 0, a_2 \neq 0, a_8 \neq 0$: Let $\epsilon_2 = \epsilon_3 = \epsilon_5 = 0, \epsilon_1 = \frac{\rho_1 a_2 a_4 - E I a_7}{E I a_2 a_8}, \epsilon_4 = \frac{\rho_1 a_4^2}{E I a_8} + \frac{E I}{k} a_6 - a_3 - \frac{E I}{2 \rho_1 a_2^2} \frac{a_7^2}{a_8}, \epsilon_6 = -\frac{a_5}{a_2}, \epsilon_7 = -\frac{a_6}{a_2}$ and $\epsilon_8 = -\frac{a_7}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_1 + \alpha X_2 + \beta X_8 \rangle$, with $\alpha, \beta \neq 0$.

Case 2 $a_1 \neq 0, a_2 \neq 0, a_8 = 0$: Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_5 = 0, \epsilon_4 = \frac{E I}{k} a_6 - a_3, \epsilon_6 = -\frac{a_5}{a_2}, \epsilon_7 = -\frac{a_6}{a_2}, \epsilon_8 = -\frac{a_7}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = 0$. The conjugacy class is $\langle X_1 + \alpha X_2 + \beta X_4 \rangle$, with $\alpha \neq 0, \beta \in \mathbb{R}$.

Case 3 $a_1 \neq 0, a_2 = 0, a_8 \neq 0$: Let $\epsilon_1 = 0, \epsilon_2 = \frac{a_7}{a_8}, \epsilon_4 = \frac{E I a_7^2}{2 k a_8} - a_3 - \frac{a_5 a_7}{a_8} + \frac{a_6 a_7^2}{2 a_8^2} - \frac{a_7^4}{8 a_8^3}$ and $\epsilon_8 = -\frac{\rho_1}{E I} a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_1 + \alpha X_5 + \beta X_6 + \gamma X_8 \rangle, \alpha, \beta \in \mathbb{R}, \gamma \neq 0$.

When $a_1 \neq 0, a_2 = 0, a_8 = 0$, we need to solve system (18) again to see what are the invariants as well the next vertices of the tree.

Solving system (18) taking into account that $a_2 = a_8 = 0$ gives $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_7, -2 a_5 a_7 + a_6^2)$, so we can consider a_7 to be the next vertex of the tree.

Case 4 $a_1 \neq 0, a_2 = 0, a_8 = 0, a_7 \neq 0$: Let $\epsilon_1 = 0, \epsilon_2 = \frac{a_6}{a_7}, \epsilon_4 = \frac{E I}{k} a_6 - a_3 - \frac{a_5 a_6}{a_7} + \frac{a_6^2}{3 a_7^2}$ and $\epsilon_8 = -\frac{\rho_1}{E I} a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_6 = \tilde{a}_8 = 0$: the conjugacy class is of the form $\langle X_1 + \alpha X_5 + \beta X_7 \rangle, \alpha \in \mathbb{R}, \beta \neq 0$.

Again, for the case $a_1 \neq 0, a_2 = 0, a_8 = 0, a_7 = 0$, we resolve system (18) to have $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_1, a_6)$, so next vertex can be a_6 :

Case 5 $a_1 \neq 0, a_2 = 0, a_8 = 0, a_7 = 0, a_6 \neq 0$: Let $\epsilon_1 = 0, \epsilon_2 = \frac{a_5}{a_6}, \epsilon_4 = -a_3 - \frac{a_5^2}{2 a_6}$ and $\epsilon_8 = -\frac{\rho_1}{E I} a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$: the corresponding conjugacy class is $\langle X_1 + \alpha X_6 \rangle$, where $\alpha \neq 0$.

Case 6 $a_1 \neq 0, a_2 = 0, a_8 = 0, a_7 = 0, a_6 = 0$: Let $\epsilon_1 = \epsilon_2 = 0, \epsilon_4 = -a_3$ and $\epsilon_8 = -\frac{\rho_1}{E I} a_4$ to make $\tilde{a}_3 = \tilde{a}_4 = 0$: the conjugacy class is $\langle X_1 + \alpha X_5 \rangle, \alpha \in \mathbb{R}$.

Case 7 $a_1 = 0, a_2 \neq 0, a_8 \neq 0$: Let $\epsilon_1 = \frac{\rho_1 a_4}{E I a_8}, \epsilon_2 = 0, \epsilon_5 = a_3 - \frac{E I}{k} a_6 - \frac{\rho_1 a_4^2}{2 E I a_8}, \epsilon_6 = -a_5, \epsilon_7 = -a_6$ and

$\epsilon_8 = -a_7$ to make $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_2 + \alpha X_8 \rangle, \alpha \neq 0$.

Case 8 $a_1 = 0, a_2 \neq 0, a_8 = 0$: Let $\epsilon_2 = 0, \epsilon_5 = a_3 - \epsilon_1 a_4 - \frac{EI}{k} a_6, \epsilon_6 = -a_5, \epsilon_7 = -a_6$ and $\epsilon_8 = -a_7$ to make $\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = 0$: the conjugacy class is $\langle X_2 + \alpha X_4 \rangle, \alpha \in \mathbb{R}$.

Case 9 $a_1 = 0, a_2 = 0, a_8 \neq 0$: Let $\epsilon_1 = \frac{\rho_1}{EI} a_4$ and $\epsilon_2 = a_7$, to get $\tilde{a}_4 = \tilde{a}_7 = 0$: the conjugacy class is of the form $\langle \alpha X_3 + \beta X_5 + \gamma X_6 + X_8 \rangle, \alpha, \beta, \gamma \in \mathbb{R}$.

By substituting $a_1 = a_2 = a_8 = 0$ in system (18) and solving it, we get $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_4, a_7, -2a_5 a_7 + a_6^2)$. So the next two vertices can be a_4 and a_7 in any order. We consider the order a_4, a_7 .

Case 10 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 \neq 0, a_7 \neq 0$: Let $\epsilon_1 = -\frac{EI}{k} a_6 + a_3 + \frac{a_5 a_6}{a_7} - \frac{a_6^3}{3a_7^2}$ and $\epsilon_2 = \frac{a_6}{a_7}$ to make $\tilde{a}_3 = \tilde{a}_6 = 0$: the conjugacy class is $\langle X_4 + \alpha X_5 + \beta X_7 \rangle, \alpha \in \mathbb{R}, \beta \neq 0$.

In case $a_1 = a_2 = a_8 = a_7 = 0$, solving system (18) yields $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_4, a_6)$ which implies, after considering a_4 and a_7 as vertices, that one can consider the invariant a_6 as a new vertex.

Case 11 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 \neq 0, a_7 = 0, a_6 \neq 0$: Let $\epsilon_1 = \frac{a_5^2}{2a_6} + a_3$ and $\epsilon_2 = \frac{a_5}{a_6}$ to make $\tilde{a}_3 = \tilde{a}_5 = 0$: the conjugacy class is $\langle X_4 + \alpha X_6 \rangle$, with $\alpha \neq 0$.

Case 12 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 \neq 0, a_7 = 0, a_6 = 0$: Let $\epsilon_1 = a_3, \epsilon_2 = 0$ to make $\tilde{a}_3 = 0$: then the conjugacy class is $\langle X_4 + \alpha X_5 \rangle, \alpha \in \mathbb{R}$.

Case 13 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 = 0, a_7 \neq 0$: Let $\epsilon_2 = a_6$ to have $\tilde{a}_6 = 0$: the conjugacy class is $\langle \alpha X_3 + \beta X_5 + X_7 \rangle, \alpha, \beta \in \mathbb{R}$.

Case 14 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 = 0, a_7 = 0, a_6 \neq 0$: Let $\epsilon_2 = a_5$ to have $\tilde{a}_5 = 0$: the conjugacy class is $\langle \alpha X_3 + X_6 \rangle, \alpha \in \mathbb{R}$.

When $a_6 = 0$ with $a_1 = a_2 = a_8 = a_4 = a_7 = 0$, solving the PDEs system (18) gives $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = F(a_5)$. So the next invariant vertex is a_5 .

Case 15 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 = 0, a_7 = 0, a_6 = 0, a_5 \neq 0$: Let $\epsilon_2 = -a_3$ to have $\tilde{a}_3 = 0$: the conjugacy class is $\langle X_5 \rangle$.

Case 16 $a_1 = 0, a_2 = 0, a_8 = 0, a_4 = 0, a_7 = 0, a_6 = 0, a_5 = 0$: the conjugacy class is $\langle X_3 \rangle$.

□

5. Optimal Reductions and Invariant Solutions

It is known that the invariant solutions for PDEs can be determined by two procedures, namely the invariant form method and the direct substitution method [14]. The idea of looking for group invariant solutions generalizes quite naturally to PDEs with any number of independent and dependent variables. A one parameter group that acts nontrivially on one or more independent variables can be used to reduce the number of independent variables by one.

In this section, we focus on the invariant form method, which requires that at least one of the infinitesimals ζ^1 and ζ^2 is not equal to zero [9,14]. Basically, the reason is that a vector field of the type $X = \eta_1 \frac{\partial}{\partial \varphi} + \eta_2 \frac{\partial}{\partial \psi}$ cannot leave the submanifold of codimension two defined by the equations $\varphi = f(t, x)$ and $\psi = g(t, x)$ invariant, otherwise the dimension of the solution space would go down. This means the transversality condition in the sense of [16] will not be satisfied. For this reason, only the generators X^1, \dots, X^8 from system (16) are considered in this section.

We solve the invariance surface conditions explicitly by solving the corresponding characteristic equation given by:

$$\frac{dt}{\xi^1(t, x, \varphi, \psi)} = \frac{dx}{\xi^2(t, x, \varphi, \psi)} = \frac{d\varphi}{\eta^1(t, x, \varphi, \psi)} = \frac{d\psi}{\eta^2(t, x, \varphi, \psi)}, \quad (19)$$

to get the corresponding invariants, which are used to reduce the number of independent variables by one. The procedure is explained in detail in the following example. Moreover, all possible invariant variables of the optimal system (16) and their corresponding reductions are given in Table 2.

Table 2. Reductions using one-dimensional optimal System (16).

Generators in (16)	Invariant Variables	The Reduced System
$X^1 = X_1 + \alpha X_2 + \beta X_8,$ $\alpha, \beta \neq 0.$	A	$(k - \alpha^2 \rho_1) Z'' + k W' + \frac{\alpha \beta \rho_1 EI}{k} \zeta + \frac{1}{6} \left(\frac{\beta k}{\alpha} - \alpha \beta \rho_1 \right) \zeta^3 = 0,$ $(\rho_2 \alpha^2 - EI) W'' + k Z' + k W + \chi (-\alpha W') - \frac{\beta EI}{2\alpha} \zeta^2 + \frac{\beta k}{24\alpha} \zeta^4 = 0.$
$X^2 = X_1 + \alpha X_2 + \beta X_4,$ $\alpha \neq 0, \beta \in \mathbb{R}.$	B	$(k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 = 0,$ $(\rho_2 \alpha^2 - EI) W'' - k Z' + k W + \chi (-\alpha W') = 0.$
$X^3 = X_1 + \alpha X_5 + \beta X_6 + \gamma X_8,$ $\alpha, \beta \in \mathbb{R}, \gamma \neq 0.$	C	$Z'' + W' = 0,$ $EI W'' - k Z' - k W - \chi \left(\frac{\gamma}{6} \zeta^3 + \beta \zeta + \alpha \right) = 0.$
$X^4 = X_1 + \alpha X_5 + \beta X_7,$ $\alpha \in \mathbb{R}, \beta \neq 0.$	D	$Z'' + W' = 0,$ $EI W'' - k Z' - k W - \chi \left(\alpha + \frac{\beta}{2} \zeta^2 \right) = 0.$
$X^5 = X_1 + \alpha X_6,$ $\alpha \neq 0.$	E	$Z'' + W' = 0,$ $EI W'' - k Z' - k W - \chi (\alpha \zeta) = 0.$
$X^6 = X_1 + \alpha X_5,$ $\alpha \in \mathbb{R}.$	F	$Z'' + W' = 0,$ $EI W'' - k Z' - k W - \chi (\alpha) = 0.$
$X^7 = X_2 + \alpha X_8,$ $\alpha \neq 0.$	G	$\rho_1 Z'' = 0,$ $\rho_2 W'' + \chi (W') + k W + \frac{\alpha EI k}{2\rho_1} \zeta^2 = 0.$
$X^8 = X_2 + \alpha X_4,$ $\alpha \in \mathbb{R}.$	H	$\rho_1 Z'' = 0,$ $\rho_2 W'' + \chi (W') + k W + \alpha k \zeta = 0.$

As another application for the optimal system based on the definition of the invariant boundary value problem given in [14,17], we classify the non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations for Timoshenko system (1) as given in Table 3. This is achieved by finding invariant conditions as arbitrary functions of the invariants up to the first order and invariant surfaces as arbitrary functions of the invariants of order zero, which depend on the independent variables for each generator in the optimal system (16).

Table 3. Invariant conditions prescribed on invariant surfaces and theirs reductions.

X^i	Invariant Condition	Invariant Surface	Reduced Invariant Condition	Reduced Invariant Surface
X^1	\hat{A}	$\omega(x - \alpha t)$	$B \left(\zeta, Z, W, \frac{\beta EI}{2k} \zeta^2 - \frac{\beta}{24} \zeta^4 - \alpha Z', Z', -\alpha W', W' \right)$	$\omega(\zeta)$
X^2	\hat{B}	$\omega(x - \alpha t)$	$B \left(\zeta, Z, W, \alpha Z', Z', \alpha W', W' \right)$	$\omega(\zeta)$
X^3	\hat{C}	$\omega(x)$	$B \left(\zeta, Z, W, -\alpha \zeta - \left(\frac{\beta}{2} + \frac{\gamma EI}{2k} \right) \zeta^2 - \frac{\gamma}{24} \zeta^4, Z', \alpha + \beta \zeta + \frac{\gamma}{6} \zeta^3, W' \right)$	$\omega(\zeta)$
X^4	\hat{D}	$\omega(x)$	$B \left(\zeta, Z, W, \left(\frac{EI}{k} \beta - \alpha \right) \zeta - \frac{\beta}{6} \zeta^3, Z', \alpha + \frac{\beta}{2} \zeta^2, W' \right)$	$\omega(\zeta)$
X^5	\hat{E}	$\omega(x)$	$B \left(\zeta, Z, W, -\frac{\alpha}{2} \zeta^2, Z', \alpha \zeta, W' \right)$	$\omega(\zeta)$
X^6	\hat{F}	$\omega(x)$	$B \left(\zeta, Z, W, -\alpha \zeta, Z', \alpha, W' \right)$	$\omega(\zeta)$
X^7	\hat{G}	$\omega(t)$	$B \left(\zeta, Z, W, Z', \frac{\alpha EI}{2\rho_1} \zeta^2, W', 0 \right)$	$\omega(\zeta)$
X^8	\hat{H}	$\omega(t)$	$B \left(\zeta, Z, W, Z', \alpha \zeta, W', 0 \right)$	$\omega(\zeta)$

Example 1. Reduction and invariant solution using X^6 :

Consider the generator $X^6 = X_1 + \alpha X_5 = \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial \varphi} + \alpha \frac{\partial}{\partial \psi}$ where $\alpha \in \mathbb{R}$, from the optimal system (16). Solving the corresponding characteristic equations of the first prolongation:

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\varphi}{-\alpha x} = \frac{d\psi}{\alpha} = \frac{d\varphi_t}{0} = \frac{d\varphi_x}{-\alpha} = \frac{d\psi_t}{0} = \frac{d\psi_x}{0},$$

gives the following invariants up to the first order:

$$I_1 = x, \quad I_2 = \varphi + \alpha tx, \quad I_3 = \psi - \alpha t, \quad I_4 = \varphi_t, \quad I_5 = \varphi_x + \alpha t, \quad I_6 = \psi_t, \quad I_7 = \psi_x. \quad (20)$$

Since the order of the Timoshenko system (1) is two, then the invariant conditions have to be functions of the first order invariants of the form:

$$B_\Gamma(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = 0, \quad \Gamma = 1, \dots, 4. \quad (21)$$

The invariant conditions (21) are prescribed on the invariant surfaces:

$$\omega_\gamma(I_1) = 0, \quad \gamma = 1, 2, \quad (22)$$

which are invariants of order zero that depend on the independent variables.

For $\alpha \neq 0$, if we restrict the invariant conditions (21) to be linear, then it will take the form:

$$B_\Gamma = A_{\Gamma,1}(x) + A_{\Gamma,2}(x)(\varphi + \alpha tx) + A_{\Gamma,3}(x)(\psi - \alpha t) + A_{\Gamma,4}(x)\varphi_t + A_{\Gamma,5}(x)(\varphi_x + \alpha t) + A_{\Gamma,6}(x)\psi_t + A_{\Gamma,7}(x)\psi_x = 0, \quad (23)$$

where $A_{\Gamma,i}(x)$ are arbitrary functions for $\Gamma = 1, \dots, 4$ and $i = 1, \dots, 7$.

Moreover, the invariants of order zero I_1, I_2 and I_3 give the invariant variables:

$$\varphi(t, x) = Z(\zeta) - \alpha tx, \quad \psi(t, x) = W(\zeta) + \alpha t, \quad \zeta = x. \quad (24)$$

The reduction of Timoshenko System (1) with the boundary conditions (23) prescribed on the surfaces $x = 0$ and $x = L$ using the invariant variables (24) is the system of ODEs of the form:

$$\begin{aligned} Z'' + W' &= 0, \\ EI W'' - k Z' - k W - \chi(\alpha) &= 0, \end{aligned} \quad (25)$$

with general boundary conditions of the form:

$$A_{\Gamma,1}(\zeta) + A_{\Gamma,2}(\zeta)Z + A_{\Gamma,3}(\zeta)W - \alpha A_{\Gamma,4}(\zeta)\zeta + A_{\Gamma,5}(\zeta)Z' + \alpha A_{\Gamma,6}(\zeta) + A_{\Gamma,7}(\zeta)W' = 0 \quad (26)$$

prescribed on the surfaces $\zeta = 0$ and $\zeta = L$.

For instance, we consider a beam that models small motions of a hinged arm, which is hinged at the origin and free at its other end. This case was considered in [18] with two control functions. The control functions are the force $f(t)$ applied at the free end and a torque $\tau(t)$ applied at the hinged end. The associated boundary conditions:

$$\varphi(t, 0) = 0, \quad \psi_x(t, 0) = \tau(t), \quad \varphi_x(t, L) - \psi(t, L) = f(t), \quad \psi_x(t, L) = 0, \quad (27)$$

can be obtained from Equation (23) for the non-zero values $A_{1,2} = 1$ at $x = 0$; $A_{2,1} = -r$, $A_{2,7} = 1$ at $x = 0$; $A_{3,1} = -q$, $A_{3,3} = -1$, $A_{3,5} = 1$ at $x = L$; $A_{4,7} = 1$ at $x = L$ with force $f(t) = q - 2\alpha t$ and torque $\tau(t) = r$.

Using the invariant variables (24), the boundary conditions (27) are reduced to:

$$Z(0) = 0, \quad W'(0) = r, \quad Z'(L) - W(L) = q, \quad W'(L) = 0. \quad (28)$$

The solution of the boundary value problem (25) and (28) is given by:

$$\begin{aligned} Z(\zeta) &= \frac{r}{6L}\zeta^3 - \frac{r}{2}\zeta^2 + \left(\frac{Lr}{2} - \frac{rEI}{2Lk} - \frac{\chi(\alpha)}{2k} + \frac{q}{2}\right)\zeta, \\ W(\zeta) &= -\frac{r}{2L}\zeta^2 + r\zeta - \frac{\chi(\alpha)}{2k} - \frac{1}{2}\left(q + r\left(\frac{EI}{Lk} + L\right)\right). \end{aligned} \quad (29)$$

Substituting back in the invariant variables (24), the system (1) with the boundary conditions (27) has the following solution:

$$\begin{aligned} \varphi(t, x) &= \frac{r}{6L}x^3 - \frac{r}{2}x^2 + \left(-\alpha t - \frac{\chi(\alpha)}{2k} - \frac{rEI}{2Lk} + \frac{q}{2} + \frac{rL}{2}\right)x, \\ \psi(t, x) &= \alpha t - \frac{r}{2L}x^2 + rx - \frac{\chi(\alpha)}{2k} - \frac{1}{2}\left(q + r\left(\frac{EI}{Lk} + L\right)\right). \end{aligned} \quad (30)$$

$$\begin{aligned} A: \quad \varphi(t, x) &= Z(\zeta) - \frac{\beta}{24}tx^4 + \frac{\beta EI}{2k}tx^2 + \frac{\alpha\beta}{12}t^2x^3 - \frac{\alpha\beta EI}{2k}t^2x + \frac{\beta EI}{6\rho_1}t^3 - \frac{\alpha^2\beta}{12}x^2t^3 + \frac{\alpha^2\beta EI}{6k}t^3 \\ &\quad + \frac{\alpha^3\beta}{24}xt^4 - \frac{\alpha^4}{120}t^5, \quad \psi(t, x) = W(\zeta) + \frac{\beta}{24\alpha}x^4, \quad \zeta = x - \alpha t. \\ B: \quad \varphi(t, x) &= Z(\zeta) + \frac{\beta}{2}t^2, \quad \psi(t, x) = W(\zeta), \quad \zeta(t, x) = x - \alpha t. \\ C: \quad \varphi(t, x) &= Z(\zeta) - \alpha tx - \frac{\beta}{2}tx^2 - \frac{\gamma}{24}tx^4 + \frac{\gamma EI}{2k}tx^2 + \frac{\gamma EI}{6\rho_1}t^3, \quad \psi(t, x) = W(\zeta) + \alpha t \\ &\quad + \beta tx + \frac{\gamma}{6}tx^3, \quad \zeta = x. \\ D: \quad \varphi(t, x) &= Z(\zeta) + \left(\frac{\beta EI}{k} - \alpha\right)tx - \frac{\beta}{6}tx^3, \quad \psi(t, x) = W(\zeta) + \alpha t + \frac{\beta}{2}tx^2, \quad \zeta = x. \\ E: \quad \varphi(t, x) &= Z(\zeta) - \frac{\alpha}{2}x^2t, \quad \psi(t, x) = W(\zeta) + \alpha xt, \quad \zeta = x. \\ F: \quad \varphi(t, x) &= Z(\zeta) - \alpha tx, \quad \psi(t, x) = W(\zeta) + \alpha t, \quad \zeta = x. \\ G: \quad \varphi(t, x) &= Z(\zeta) + \frac{\alpha EI}{2\rho_1}t^2x + \frac{\alpha EI}{6k}x^3, \quad \psi(t, x) = W(\zeta) + \frac{1}{24}\alpha x^4 - \frac{\alpha}{120}x^5, \quad \zeta = t. \\ H: \quad \varphi(t, x) &= Z(\zeta) + \alpha tx, \quad \psi(t, x) = W(\zeta), \quad \zeta = t. \end{aligned} \quad (31)$$

$$\begin{aligned} \hat{A}: \quad B &\left(x - \alpha t, \varphi + \frac{\alpha^4\beta}{120}t^5 - \frac{\alpha^3\beta}{24}t^4x - \frac{\alpha^2\beta EI}{6k}t^3 + \frac{\alpha^2\beta}{12}t^3x^2 - \frac{\beta EI}{6\rho_1}t^3 - \frac{\alpha\beta}{12}t^2x^3 + \frac{\alpha\beta EI}{2k}t^2x + \frac{\beta}{24}tx^4 - \frac{\beta EI}{2k}tx^2\right. \\ &\quad \left., \psi - \frac{\beta}{24\alpha}x^4, \varphi_t - \frac{\beta EI}{2\rho_1}t^2, \varphi_x - \frac{\beta\alpha^3}{24}t^4 + \frac{\beta\alpha^2}{6}t^3x + \frac{\alpha\beta EI}{2k}t^2 - \frac{\alpha\beta}{4}t^2x^2 + \frac{\beta}{6}tx^3 - \frac{\beta EI}{k}tx, \psi_t, \psi_x - \frac{\beta}{6\alpha}x^3\right). \\ \hat{B}: \quad B &\left(x - \alpha t, \varphi - \frac{\beta}{2}t^2, \psi, \varphi_t - \beta t, \varphi_x, \psi_t, \psi_x\right). \\ \hat{C}: \quad B &\left(x, \varphi + \alpha tx + \frac{\beta}{2}tx^2 + \frac{\gamma}{24}tx^4 - \frac{\gamma EI}{2k}tx^2 - \frac{\gamma EI}{6\rho_1}t^3, \psi - \alpha t - \beta tx - \frac{\gamma}{6}tx^3, \varphi_t - \frac{\gamma EI}{2\rho_1}t^2, \varphi_x + \alpha t + \beta tx + \frac{\gamma}{6}tx^3\right. \\ &\quad \left.- \frac{\gamma EI}{k}tx, \psi_t, \psi_x - \beta t - \frac{\gamma}{2}tx^2\right). \\ \hat{D}: \quad B &\left(x, \varphi + \alpha tx + \frac{\beta}{6}tx^3 - \frac{\beta EI}{k}tx, \psi - \alpha t - \frac{\beta}{2}tx^2, \varphi_t, \varphi_x + \alpha t + \frac{\beta}{2}tx^2 - \frac{\beta EI}{k}t, \psi_t, \psi_x - \beta tx\right). \\ \hat{E}: \quad B &\left(x, \varphi + \frac{\alpha}{2}tx^2, \psi - \alpha tx, \varphi_t, \varphi_x + \alpha tx, \psi_t, \psi_x - \alpha t\right). \\ \hat{F}: \quad B &\left(x, \varphi + \alpha tx, \psi - \alpha t, \varphi_t, \varphi_x + \alpha t, \psi_t, \psi_x\right). \\ \hat{G}: \quad B &\left(t, \varphi + \frac{\alpha}{120}x^5 - \frac{\alpha EI}{6k}x^3 - \frac{\alpha EI}{2\rho_1}t^2x, \psi - \frac{\alpha}{24}x^4, \varphi_t - \frac{\alpha EI}{\rho_1}tx, \varphi_x + \frac{\alpha}{24}x^4 - \frac{\alpha EI}{2k}x^2, \psi_t, \psi_x - \frac{\alpha}{6}x^3\right). \\ \hat{H}: \quad B &\left(t, \varphi - \alpha tx, \psi, \varphi_t - \alpha x, \varphi_x, \psi_t, \psi_x\right). \end{aligned} \quad (32)$$

6. Discussion and Concluding Remarks

The Lie group study of a nonlinear Timoshenko system of PDEs with the frictional damping term in rotational angle is performed. Lie symmetry generators and their Lie group transformations for the Timoshenko system are presented. A systematic approach and a formula for computing invariants in the adjoint representation is illustrated. Furthermore, the one-dimensional optimal system is derived for the corresponding Lie algebra. All possible invariant variables and their corresponding reductions for each vector field in the optimal system are found. The reductions to system of ordinary differential equations (ODEs) are given in Table 2. They are described by optimal reductions where all non-similar invariant solutions under symmetry transformations can be given from the solution of the reduced system of ODEs. The hinged-free beam with two control functions, constant torque applied at the hinged end and linear force applied at the free end, has been considered as an example. Finally, all possible non-similar invariant conditions prescribed on invariant surfaces under symmetry transformations are given in Table 3.

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