mathematics

MDPI

Article

# Some Types of Subsemigroups Characterized in Terms of Inequalities of Generalized Bipolar Fuzzy Subsemigroups

**Pannawit Khamrot [1] and Manoj Siripitukdet [1,2,*]**

1 Departament of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand; pannawit.k@gmail.com
2 Research Center for Academic Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
* Correspondence: manojs@nu.ac.th

Received: 14 November2017; Accepted: 23 November 2017; Published: 27 November 2017

**Abstract:** In this paper, we introduce a generalization of a bipolar fuzzy (BF) subsemigroup, namely, a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup. The notions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi(generalized bi-, bi-) ideals are discussed. Some inequalities of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi(generalized bi-, bi-) ideals are obtained. Furthermore, any regular semigroup is characterized in terms of generalized BF semigroups.

**Keywords:** generalized bipolar fuzzy (BF) semigroup; $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup; fuzzy quasi(generalized bi-, bi-) ideal; regular semigroup

## 1. Introduction

Most of the bipolarities separate positive and negative information response; positive information representations are compiled to be possible, while negative information representations are impossible [1]. The bipolar information of evaluation can help to evaluate decisions. Sometimes, decisions are not only influenced by the positive decision criterion, but also with the negative decision criterion, for example, environmental and social impact assessment. Evaluated alternative consideration should weigh the negative effects to select the optimal choice. Therefore bipolar information affects the effectiveness and efficiency of decision making. It is used in decision-making problems, organization problems, economic problems, evaluation, risk management, environmental and social impact assessment, and so forth. Thus, the concept of bipolar fuzzy (BF) sets are more relevant in mathematics.

In 1965, Zadeh [2] introduced the fuzzy set theory, which can be applied to many areas, such as mathematics, statistics, computers, electrical instruments, the industrial industry, business, engineering, social applications, and so forth. In 2003, Bucolo et al. [3] proposed small-world networks of fuzzy chaotic oscillators. The fuzzy set was used to establish the mathematical method for dealing with imprecise and uncertain environments. In 1971, Rosenfeld [4] applied fuzzy sets to group structures. Then, the fuzzy set was used in the theory of semigroups in 1979. Kuroki [5] initiated fuzzy semigroups based on the notion of fuzzy ideals in semigroups and introduced some properties of fuzzy ideals and fuzzy bi-ideals of semigroups. The fundamental concepts of BF sets were initiated by Zhang [6] in 1994. He innovated the BF set as BF logic, which has been widely applied to solve many real-world problems. In 2000, Lee [7] studied the notion of bipolar-valued fuzzy sets. Kim et al. [8] studied the notions of BF subsemigroups, BF left (right, bi-) ideals. He provided some necessary and sufficient conditions for a BF subsemigroup and BF left (right, bi-) ideals of semigroups.

In this paper, generalizations of BF semigroups are introduced. Definitions and properties of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi (generalized bi-, bi-) ideals are obtained. Some inequalities of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi (generalized bi-, bi-) ideals are obtained. Finally, we characterize a regular semigroup in terms of generalized BF semigroups.

## 2. Preliminaries

In this section, we give definitions and examples that are used in this paper. By a subsemigroup of a semigroup $S$ we mean a non-empty subset $A$ of $S$ such that $A^2 \subseteq A$, and by a left (right) ideal of $S$ we mean a non-empty subset $A$ of $S$ such that $SA \subseteq A(AS \subseteq A)$. By a two-sided ideal or simply an ideal, we mean a non-empty subset of a semigroup $S$ that is both a left and a right ideal of $S$. A non-empty subset $A$ of $S$ is called an **interior ideal** of $S$ if $SAS \subseteq A$, and a **quasi-ideal** of $S$ if $AS \cap SA \subseteq A$. A subsemigroup $A$ of $S$ is called a **bi-ideal** of $S$ if $ASA \subseteq A$. A non-empty subset $A$ is called a **generalized bi-ideal** of $S$ if $ASA \subseteq A$ [9].

By the definition of a left (right) ideal of a semigroup $S$, it is easy to see that every left (right) ideal of $S$ is a quasi-ideal of $S$.

**Definition 1.** *A semigroup $S$ is called **regular** if for all $a \in S$ there exists $x \in S$ such that $a = axa$.*

**Theorem 1.** *For a semigroup $S$, the following conditions are equivalent.*

(1) *$S$ is regular.*
(2) *$R \cap L = RL$ for every right ideal $R$ and every left ideal $L$ of $S$.*
(3) *$ASA = A$ for every quasi-ideal $A$ of $S$.*

**Definition 2.** *Let $X$ be a set; a fuzzy set (or fuzzy subset) $f$ on $X$ is a mapping $f : X \to [0,1]$, where $[0,1]$ is the usual interval of real numbers.*

The symbols $f \wedge g$ and $f \vee g$ will denote the following fuzzy sets on $S$:

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

$$(f \vee g)(x) = f(x) \vee g(x)$$

for all $x \in S$.

A product of two fuzzy sets $f$ and $g$ is denoted by $f \circ g$ and is defined as

$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz}\{f(y) \wedge g(z)\}, & \text{if } \; x = yz \; \text{ for some } \; y,z \in S \\ 0, & \text{otherwise} \end{cases}$$

**Definition 3.** *Let $S$ be a non-empty set. A BF set $f$ on $S$ is an object having the following form:*

$$f := \{(x, f_p(x), f_n(x)) : x \in S\}$$

*where $f_p : S \to [0,1]$ and $f_n : S \to [-1,0]$.*

**Remark 1.** *For the sake of simplicity we use the symbol $f = (S; f_p, f_n)$ for the BF set $f = \{(x, f_p(x), f_n(x)) : x \in S\}$.*

**Definition 4.** *Given a BF set $f = (S; f_p, f_n)$, $\alpha \in [0,1]$ and $\beta \in [-1,0]$, the sets*

$$P(f;\alpha) := \{x \in S | f_p(x) \geq \alpha\}$$

*and*

$$N(f;\beta) := \{x \in S | f_n(x) \leq \beta\}$$

*are called the* ***positive*** $\alpha$***-cut*** *and* ***negative*** $\beta$***-cut*** *of* $f$*, respectively. The set* $C(f;(\alpha,\beta)) := P(f;\alpha) \cap N(f;\beta)$ *is called the* ***bipolar*** $(\alpha,\beta)$***-cut*** *of* $f$*.*

We give the generalization of a BF subsemigroup, which is defined by Kim et al. (2011).

**Definition 5.** *A BF set* $f = (S; f_p, f_n)$ *on* $S$ *is called a* $(\alpha_1, \alpha_2; \beta_1, \beta_2)$***-BF subsemigroup*** *on* $S$*, where* $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$ *if it satisfies the following conditions:*

(1) $f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$
(2) $f_n(xy) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

*for all* $x, y \in S$*.*

We note that every BF subsemigroup is a $(0,1,-1,0)$-BF subsemigroup.

The following examples show that $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup on $S$ but $f = (S; f_p, f_n)$ is not a BF subsemigroup on $S$.

**Example 1.** *The set* $S = \{2, 3, 4, ...\}$ *is a semigroup under the usual multiplication. Let* $f = (S; f_p, f_n)$ *be a BF set on* $S$ *defined as follows:*

$$f_p(x) := \frac{1}{x+2} \text{ and } f_n(x) := -\frac{1}{x+2}$$

*for all* $x \in S$*.*

*Let* $x, y \in S$*. Then*

$$f_p(xy) = \frac{1}{xy+2} < \frac{1}{x+2} = f_p(x)$$

*and*

$$f_p(xy) = \frac{1}{xy+2} < \frac{1}{y+2} = f_p(y)$$

*Thus,* $f_p(xy) < f_p(x) \wedge f_p(y)$*. Therefore* $f = (S; f_p, f_n)$ *is not a BF subsemigroup on* $S$*.*

*Let* $\alpha_2 \in [0,1]$*,* $\beta_1 \in [-1,0]$*,* $\alpha_1 = \frac{1}{4}$ *and* $\beta_2 = -\frac{1}{4}$*. Thus for all* $x, y \in S$*,*

$$f_p(xy) \vee \frac{1}{4} \geq \frac{1}{x+2} \wedge \frac{1}{y+2} \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$$

*and*

$$f_n(xy) \wedge -\frac{1}{4} \leq -\frac{1}{x+2} \vee -\frac{1}{y+2} \leq f_n(x) \vee f_n(y) \vee \beta_1$$

*Hence* $f = (S; f_p, f_n)$ *is a* $(\frac{1}{4}, \alpha_2; \beta_1, -\frac{1}{4})$*-BF subsemigroup on* $S$*.*

*We note that* $f = (S; f_p, f_n)$ *is a* $(\alpha_1, \alpha_2; \beta_1, \beta_2)$*-BF subsemigroup on* $S$ *for all* $\alpha_1 \geq \dfrac{1}{4}$ *and* $\beta_2 \leq -\dfrac{1}{4}$*.*

**Definition 6.** *A BF set* $f = (S; f_p, f_n)$ *on* $S$ *is called a* $(\alpha_1, \alpha_2; \beta_1, \beta_2)$***-BF left (right) ideal*** *on* $S$*, where* $\alpha_1, \alpha_2 \in [0,1]$*, and* $\beta_1, \beta_2 \in [-1,0]$ *if it satisfies the following conditions:*

(1) $f_p(xy) \vee \alpha_1 \geq f_p(y) \wedge \alpha_2$ $(f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge \alpha_2)$
(2) $f_n(xy) \wedge \beta_2 \leq f_n(y) \vee \beta_1$ $(f_n(xy) \wedge \beta_2 \leq f_n(x) \vee \beta_1)$

*for all* $x, y \in S$*.*

A BF set $f = (S; f_p, f_n)$ on $S$ is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$**-BF ideal** on $S$ $(\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0])$ if it is both a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal on $S$.

By Definition 6, every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF ideal on a semigroup $S$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup on $S$.

We note that a $(0,1,-1,0)$-BF left (right) ideal is a BF left (right) ideal.

**Definition 7.** *A $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup $f = (S; f_p, f_n)$ on a subsemigroup $S$ is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-**BF bi-ideal** on $S$, where $\alpha_1, \alpha_2 \in [0,1]$, and $\beta_1, \beta_2 \in [-1,0]$ if it satisfies the following conditions:*

*(1)* $f_p(xay) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$
*(2)* $f_n(xay) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

*for all $x, y \in S$.*

We note that every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on a semigroup is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup on the semigroup.

## 3. Generalized Bi-Ideal and Quasi-Ideal

In this section, we introduce a product of BF sets and characterize a regular semigroup by generalized BF subsemigroups.

We let $f = (S : f_p, f_n)$ and $g = (S : g_p, g_n)$ be two BF sets on a semigroup $S$ and let $\alpha_1, \alpha_2 \in [0,1]$, and $\beta_1, \beta_2 \in [-1,0]$. We define two fuzzy sets $f_p^{(\alpha_1,\alpha_2)}$and $f_n^{(\beta_1,\beta_2)}$ on $S$ as follows:

$$f_p^{(\alpha_1,\alpha_2)}(x) = (f_p(x) \wedge \alpha_1) \vee \alpha_2$$

$$f_n^{(\beta_1,\beta_2)}(x) = (f_n(x) \vee \beta_2) \wedge \beta_1$$

for all $x \in S$.

We define two operations $\overset{(\alpha_1,\alpha_2)}{\wedge}$ and $\underset{(\beta_1,\beta_2)}{\vee}$ on $S$ as follows:

$$(f_p \overset{(\alpha_1,\alpha_2)}{\wedge} g_p)(x) = ((f_p \wedge g_p)(x) \wedge \alpha_1) \vee \alpha_2$$

$$(f_n \underset{(\beta_1,\beta_2)}{\wedge} g_n)(x) = ((f_n \vee g_n)(x) \vee \alpha_2) \wedge \alpha_1$$

for all $x \in S$, and we define products $f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p$ and $f_n \underset{(\beta_1,\beta_2)}{\circ} g_n$ as follows:

For all $x \in S$,

$$(f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p)(x) = ((f_p \bar{\circ} g_p)(x) \wedge \alpha_1) \vee \alpha_2$$
$$(f_n \underset{(\beta_1,\beta_2)}{\circ} g_n)(x) = ((f_n \underline{\circ} g_n)(x) \vee \alpha_2) \wedge \alpha_1$$

where

$$(f_p \bar{\circ} g_p)(x) = \begin{cases} \bigvee_{x=yz}\{f_p(y) \wedge g_p(z)\} & \text{if } \ x = yz \ \text{ for some } \ y,z \in S \\ 0 & \text{otherwise} \end{cases}$$

$$(f_n \underline{\circ} g_n)(x) = \begin{cases} \bigwedge_{y=yz}\{f_n(y) \vee g_n(z)\} & \text{if } \ x = yz \ \text{ for some } \ y,z \in S \\ 0 & \text{otherwise} \end{cases}$$

We set

$$f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g := (S; f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p, f_n \underset{(\beta_1,\beta_2)}{\circ} g_n)$$

Then it is a BF set.

We note that

(1) $f_p^{(1,0)}(x) = f_p(x)$,
(2) $f_n^{(0,-1)}(x) = f_n(x)$,

(3) $f = (S; f_p, f_n) = (S; f_p^{(1,0)}, f_n^{(0,-1)})$,

(4) $(f \overset{(\alpha_1,\alpha_2)}{\underset{(\beta_1,\beta_2)}{\circ}} g)_p = f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p$ and $(f \overset{(\alpha_1,\alpha_2)}{\underset{(\beta_1,\beta_2)}{\circ}} g)_n = f_n \underset{(\beta_1,\beta_2)}{\circ} g_n$.

**Definition 8.** *A BF set $f = (S; f_p, f_n)$ on $S$ is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-**BF generalized bi-ideal** on $S$, where $\alpha_1, \alpha_2 \in [0,1]$, and $\beta_1, \beta_2 \in [-1,0]$ if it satisfies the following conditions:*

*(1)* $f_p(xay) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$

*(2)* $f_n(xay) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

*for all $x, y, a \in S$.*

**Definition 9.** *A BF set $f = (S; f_p, f_n)$ on $S$ is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-**BF quasi-ideal** on $S$, where $\alpha_1, \alpha_2 \in [0,1]$, and $\beta_1, \beta_2 \in [-1,0]$ if it satisfies the following conditions:*

*(1)* $f_p(x) \vee \alpha_1 \geq (f_p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} f_p)(x) \wedge \alpha_2$

*(2)* $f_n(x) \wedge \beta_2 \leq (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) \vee \beta_1$

*for all $x, y \in S$.*

In the following theorem, we give a relation between a bipolar $(\alpha, \beta)$-cut of $f$ and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$.

**Theorem 2.** *Let $f = (S; f_p, f_n)$ be a BF set on a semigroup $S$ with $Im(f_p) \subseteq \Delta^+ \subseteq [0,1]$ and $Im(f_n) \subseteq \Delta^- \subseteq [-1,0]$. Then $C(f; (\alpha, \beta))(\neq\emptyset)$ is a generalized bi-ideal of $S$ for all $\alpha \in \Delta^+$ and $\beta \in \Delta^-$ if and only if $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$.*

**Proof.** Let $\alpha_1, \alpha_2 \in [0,1], \beta_1, \beta_2 \in [-1,0]$. Suppose on the contrary that $f$ is not a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$. Then there exists $x, y, a \in S$ such that

$$f_p(xay) \vee \alpha_1 < f_p(x) \wedge f_p(y) \wedge \alpha_2 \text{ or } f_n(xay) \wedge \beta_2 > f_n(x) \vee f_n(y) \vee \beta_1 \tag{1}$$

Let $\alpha' = f_p(x) \wedge f_p(y)$ and $\beta' = f_n(x) \vee f_n(y)$. Then $x, y \in C(f; (\alpha', \beta'))$. By assumption, we have $xay \in C(f; (\alpha', \beta'))$. By Equation (1), $f_p(xay) \leq f_p(xay) \wedge \alpha_1 < f_p(x) \wedge f_p(y) \wedge \alpha_2 \leq f_p(x) \wedge f_p(y) = \alpha'$ or $f_n(xay) \geq f_n(xay) \wedge \beta_2 > f_n(x) \vee f_n(y) \vee \beta_1 \geq f_n(x) \vee f_n(y) = \beta'$. Thus, $xay \notin C(f; (\alpha', \beta'))$. This is a contradiction. Therefore $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$.

Conversely, let $\alpha \in \Delta^+$, and $\beta \in \Delta^-$, and suppose that $C(f; (\alpha, \beta)) \neq \emptyset$. Let $a \in S$ and $x, y \in C(f; (\alpha, \beta))$. Then $f_p(x) \geq \alpha, f_p(y) \geq \alpha, f_n(x) \leq \beta$ and $f_n(y) \leq \beta$. By assumption, $f$ is a $(\alpha, f_p(xay); f_n(xay), \beta)$-BF generalized bi-ideal on $S$, and thus $f_p(xay) \vee f_p(xay) \geq f_p(x) \wedge f_p(y) \wedge \alpha$ and $f_n(xay) \wedge f_n(xay) \leq f_n(x) \vee f_n(y) \vee \beta$. Then $f_p(xay) \geq f_p(x) \wedge f_p(y) \wedge \alpha \geq \alpha \wedge \alpha = \alpha$ and $f_n(xay) \leq f_n(x) \vee f_n(x) \vee \beta \leq \beta \vee \beta = \beta$. Hence, $xay \in C(f; (\alpha, \beta))$. Therefore $C(f; (\alpha, \beta))$ is a generalized bi-ideal of $S$. □

**Corollary 1.** *Let $f = (S; f_n, f_p)$ be a BF set on a semigroup. Then the following statements hold:*

*(1)* *$f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$ if and only if $C(f; (\alpha, \beta))(\neq\emptyset)$ is a generalized bi-ideal of $S$ for all $\alpha \in Im(f_p)$, and $\beta \in Im(f_n)$;*

*(2)* *$f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$, and $\beta_1, \beta_2 \in [-1,0]$ if and only if $C(f; (\alpha, \beta))(\neq\emptyset)$ is a generalized bi-ideal of $S$ for all $\alpha \in [0,1]$, and $\beta \in [-1,0]$.*

**Proof.** (1) Set $\Delta^+ = [0,1]$ and $\Delta^- = [-1,0]$, and apply Theorem 2.

(2) Set $\Delta^+ = Im(f_p)$ and $\Delta^- = Im(f_n)$, and apply Theorem 2. □

**Lemma 1.** *Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on a regular semigroup $S$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on $S$.*

**Proof.** Let $S$ be a regular semigroup and $f = (S; f_p, f_n)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$. Let $a, b \in S$; then there exists $x \in S$ such that $b = bxb$. Thus we have $f_p(ab) \vee \alpha_1 = f_p(a(bxb)) \vee \alpha_1 = f_p(a(bx)b) \vee \alpha_1 \geq f_p(a) \wedge f_p(b) \wedge \alpha_2$ and $f_n(ab) \wedge \beta_2 = f_n(a(bxb)) \wedge \beta_2 = f_n(a(bx)b) \wedge \beta_2 \leq f_n(a) \vee f_n(b) \vee \beta_1$. This shows that $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF subsemigroup on $S$, and thus $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on $S$. □

Let $S$ be a semigroup and $\emptyset \neq I \subseteq S$. A positive characteristic function and a negative characteristic function are respectively defined by

$$C_I^p : S \to [0,1], x \mapsto C_I^p(x) := \begin{cases} 1, & x \in I \\ 0, & x \notin I \end{cases}$$

and

$$C_I^n : S \to [-1,0], x \mapsto C_I^n(x) := \begin{cases} -1, & x \in I \\ 0, & x \notin I \end{cases}$$

**Remark 2.**

(1) *For the sake of simplicity, we use the symbol* $C_I = (S; C_I^p, C_I^n)$ *for the BF set. That is,* $C_I = (S; C_I^p, C_I^n) = (S; (C_I)_p, (C_I)_n)$. *We call this a bipolar characteristic function.*

(2) *If* $I = S$, *then* $C_S = (S; C_S^p, C_S^n)$. *In this case, we denote* $\mathcal{S} = (S, \mathcal{S}_p, \mathcal{S}_n)$.

In the following theorem, some necessary and sufficient conditions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideals are obtained.

**Theorem 3.** *Let* $f = (S; f_p, f_n)$ *be a BF set on a semigroup S. Then the following statements are equivalent:*

(1) $f$ *is a* $(\alpha_1, \alpha_2; \beta_1, \beta_2)$*-BF generalized bi-ideal on S.*

(2) $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ *and* $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$.

**Proof.** $(\Rightarrow)$ Let $a$ be any element of $S$. In the case for which $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) = 0$, it is clear that $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$. Otherwise, there exist $x, y, r, s \in S$ such that $a = xy$ and $x = rs$. Because $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$, we have $f_p(rsy) \vee \alpha_1 \geq f_p(r) \wedge f_p(s) \wedge \alpha_2$ and $f_n(rsy) \wedge \beta_2 \leq f_n(r) \vee f_n(s) \vee \beta_1$. Consider

$$\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) &= ((f_p \bar{\circ} \mathcal{S}_p \bar{\circ} f_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=xy} \{(f_p \bar{\circ} \mathcal{S}_p)(x) \wedge f_p(y)\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=xy} \{\bigvee_{x=rs} \{f_p(r) \wedge \mathcal{S}_p(s)\} \wedge f_p(y)\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=xy} \{\bigvee_{x=rs} \{f_p(r) \wedge 1\} \wedge f_p(y)\} \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=rsy} \{f_p(r) \wedge f_p(y) \wedge \alpha_2\} \wedge \alpha_2) \vee \alpha_1 \\
&\leq (\bigvee_{a=rsy} \{f_p(rsy) \vee \alpha_1\} \wedge \alpha_2) \vee \alpha_1 \\
&\leq (\bigvee_{a=rsy} \{f_p(a) \vee \alpha_1\} \wedge \alpha_2) \vee \alpha_1 \\
&\leq ((f_p(a) \vee \alpha_1) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= f_p^{(\alpha_2,\alpha_1)}(a)
\end{aligned}$$

Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$.

Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$.

$(\Leftarrow)$ Conversely, let $a, x, y, z \in S$ such that $a = xyz$. Then we have

$$\begin{aligned}
f_p(xyz) \vee \alpha_1 &\geq (f_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= f_p^{(\alpha_2,\alpha_1)}(a) \\
&\geq (f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) \\
&= ((f_p \bar{\circ} \mathcal{S}_p \bar{\circ} f_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=bc} \{(f_p \bar{\circ} \mathcal{S}_p)(b) \wedge f_p(c)\} \wedge \alpha_2) \vee \alpha_1 \\
&\geq ((f_p \bar{\circ} \mathcal{S}_p)(xy) \wedge f_p(z) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{xy=rs} \{f_p(r) \wedge \mathcal{S}_p(s)\} \wedge f_p(z) \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(x) \wedge \mathcal{S}_p(y) \wedge f_p(z) \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(x) \wedge f_p(z) \wedge \alpha_2) \vee \alpha_1 \\
&\geq f_p(x) \wedge f_p(z) \wedge \alpha_2
\end{aligned}$$

Similarly, we can show that $f_n(xyz) \wedge \beta_2 \leq f_n(x) \vee f_n(z) \vee \beta_1$ for all $x, y, z \in S$. Therefore $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$. □

**Theorem 4.** *Let $f = (S; f_p, f_n)$ be a BF set on a semigroup $S$. Then the following statements are equivalent:*

*(1)* $f$ *is a* $(\alpha_1, \alpha_2; \beta_1, \beta_2)$*-BF bi-ideal on* $S$ *.*

*(2)* $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ *and* $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$.

**Proof.** The proof is similar to the proof of Theorem 3. □

In the following theorem, we give a relation between a bipolar $(\alpha, \beta)$-cut of $f$ and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$.

**Theorem 5.** *Let $f = (S; f_p, f_n)$ be a BF set on a semigroup $S$ with $Im(f_p) \subseteq \Delta^+ \subseteq [0, 1]$ and $Im(f_n) \subseteq \Delta^- \subseteq [-1, 0]$. Then $C(f; (\alpha, \beta))(\neq\emptyset)$ is a quasi-ideal of $S$ for all $\alpha \in \Delta^+$ and $\beta \in \Delta^-$ if and only if $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$.*

**Proof.** $(\Rightarrow)$ Let $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$. Suppose on the contrary that $f$ is not a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$. Then there exists $x \in S$ such that

$$f_p(x) \vee \alpha_1 < (f_p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} f_p)(x) \wedge \alpha_2$$

or

$$f_n(x) \wedge \beta_2 > (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) \vee \beta_1 \tag{2}$$

Case 1: $f_p(x) \vee \alpha_1 < (f_p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} f_p)(x) \wedge \alpha_2$. Let $\alpha' = (f_p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} f_p)(x)$. Then $\alpha' \leq (f_p \bar{\circ} \mathcal{S}_p)(x), \alpha' \leq (\mathcal{S}_p \bar{\circ} f_p)(x)$. This implies that there exist $a, b, c, d \in S$ such that $x = ab = cd$. Then

$$\alpha' \leq (f_p \bar{\circ} \mathcal{S}_p)(x) = \bigvee_{x=yz} \{f_p(y) \wedge \mathcal{S}_p(z)\} \leq f_p(a) \wedge \mathcal{S}_p(b) = f_p(a)$$

$$\alpha' \leq (\mathcal{S}_p \overline{\circ} f_p)(x) = \bigvee_{x=yz} \{\mathcal{S}_p(z) \wedge f_p(y)\} \leq \mathcal{S}_p(c) \wedge f_p(d) = f_p(d)$$

Let $\beta' = f_n(a) \vee f_n(d)$. Then $f_n(a) \leq \beta'$ and $f_n(d) \leq \beta'$.

Thus $a, d \in C(f; (\alpha', \beta'))$, and so $ad \in C(f; (\alpha', \beta'))S$ and $ad \in SC(f; (\alpha', \beta'))$. Hence $x \in C(f; (\alpha', \beta'))S$ and $x \in SC(f; (\alpha', \beta'))$, and it follows that $x \in C(f; (\alpha', \beta'))S \cap SC(f; (\alpha', \beta'))$. By hypothesis, $x \in C(f; (\alpha', \beta'))$.

Case 2: $f_n(x) \wedge \beta_2 > (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) \vee \beta_1$. Let $\beta' = (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x)$. Then $\beta' \geq (f_n \underline{\circ} \mathcal{S}_n)(x)$ and $\beta' \geq (\mathcal{S}_n \underline{\circ} f_n)(x)$. This implies that there exist $a', b', c', d' \in S$ such that $x = a'b' = c'd'$. Then

$$\beta' \geq (f_n \underline{\circ} \mathcal{S}_n)(x) = \bigwedge_{x=yz} \{f_n(y) \vee \mathcal{S}_n(z)\} \geq f_n(a') \vee \mathcal{S}_n(b') \geq f_n(a')$$

$$\beta' \geq (\mathcal{S}_n \underline{\circ} f_n)(x) = \bigwedge_{x=yz} \{\mathcal{S}_n(z) \vee f_n(y)\} \geq \mathcal{S}_n(c') \vee f_n(d') \geq f_n(d')$$

Let $\alpha' = f_p(a') \wedge f_p(d')$. Then $f_p(a') \geq \alpha'$ and $f_p(d') \geq \alpha'$. Thus $a', d' \in C(f; (\alpha', \beta'))$, and so $a'd' \in C(f; (\alpha', \beta'))S$ and $a'd' \in SC(f; (\alpha', \beta'))$. Hence $x \in C(f; (\alpha', \beta'))S$ and $x \in SC(f; (\alpha', \beta'))$, and it follows that $x \in C(f; (\alpha', \beta'))S \cap SC(f; (\alpha', \beta'))$. By hypothesis, $x \in C(f; (\alpha', \beta'))$. Therefore $x \in C(f; (\alpha', \beta'))$. By Equation (2),

$$f_p(x) \leq f_p(x) \vee \alpha_1 < (f_p \overline{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \overline{\circ} f_p)(x) \wedge \alpha_2 \leq (f_p \overline{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \overline{\circ} f_p)(x) = \alpha'$$

or

$$f_n(x) \geq f_n(x) \wedge \beta_2 > (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) \vee \beta_1 \geq (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) = \beta'$$

and it follows that $x \notin C(f; (\alpha', \beta'))$. This is a contradiction. Therefore $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$.

$(\Leftarrow)$ Conversely, let $\alpha \in \Delta^+$ and $\beta \in \Delta^-$, and suppose that $C(f; (\alpha, \beta)) \neq \emptyset$. Let $x \in S$ be such that $x \in C(f; (\alpha, \beta))S \cap SC(f; (\alpha, \beta))$. Then $x \in C(f; (\alpha, \beta))S$ and $x \in SC(f; (\alpha, \beta))$. Thus there exist $y, z' \in C(f; (\alpha, \beta))$ and $z, y' \in S$ such that $x = yz$ and $x = y'z'$.

By assumption, $f$ is a $(f_p(x), \alpha; \beta, f_n(x))$-BF quasi-ideal on $S$, and thus

$$\begin{aligned}
f_p(x) &= f_p(x) \vee f_p(x) \\
&\geq (f_p \overline{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \overline{\circ} f_p)(x) \wedge \alpha \\
&= \bigvee_{x=yz} \{f_p(y) \wedge \mathcal{S}_p(z)\} \wedge \bigvee_{x=y'z'} \{\mathcal{S}_p(y') \wedge f_p(z')\} \wedge \alpha \\
&= \bigvee_{x=yz} \{f_p(y) \wedge 1\} \wedge \bigvee_{x=y'z'} \{1 \wedge f_p(z')\} \wedge \alpha \\
&= \bigvee_{x=yz} \{f_p(y)\} \wedge \bigvee_{x=y'z'} \{f_p(z')\} \wedge \alpha \\
&\geq f_p(y) \wedge f_p(z') \wedge \alpha
\end{aligned}$$

Because $y, z' \in C(f; (\alpha, \beta))$, we have $f_p(y) \geq \alpha$ and $f_p(z') \geq \alpha$. Then $f_p(x) \geq \alpha$. Similarly, we can show that $f_n(x) \leq \beta$. Hence, $x \in C(f; (\alpha, \beta))$. Therefore $C(f; (\alpha, \beta))$ is a quasi-ideal of $S$. □

**Corollary 2.** *Let $f = (S; f_n, f_p)$ be a BF set on a semigroup $S$. Then*

*(1)* *$f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$ if and only if $C(f; (\alpha, \beta))(\neq \emptyset)$ is a quasi-ideal of $S$ for all $\alpha \in Im(f_p)$ and $\beta \in Im(f_n)$;*

*(2)* *$f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$ if and only if $C(f; (\alpha, \beta))(\neq \emptyset)$ is a quasi-ideal of $S$ for all $\alpha \in [0, 1]$ and $\beta \in [-1, 0]$.*

**Proof.** (1) Set $\Delta^+ = [0,1]$ and $\Delta^- = [-1,0]$, and apply Theorem 5.

(2) Set $\Delta^+ = Im(f_p)$ and $\Delta^- = Im(f_n)$, and apply Theorem 5. □

In the following theorem, we discuss a quasi-ideal of a semigroup $S$ in terms of the bipolar characteristic function being a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$.

**Theorem 6.** *Let $S$ be a semigroup. Then a non-empty subset $I$ is a quasi-ideal of $S$ if and only if the bipolar characteristic function $C_I = (S; C_I^p, C_I^n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$.*

**Proof.** $(\Rightarrow)$ Let $I$ be a quasi-ideal of $S$ and $x \in S$. Let $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$.

Case 1: $x, y \in I$. Then

$$C_I^p(x) \vee \alpha_1 = 1 \geq (C_I^p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} C_I^p)(x) \wedge \alpha_2$$

$$C_I^n(x) \wedge \beta_2 = -1 \leq (C_I^n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} C_I^n)(x) \vee \beta_1$$

Case 2: $x \notin I$. Then $x \notin SI$ or $x \notin IS$. If $x \notin SI$, then $(C_I^p \bar{\circ} \mathcal{S}_p)(x) = 0$ and $(C_I^n \underline{\circ} \mathcal{S}_n)(x) = 0$. Thus

$$C_I^p(x) \vee \alpha_1 \geq 0 = (C_I^p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} C_I^p)(x) \wedge \alpha_2$$

$$C_I^n(x) \wedge \beta_2 \leq 0 = (C_I^n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} C_I^n)(x) \vee \beta_1$$

Therefore $C_I = (S; C_I^p, C_I^n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$.
$(\Leftarrow)$ Conversely, let $C_I$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$. Let $a \in IS \cap SI$. Then there exist $a, b \in S$ and $x, y \in I$ such that $a = xb = cy$. Then $(C_I)_p(x) = 1 = (C_I)_p(y)$ and $(C_I)_n(x) = -1 = (C_I)_n(y)$. Hence $x, y \in C(C_I; (1,-1))$, and so $xb \in C(C_I; (1,-1))S$ and $cy \in SC(C_I; (1,-1))$. Hence $a \in C(C_I; (1,-1))S$ and $a \in SC(C_I; (1,-1))$, and it follows that $a \in C(C_I; (1,-1))S \cap SC(C_I; (1,-1))$. By Corollary 2, $C(C_I; (1,-1))$ is a quasi-ideal. Thus $a \in C(C_I; (1,-1))$, and so $C_I^p(a) \geq 1$. This implies that $a \in I$. Therefore $I$ is a quasi-ideal on $S$. □

**Theorem 7.** *Let $S$ be a semigroup. Then $I$ is a generalized bi-ideal of $S$ if and only if the bipolar characteristic function $C_I = (S; C_I^p, C_I^n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$.*

**Proof.** $(\Rightarrow)$ Let $I$ be a generalized bi-ideal of $S$ and $x, y, a \in S$. Let $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$.

Case 1: $x, y \in I$. Then $xay \in I$; thus

$$C_I^p(xay) \vee \alpha_1 = 1 \geq C_I^p(x) \wedge C_I^p(y) \wedge \alpha_2$$

and

$$C_I^n(xay) \wedge \beta_2 = -1 \leq C_I^n(x) \vee C_I^n(y) \vee \beta_1$$

Case 2: $x \notin I$ or $y \notin I$. Then

$$C_I^p(xay) \vee \alpha_1 \geq 0 = C_I^p(x) \wedge C_I^p(y) \wedge \alpha_2$$

and

$$C_I^n(xay) \wedge \beta_2 \leq 0 = C_I^n(x) \vee C_I^n(y) \vee \beta_1$$

Therefore $C_I = (S; C_I^p, C_I^n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$.
$(\Leftarrow)$ Conversely, let $C_I$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ for all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [-1,0]$. Let $a \in S$ and $x, y \in I$. Then $(C_I)_p(x) = 1 = (C_I)_p(y)$ and $(C_I)_n(x) = -1 = (C_I)_n(y)$. Hence, $x, y \in C(C_I; (1,-1))$. By Corollary 1, $C(C_I; (1,-1))$ is a generalized bi-ideal. Thus $xay \in C(C_I; (1,-1))$, and so $C_I^p(xay) \geq 1$. This implies that $xay \in I$. Therefore $I$ is a generalized bi-ideal on $S$. □

**Theorem 8.** *Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left (right) ideal on a semigroup $S$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$.*

**Proof.** Let $f = (S; f_p, f_n)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal on $S$ and $x, y \in S$. Then

$$\begin{aligned}
(\mathcal{S}\bar{\circ}f)_p(xy) \wedge \alpha_2 &= (\mathcal{S}_p\bar{\circ}f_p)(xy) \wedge \alpha_2 \\
&= (\bigvee_{x=yz} \{\mathcal{S}_p(y) \wedge f_p(z)\}) \wedge \alpha_2 \\
&= \bigvee_{x=yz} \{f_p(z)\} \wedge \alpha_2 \\
&\leq \bigvee_{x=yz} \{f_p(yz)\} \vee \alpha_1 \\
&\leq f_p(x) \vee \alpha_1
\end{aligned}$$

Thus $f_p(x) \vee \alpha_1 \geq (\mathcal{S}_p\bar{\circ}f_p)(xy) \wedge \alpha_2$. Hence $f_p(x) \vee \alpha_1 \geq (\mathcal{S}_p\bar{\circ}f_p)(xy) \wedge \alpha_2 \geq (\mathcal{S}_p\bar{\circ}f_p)(xy) \wedge (f_p\bar{\circ}\mathcal{S}_p)(xy) \wedge \alpha_2$. Similarly, we can show that $f_n(x) \wedge \beta_2 \leq (f_n\underline{\circ}\mathcal{S}_n)(x) \vee (\mathcal{S}_n\underline{\circ}f_n)(x) \vee \beta_1$. Therefore $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$. □

**Lemma 2.** *Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on a semigroup $S$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on $S$.*

**Proof.** Let $f = (S; f_p, f_n)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal on $S$ and $x, y, z \in S$. Then

$$\begin{aligned}
f_p(xy) \vee \alpha_1 &\geq (f_p\bar{\circ}\mathcal{S}_p)(xy) \wedge (\mathcal{S}_p\bar{\circ}f_p)(xy) \wedge \alpha_2 \\
&= \bigvee_{xy=ab} \{f_p(a) \wedge \mathcal{S}_p(b)\} \wedge \bigvee_{xy=rs} \{\mathcal{S}_p(r) \wedge f_p(s)\} \wedge \alpha_2 \\
&\geq f_p(x) \wedge \mathcal{S}_p(y) \wedge \mathcal{S}_p(x) \wedge f_p(y) \wedge \alpha_2 \\
&\geq f_p(x) \wedge 1 \wedge 1 \wedge f_p(y) \wedge \alpha_2 \\
&= f_p(x) \wedge f_p(y) \wedge \alpha_2
\end{aligned}$$

Hence, $f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$. Additionally,

$$\begin{aligned}
f_p(xyz) \vee \alpha_1 &\geq (f_p\bar{\circ}\mathcal{S}_p)(xyz) \wedge (\mathcal{S}_p\bar{\circ}f_p)(xyz) \wedge \alpha_2 \\
&= \bigvee_{xyz=ab} \{f_p(a) \wedge \mathcal{S}_p(b)\} \wedge \bigvee_{xyz=rs} \{\mathcal{S}_p(r) \wedge f_p(s)\} \wedge \alpha_2 \\
&\geq f_p(x) \wedge \mathcal{S}_p(yz) \wedge \mathcal{S}_p(xy) \wedge f_p(z) \wedge \alpha_2 \\
&\geq f_p(x) \wedge 1 \wedge 1 \wedge f_p(z) \wedge \alpha_2 \\
&= f_p(x) \wedge f_p(z) \wedge \alpha_2
\end{aligned}$$

Hence, $f_p(xyz) \vee \alpha_1 \geq f_p(x) \wedge f_p(z) \wedge \alpha_2$. Similarly, we can show that $f_n(xy) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$ and $f_n(xyz) \wedge \beta_2 \leq f_n(x) \vee f_n(z) \vee \beta_1$. Therefore $f$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on $S$. □

**Lemma 3.** *Let $A$ and $B$ be non-empty subsets of a semigroup $S$. Then the following conditions hold:*

*(1)* $(C_A)_p \overset{(\alpha_2,\alpha_1)}{\wedge} (C_B)_p = (C_{A\cap B})_p^{(\alpha_2,\alpha_1)}$.

*(2)* $(C_A)_n \overset{(\beta_2,\beta_1)}{\vee} (C_B)_n = (C_{A\cup B})_n^{(\beta_2,\beta_1)}$.

*(3)* $(C_A)_p \overset{(\alpha_2,\alpha_1)}{\circ} (C_B)_p = (C_{AB})_p^{(\alpha_2,\alpha_1)}$.

*(4)* $(C_A)_n \overset{(\beta_2,\beta_1)}{\circ} (C_B)_n = (C_{AB})_n^{(\beta_2,\beta_1)}$.

**Lemma 4.** *If $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal and $g = (S; g_p, g_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal on a semigroup $S$, then $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.*

**Theorem 9.** *For a semigroup $S$, the following are equivalent.*
(1) *$S$ is regular.*
(2) *$f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*

Next, we characterize a regular semigroup by generalizations of BF subsemigroups.

**Theorem 10.** *For a semigroup $S$, the following are equivalent.*

(1) *$S$ is regular.*
(2) *$f_p \overset{(\alpha_2,\alpha_1)}{\wedge} h_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} h_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} h_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} h_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal $f = (S; f_p, f_n)$, every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal $h = (S; h_p, h_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*
(3) *$f_p \overset{(\alpha_2,\alpha_1)}{\wedge} h_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} h_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} h_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} h_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal $f = (S; f_p, f_n)$ , every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal $h = (S; h_p, h_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*
(4) *$f_p \overset{(\alpha_2,\alpha_1)}{\wedge} h_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} h_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} h_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} h_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal $f = (S; f_p, f_n)$ , every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal $h = (S; h_p, h_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*

**Proof.** $(1 \Rightarrow 2)$. Let $f, h$ and $g$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal, a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal on $S$, respectively. Let $a \in S$. Because $S$ is regular, there exists $x \in S$ such that $a = axa$. Thus

$$
\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} h_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) &= ((f_p \bar{\circ} h_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=yz} \{f_p(y) \wedge (h_p \bar{\circ} g_p)(z) \wedge \alpha_2\}) \vee \alpha_1 \\
&\geq (f_p(ax) \wedge (h_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(ax) \vee \alpha_1) \wedge ((h_p \bar{\circ} g_p)(a) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq (f_p(a) \wedge \alpha_2) \wedge (\bigvee_{a=rs} \{(h_p(r) \wedge g_p(s)) \vee \alpha_1\}) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq (f_p(a) \wedge \alpha_2) \wedge ((h_p(a) \vee \alpha_1) \wedge (g_p(xa) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq (f_p(a) \wedge \alpha_2) \wedge ((h_p(a) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq (f_p(a) \wedge h_p(a) \wedge g_p(a) \wedge \alpha_2) \wedge \alpha_1 \\
&= ((f_p \wedge h_p \wedge g_p)(a) \wedge \alpha_2) \wedge \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} h_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a)
\end{aligned}
$$

Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\vee} h_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} h_n \underset{(\beta_2,\beta_1)}{\circ} g_n$.

$(2 \Rightarrow 3 \Rightarrow 4)$. This is straightforward, because every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal on $S$.

$(4 \Rightarrow 1)$. Let $f$ and $g$ be any $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal on $S$, respectively. Let $a \in S$. By Theorem 8, $\mathcal{S} = (S, \mathcal{S}_p, \mathcal{S}_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi ideal, and we have

$$\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a) &= ((f_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p \wedge \mathcal{S}_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a) \\
&\leq (f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) \\
&\leq (f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a)
\end{aligned}$$

Thus $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF right ideal $f$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF left ideal $g$ on $S$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. By Lemma 4, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n = f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Therefore by Theorem 9, $S$ is regular. □

**Theorem 11.** *For a semigroup $S$, the following are equivalent.*

*(1)* *$S$ is regular.*

*(2)* *$f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal $f = (S; f_p, f_n)$ on $S$.*

*(3)* *$f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF bi-ideal $f = (S; f_p, f_n)$ on $S$.*

*(4)* *$f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi-ideal $f = (S; f_p, f_n)$ on $S$.*

**Proof.** $(1 \Rightarrow 2)$. Let $f$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF generalized bi-ideal on $S$ and $a \in S$. Because $S$ is regular, there exists $x \in S$ such that $a = axa$. Hence we have

$$\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) &= ((f_p \bar{\circ} \mathcal{S}_p \bar{\circ} f_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=yz} \{(f_p \bar{\circ} \mathcal{S}_p)(y) \wedge f_p(z)\} \wedge \alpha_2) \vee \alpha_1 \\
&\geq ((f_p \bar{\circ} \mathcal{S}_p)(ax) \wedge f_p(a)) \wedge \alpha_2) \vee \alpha_1 \\
&= ((\bigvee_{ax=rs} \{f_p(r) \wedge \mathcal{S}_p(s)\}) \wedge f_p(a)) \wedge \alpha_2) \vee \alpha_1 \\
&\geq (((f_p(a) \wedge \mathcal{S}_p(x)) \wedge f_p(a)) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p(a) \wedge 1)) \wedge f_p(a)) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= f_p^{(\alpha_2,\alpha_1)}(a)
\end{aligned}$$

Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \geq f_p^{(\alpha_2,\alpha_1)}$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \leq f_n^{(\beta_2,\beta_1)}$. By Theorem 3, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$. Therefore, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n^{(\beta_2,\beta_1)}$.

$(2 \Rightarrow 3 \Rightarrow 4)$. Obvious.

$(4 \Rightarrow 1)$. Let $Q$ be any quasi-ideal of $S$. By Theorem 6 and Lemma 3, we have

$$\begin{aligned}(C_Q)_p^{(\alpha_2,\alpha_1)} &= (C_Q)_p \overset{(\alpha_2,\alpha_1)}{\circ} (\mathcal{S})_p \overset{(\alpha_2,\alpha_1)}{\circ} (C_Q)_p \\ &= (C_{QSQ})_p^{(\alpha_2,\alpha_1)}\end{aligned}$$

Thus, $Q = QSQ$. Therefore it follows from Theorem 1 that $S$ is regular. □

**Theorem 12.** *For a semigroup $S$, the following are equivalent.*

*(1)* *$S$ is regular.*

*(2)* $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ *and* $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ *for every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF generalized bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*

*(3)* $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ *and* $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ *for every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*

*(4)* $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ *and* $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ *for every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF quasi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF left ideal $g = (S; g_p, g_n)$ on $S$.*

**Proof.** $(1 \Rightarrow 2)$. Let $f$ and $g$ be any $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF generalized bi-ideal and any $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF left ideal on $S$, respectively. Let $a \in S$. Because $S$ is regular, there exists $x \in S$ such that $a = axa$. Thus we have

$$\begin{aligned}(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) &= ((f_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\ &= (\bigvee_{a=yz} \{f_p(y) \wedge g_p(z)\} \wedge \alpha_2) \vee \alpha_1 \\ &\geq (f_p(a) \wedge g_p(xa)) \wedge \alpha_2) \vee \alpha_1 \\ &\geq (f_p(a) \vee \alpha_1) \wedge (g_p(xa) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\ &\geq (f_p(a) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\ &= (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\ &= ((f_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\ &= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a)\end{aligned}$$

Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

$(2 \Rightarrow 3 \Rightarrow 4)$. Obvious.

$(4 \Rightarrow 1)$. Let $f$ and $g$ be any $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF right ideal and $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF left ideal on $S$, respectively. By Theorem 8, $f$ is a $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BF quasi ideal. Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$

and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. By Lemma 4, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n = f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Therefore by Theorem 9, $S$ is regular. $\square$

## 4. Conclusions

In this paper, we propose the generalizations of BF sets. In particular, we introduce several concepts of generalized BF sets and study the relationship between such sets and semigroups. In other words, we propose generalized BF subsemigroups. This under consideration, the results obtained in this paper are some inequalities of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BF quasi(generalized bi-, bi-) ideals and characterize a regular semigroup in terms of generalized BF semigroups. The importance of BF sets has positive and negative components frequently found in daily life, for example, in organizations, economics, performance, development, evaluation, risk management or decisions, and so forth. Therefore we establish generalized BF sets on semigroups, which enhances the structure of the algebra. We hope that the study of some types of subsemigroups characterized in terms of inequalities of generalized BF subsemigroups is a useful mathematical tool.

**Author Contributions:** Both authors contributed equally to this manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Bloch, I. Bipolar Fuzzy Mathematical Morphology for Spatial Reasoning. *Int. J. Approx. Reason.* **2012**, *53*, 1031–1070.
2. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
3. Bucolo, M.; Fazzino, S.; La Rosa, M.; Fortuna, L. Small-world networks of fuzzy chaotic oscillators. *Chaos Solitons Fractals* **2003**, *17*, 557–565.
4. Rosenfeld, A. Fuzzy group. *J. Math. Anal. Appl.* **1971**, *35*, 338–353.
5. Kuroki, N. On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets Syst.* **1981**, *5*, 203–215.
6. Zhang, W.R. Bipolar Fuzzy Sets and Relations: A Computational Framework Forcognitive Modeling and Multiagent Decision Analysis. In Proceedings of the 1994 Industrial Fuzzy Control and Intelligent Systems Conference and the NASA Joint Technology Workshop on Neural Networks and Fuzzy Logic Fuzzy Information Processing Society Biannual Conference, San Antonio, TX, USA, 18–21 December 1994; pp. 305–309.
7. Lee, K.M. Bipolar-Valued Fuzzy Sets and Their Operations. In Proceedings of the International Conference on Intelligent Technologies, Bangkok, Thailand, 13–15 December 2000; pp. 307–312.
8. Kim, C.S.; Kang, J.G.; Kang, J.M. Ideal theory of semigroups based on the bipolar valued fuzzy set theory. *Ann. Fuzzy Math. Inform.* **2011**, *2*, 193–206.
9. Mordeson, J.N.; Malik, D.S.; Kuroki, N. *Fuzzy Semigroups*; Springer: Berlin, Germany, 2003.