mathematics

Article

Length-Fuzzy Subalgebras in BCK/BCI-Algebras

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Received: 1 December 2017; Accepted: 5 January 2018; Published: 12 January 2018

Abstract: As a generalization of interval-valued fuzzy sets and fuzzy sets, the concept of hyperfuzzy sets was introduced by Ghosh and Samanta in the paper [J. Ghosh and T.K. Samanta, Hyperfuzzy sets and hyperfuzzy group, Int. J. Advanced Sci Tech. 41 (2012), 27–37]. The aim of this manuscript is to introduce the length-fuzzy set and apply it to BCK/BCI-algebras. The notion of length-fuzzy subalgebras in BCK/BCI-algebras is introduced, and related properties are investigated. Characterizations of a length-fuzzy subalgebra are discussed. Relations between length-fuzzy subalgebras and hyperfuzzy subalgebras are established.

Keywords: hyperfuzzy set; hyperfuzzy subalgebra; length of hyperfuzzy set; length-fuzzy subalgebra

MSC: 06F35; 03G25; 03B52

1. Introduction

Fuzzy set theory was firstly introduced by Zadeh [1] and opened a new path of thinking to mathematicians, physicists, chemists, engineers and many others due to its diverse applications in various fields. Algebraic hyperstructure, which was introduced by the French mathematician Marty [2], represents a natural extension of classical algebraic structures. Since then, many papers and several books have been written in this area. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer science. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The study of fuzzy hyperstructures is an interesting research area of fuzzy sets. As a generalization of fuzzy sets and interval-valued fuzzy sets, Ghosh and Samanta [3] introduced the notion of hyperfuzzy sets, and applied it to group theory. Jun et al. [4] applied the hyperfuzzy sets to BCK/BCI-algebras, and introduced the notion of k-fuzzy substructures for $k \in \{1,2,3,4\}$. They introduced the concepts of hyperfuzzy substructures of several types by using k-fuzzy substructures, and investigated their basic properties. They also defined hyperfuzzy subalgebras of type (i,j) for $i,j \in \{1,2,3,4\}$, and discussed relations between the hyperfuzzy substructure/subalgebra and its length. They investigated the properties of hyperfuzzy subalgebras related to upper- and lower-level subsets.

In this paper, we introduce the length-fuzzy subalgebra in BCK/BCI-algebras based on hyperfuzzy structures, and investigate several properties.

2. Preliminaries

By a BCI-*algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x*y)*(x*z))*(z*y) = 0$,
(II) $(x*(x*y))*y = 0$,

(III) $x * x = 0$,
(IV) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [5,6] for further information regarding BCK/BCI-algebras.

An ordered pair (X, ρ) of a nonempty set X and a fuzzy set ρ on X is called a *fuzzy structure* over X.

Let X be a nonempty set. A mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is called a *hyperfuzzy set* over X (see [3]), where $\tilde{\mathcal{P}}([0,1])$ is the family of all nonempty subsets of $[0,1]$. An ordered pair $(X, \tilde{\mu})$ is called a *hyper structure* over X.

Given a hyper structure $(X, \tilde{\mu})$ over a nonempty set X, we consider two fuzzy structures $(X, \tilde{\mu}_{\inf})$ and $(X, \tilde{\mu}_{\sup})$ over X in which

$$\tilde{\mu}_{\inf} : X \to [0,1],\ \ x \mapsto \inf\{\tilde{\mu}(x)\},$$
$$\tilde{\mu}_{\sup} : X \to [0,1],\ \ x \mapsto \sup\{\tilde{\mu}(x)\}.$$

Given a nonempty set X, let $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also, $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

Definition 1. *[4] For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a*

- *fuzzy subalgebra of $(X, *, 0)$ with type 1 (briefly, 1-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{3}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 2 (briefly, 2-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{4}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 3 (briefly, 3-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{5}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 4 (briefly, 4-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\,(\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{6}$$

It is clear that every 3-fuzzy subalgebra is a 1-fuzzy subalgebra and every 2-fuzzy subalgebra is a 4-fuzzy subalgebra.

Definition 2. *[4] For any $(X, *, 0) \in \mathcal{B}(X)$ and $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called an (i, j)-hyperfuzzy subalgebra of $(X, *, 0)$ if $(X, \tilde{\mu}_{\inf})$ is an i-fuzzy subalgebra of $(X, *, 0)$ and $(X, \tilde{\mu}_{\sup})$ is a j-fuzzy subalgebra of $(X, *, 0)$.*

3. Length-Fuzzy Subalgebras

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ unless otherwise specified.

Definition 3. *[4] Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, we define*

$$\tilde{\mu}_\ell : X \to [0, 1],\ x \mapsto \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x), \tag{7}$$

which is called the length of $\tilde{\mu}$.

Definition 4. *A hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called a length 1-fuzzy (resp. 2-fuzzy, 3-fuzzy and 4-fuzzy) subalgebra of $(X, *, 0)$ if $\tilde{\mu}_\ell$ satisfies the condition (3) (resp. (4)–(6)).*

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [6]).*

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]),\ \ x \mapsto \begin{cases} [0.2, 0.4) \cup [0.5, 0.8) & \text{if } x = 0, \\ (0.5, 0.9] & \text{if } x = 1, \\ [0.1, 0.3] \cup (0.4, 0.6] & \text{if } x = 2, \\ [0.6, 0.9] & \text{if } x = 3, \\ [0.3, 0.5] & \text{if } x = 4. \end{cases}$$

Then, the length of $\tilde{\mu}$ is given by Table 2.

Table 2. The length of $(X, \tilde{\mu})$.

X	**0**	**1**	**2**	**3**	**4**
$\tilde{\mu}_\ell$	0.6	0.4	0.5	0.3	0.2

It is routine to verify that $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$.

Proposition 1. *If $(X, \tilde{\mu})$ is a length k-fuzzy subalgebra of $(X, *, 0)$ for $k = 1, 3$, then $\tilde{\mu}_\ell(0) \geq \tilde{\mu}_\ell(x)$ for all $x \in X$.*

Proof. Let $(X, \tilde{\mu})$ be a length 1-fuzzy subalgebra of $(X, *, 0)$. Then,

$$\tilde{\mu}_\ell(0) = \tilde{\mu}_\ell(x * x) \geq \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(x)\} = \tilde{\mu}_\ell(x) \tag{8}$$

for all $x \in X$. If $(X, \tilde{\mu})$ is a length 3-fuzzy subalgebra of $(X, *, 0)$, then

$$\tilde{\mu}_\ell(0) = \tilde{\mu}_\ell(x * x) \geq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(x)\} = \tilde{\mu}_\ell(x) \tag{9}$$

for all $x \in X$. □

Proposition 2. *If $(X, \tilde{\mu})$ is a length k-fuzzy subalgebra of $(X, *, 0)$ for $k = 2, 4$, then $\tilde{\mu}_\ell(0) \leq \tilde{\mu}_\ell(x)$ for all $x \in X$.*

Proof. It is similar to the proof of Proposition 1. □

Theorem 1. *Given a subalgebra A of $(X, *, 0)$ and $B_1, B_2 \in \tilde{\mathcal{P}}([0, 1])$, let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ given by*

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]),\ x \mapsto \begin{cases} B_2 & \text{if } x \in A, \\ B_1 & \text{otherwise.} \end{cases} \tag{10}$$

If $B_1 \subsetneq B_2$, then $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$. Also, if $B_1 \supsetneq B_2$, then $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$.

Proof. If $x \in A$, then $\tilde{\mu}(x) = B_2$ and so

$$\tilde{\mu}_\ell(x) = \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \sup\{\tilde{\mu}(x)\} - \inf\{\tilde{\mu}(x)\} = \sup\{B_2\} - \inf\{B_2\}.$$

If $x \notin A$, then $\tilde{\mu}(x) = B_1$ and so

$$\tilde{\mu}_\ell(x) = \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \sup\{\tilde{\mu}(x)\} - \inf\{\tilde{\mu}(x)\} = \sup\{B_1\} - \inf\{B_1\}.$$

Assume that $B_1 \subsetneq B_2$. Then, $\sup\{B_2\} - \inf\{B_2\} \geq \sup\{B_1\} - \inf\{B_1\}$. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ and so

$$\tilde{\mu}_\ell(x * y) = \sup\{B_2\} - \inf\{B_2\} = \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}.$$

If $x, y \notin A$, then $\tilde{\mu}_\ell(x * y) \geq \sup\{B_1\} - \inf\{B_1\} = \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$. Suppose that $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$). Then,

$$\tilde{\mu}_\ell(x * y) \geq \sup B_1 - \inf B_1 = \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}.$$

Therefore, $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$.

Assume that $B_1 \supsetneq B_2$. Then,

$$\sup\{B_2\} - \inf\{B_2\} \leq \sup\{B_1\} - \inf\{B_1\},$$

and so

$$\tilde{\mu}_\ell(x * y) = \sup\{B_2\} - \inf\{B_2\} = \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$$

for all $x, y \in A$. If $x \notin A$ or $y \notin A$, then $\tilde{\mu}_\ell(x * y) \leq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$. Hence, $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$. □

It is clear that every length 3-fuzzy subalgebra is a length 1-fuzzy subalgebra and every length 2-fuzzy subalgebra is a length 4-fuzzy subalgebra. However, the converse is not true, as seen in the following example.

Example 2. *Consider the BCK-algebra $(X,*,0)$ in Example 1. Given a subalgebra $A=\{0,1,2\}$ of $(X,*,0)$, let $(X,\tilde{\mu})$ be a hyper structure over $(X,*,0)$ given by*

$$\tilde{\mu}: X \rightarrow \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} \{0.2n \mid n \in [0.2,0.9)\} & \text{if } x \in A, \\ \{0.2n \mid n \in (0.3,0.7]\} & \text{otherwise.} \end{cases}$$

Then, $(X,\tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X,*,0)$ by Theorem 1. Since

$$\begin{aligned} \tilde{\mu}_\ell(2) &= \tilde{\mu}_{\sup}(2) - \tilde{\mu}_{\inf}(2) \\ &= \sup\{0.2n \mid n \in [0.2,0.9)\} - \inf\{0.2n \mid n \in [0.2,0.9)\} \\ &= 0.18 - 0.04 = 0.14 \end{aligned}$$

and

$$\begin{aligned} \tilde{\mu}_\ell(3*2) = \tilde{\mu}_\ell(3) &= \tilde{\mu}_{\sup}(3) - \tilde{\mu}_{\inf}(3) \\ &= \sup\{0.2n \mid n \in (0.3,0.7]\} - \inf\{0.2n \mid n \in (0.3,0.7]\} \\ &= 0.14 - 0.06 = 0.08, \end{aligned}$$

we have $\tilde{\mu}_\ell(3*2) = 0.08 < 0.14 = \max\{0.08, 0.14\} = \max\{\tilde{\mu}_\ell(3), \tilde{\mu}_\ell(2)\}$. Therefore, $(X,\tilde{\mu})$ is not a length 3-fuzzy subalgebra of $(X,*,0)$.

Give a subalgebra $A=\{0,1,2,3\}$ of $(X,*,0)$, let $(X,\tilde{\mu})$ be a hyper structure over $(X,*,0)$ given by

$$\tilde{\mu}: X \rightarrow \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} (0.4,0.7) & \text{if } x \in A, \\ [0.3,0.9) & \text{otherwise.} \end{cases}$$

Then, $(X,\tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X,*,0)$ by Theorem 1. However, it is not a length 2-fuzzy subalgebra of $(X,*,0)$, since

$$\tilde{\mu}_\ell(4*2) = \tilde{\mu}_\ell(4) = 0.6 > 0.3 = \min\{\tilde{\mu}_\ell(4), \tilde{\mu}_\ell(2)\}.$$

Theorem 2. *A hyper structure $(X,\tilde{\mu})$ over $(X,*,0)$ is a length 1-fuzzy subalgebra of $(X,*,0)$ if and only if the set*

$$U_\ell(\tilde{\mu};t) := \{x \in X \mid \tilde{\mu}_\ell(x) \geq t\} \tag{11}$$

is a subalgebra of $(X,*,0)$ for all $t \in [0,1]$ with $U_\ell(\tilde{\mu};t) \neq \emptyset$.

Proof. Assume that $(X,\tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X,*,0)$ and let $t \in [0,1]$ be such that $U_\ell(\tilde{\mu};t)$ is nonempty. If $x,y \in U_\ell(\tilde{\mu};t)$, then $\tilde{\mu}_\ell(x) \geq t$ and $\tilde{\mu}_\ell(y) \geq t$. It follows from (3) that

$$\tilde{\mu}_\ell(x*y) \geq \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\} \geq t,$$

and so $x*y \in U_\ell(\tilde{\mu};t)$. Hence, $U_\ell(\tilde{\mu};t)$ is a subalgebra of $(X,*,0)$.

Conversely, suppose that $U_\ell(\tilde{\mu};t)$ is a subalgebra of $(X,*,0)$ for all $t \in [0,1]$ with $U_\ell(\tilde{\mu};t) \neq \emptyset$. Assume that there exist $a,b \in X$ such that

$$\tilde{\mu}_\ell(a*b) < \min\{\tilde{\mu}_\ell(a), \tilde{\mu}_\ell(b)\}.$$

If we take $t := \min\{\tilde{\mu}_\ell(a), \tilde{\mu}_\ell(b)\}$, then $a,b \in U_\ell(\tilde{\mu};t)$ and so $a*b \in U_\ell(\tilde{\mu};t)$. Thus, $\tilde{\mu}_\ell(a*b) \geq t$, which is a contradiction. Hence,

$$\tilde{\mu}_\ell(x*y) \geq \min\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$$

for all $x, y \in X$. Therefore, $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$. $\square$

Corollary 1. *If $(X, \tilde{\mu})$ is a length 3-fuzzy subalgebra of $(X, *, 0)$, then the set $U_\ell(\tilde{\mu}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [0, 1]$ with $U_\ell(\tilde{\mu}; t) \neq \emptyset$.*

The converse of Corollary 1 is not true, as seen in the following example.

Example 3. *Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation $*$, which is given in Table 3 (see [6]).*

Table 3. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	$\boldsymbol{a}$	$\boldsymbol{b}$
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.4) \cup [0.6, 0.9) & \text{if } x = 0, \\ (0.5, 0.7] & \text{if } x = 1, \\ [0.1, 0.3] \cup (0.5, 0.6] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.3, 0.5] & \text{if } x = b. \end{cases}$$

Then, the length of $\tilde{\mu}$ is given by Table 4.

Table 4. The length of $(X, \tilde{\mu})$.

X	**0**	**1**	**2**	$\boldsymbol{a}$	$\boldsymbol{b}$
$\tilde{\mu}_\ell$	0.6	0.2	0.5	0.3	0.2

Hence, we have

$$U_\ell(\tilde{\mu}; t) = \begin{cases} \emptyset & \text{if } t \in (0.6, 1], \\ \{0\} & \text{if } t \in (0.5, 0.6], \\ \{0, 2\} & \text{if } t \in (0.3, 0.5], \\ \{0, 2, a\} & \text{if } t \in (0.2, 0.3], \\ X & \text{if } t \in [0, 0.2], \end{cases}$$

and so $U_\ell(\tilde{\mu}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [0, 1]$ with $U_\ell(\tilde{\mu}; t) \neq \emptyset$. Since

$$\tilde{\mu}_\ell(b * 2) = \tilde{\mu}_\ell(b) = 0.2 \ngeq 0.5 = \max\{\tilde{\mu}_\ell(b), \tilde{\mu}_\ell(2)\},$$

$(X, \tilde{\mu})$ is not a length 3-fuzzy subalgebra of $(X, *, 0)$.

Theorem 3. *A hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is a length 4-fuzzy subalgebra of $(X, *, 0)$ if and only if the set*

$$L_\ell(\tilde{\mu}; t) := \{x \in X \mid \tilde{\mu}_\ell(x) \leq t\} \tag{12}$$

is a subalgebra of $(X, *, 0)$ *for all* $t \in [0, 1]$ *with* $L_\ell(\tilde{\mu}; t) \neq \emptyset$.

Proof. Suppose that $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$ and $L_\ell(\tilde{\mu}; t) \neq \emptyset$ for all $t \in [0, 1]$. Let $x, y \in L_\ell(\tilde{\mu}; t)$. Then, $\tilde{\mu}_\ell(x) \leq t$ and $\tilde{\mu}_\ell(y) \leq t$, which implies from (6) that

$$\tilde{\mu}_\ell(x * y) \leq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\} \leq t.$$

Hence, $x * y \in L_\ell(\tilde{\mu}; t)$, and so $L_\ell(\tilde{\mu}; t)$ is a subalgebra of $(X, *, 0)$.

Conversely, assume that $L_\ell(\tilde{\mu}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [0, 1]$ with $L_\ell(\tilde{\mu}; t) \neq \emptyset$. If there exist $a, b \in X$ such that

$$\tilde{\mu}_\ell(a * b) > \max\{\tilde{\mu}_\ell(a), \tilde{\mu}_\ell(b)\},$$

then $a, b \in L_\ell(\tilde{\mu}; t)$ by taking $t = \max\{\tilde{\mu}_\ell(a), \tilde{\mu}_\ell(b)\}$. It follows that $a * b \in L_\ell(\tilde{\mu}; t)$, and so $\tilde{\mu}_\ell(a * b) \leq t$, which is a contradiction. Hence,

$$\tilde{\mu}_\ell(x * y) \leq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$$

for all $x, y \in X$, and therefore $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 2. *If* $(X, \tilde{\mu})$ *is a length* 2*-fuzzy subalgebra of* $(X, *, 0)$*, then the set* $L_\ell(\tilde{\mu}; t)$ *is a subalgebra of* $(X, *, 0)$ *for all* $t \in [0, 1]$ *with* $L_\ell(\tilde{\mu}; t) \neq \emptyset$.

The converse of Corollary 2 is not true, as seen in the following example.

Example 4. *Consider the BCI-algebra* $X = \{0, 1, 2, a, b\}$ *in Example 3 and let* $(X, \tilde{\mu})$ *be a hyper structure over* $(X, *, 0)$ *in which* $\tilde{\mu}$ *is given as follows:*

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]), \quad x \mapsto \begin{cases} [0.6, 0.8) & \text{if } x = 0, \\ (0.3, 0.7] & \text{if } x = 1, \\ [0.4, 0.6) \cup (0.6, 0.7] & \text{if } x = 2, \\ [0.1, 0.7] & \text{if } x = a, \\ (0.2, 0.8] & \text{if } x = b. \end{cases}$$

Then, the length of $\tilde{\mu}$ *is given by Table 5.*

Table 5. The length of $(X, \tilde{\mu})$.

X	0	1	2	a	b
$\tilde{\mu}_\ell$	0.2	0.4	0.3	0.6	0.6

Hence, we have

$$L_\ell(\tilde{\mu}; t) = \begin{cases} X & \text{if } t \in [0.6, 1], \\ \{0, 1, 2\} & \text{if } t \in [0.4, 0.6), \\ \{0, 2\} & \text{if } t \in [0.3, 0.4), \\ \{0\} & \text{if } t \in [0.2, 0.3), \\ \emptyset & \text{if } t \in [0, 0.2), \end{cases}$$

and so $L_\ell(\tilde{\mu}; t)$ *is a subalgebra of* $(X, *, 0)$ *for all* $t \in [0, 1]$ *with* $L_\ell(\tilde{\mu}; t) \neq \emptyset$*. However,* $(X, \tilde{\mu})$ *is not a length* 2*-fuzzy subalgebra of* $(X, *, 0)$ *since*

$$\tilde{\mu}_\ell(a * 1) = 0.6 \not\leq 0.4 = \min\{\tilde{\mu}_\ell(a), \tilde{\mu}_\ell(1)\}.$$

Theorem 4. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (4). If $(X, \tilde{\mu})$ is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is a length 1-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Assume that $(X, \tilde{\mu})$ is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (4). Then, $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(x)$ and $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(y)$ for all $x, y \in X$, and $(X, \tilde{\mu}_{\sup})$ is a 1-fuzzy subalgebra of X. It follows from (3) that

$$\begin{aligned}
\tilde{\mu}_{\ell}(x * y) &= \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \\
&\geq \min\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} - \tilde{\mu}_{\inf}(x * y) \\
&= \min\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x * y),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(x * y)\} \\
&\geq \min\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y)\} \\
&= \min\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}
\end{aligned}$$

for all $x, y \in X$. Therefore $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 3. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (4). If $(X, \tilde{\mu})$ is a $(k, 3)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is a length 1-fuzzy subalgebra of $(X, *, 0)$.*

Corollary 4. *For $j \in \{1, 3\}$, every $(2, j)$-hyperfuzzy subalgebra is a length 1-fuzzy subalgebra.*

In general, any length 1-fuzzy subalgebra may not be a $(k, 1)$-hyperfuzzy subalgebra for $k \in \{1, 2, 3, 4\}$, as seen in the following example.

Example 5. *Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$, which is given in Table 6 (see [6]).*

Table 6. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1]),\ \ x \mapsto \begin{cases} [0.1, 0.9) & \text{if } x = 0, \\ (0.1, 0.8] & \text{if } x = 1, \\ [0.4, 0.9] & \text{if } x = a, \\ [0.3, 0.6) & \text{if } x \in \{b, c\}. \end{cases}$$

The length of $\tilde{\mu}$ is given by Table 7 and it is routine to verify that $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$.

Table 7. The length of $(X, \tilde{\mu})$.

$\boldsymbol{X}$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\tilde{\mu}_{\ell}$	0.8	0.7	0.5	0.3	0.3

However, it is not a $(k,1)$*-hyperfuzzy subalgebra of* X *since*

$$\begin{aligned}
\tilde{\mu}_{\inf}(a*a) &= \tilde{\mu}_{\inf}(0) = 0.1 < 0.4 = \min\{\tilde{\mu}_{\inf}(a), \tilde{\mu}_{\inf}(a)\},\\
\tilde{\mu}_{\inf}(b*c) &= \tilde{\mu}_{\inf}(a) = 0.4 > 0.3 = \min\{\tilde{\mu}_{\inf}(b), \tilde{\mu}_{\inf}(c)\},\\
\tilde{\mu}_{\inf}(b*b) &= \tilde{\mu}_{\inf}(0) = 0.1 < 0.3 = \max\{\tilde{\mu}_{\inf}(b), \tilde{\mu}_{\inf}(b)\},\\
\tilde{\mu}_{\inf}(b*c) &= \tilde{\mu}_{\inf}(a) = 0.4 > 0.3 = \max\{\tilde{\mu}_{\inf}(b), \tilde{\mu}_{\inf}(c)\}.
\end{aligned}$$

We provide a condition for a length 1-fuzzy subalgebra to be a $(k,1)$-hyperfuzzy subalgebra for $k \in \{1,2,3,4\}$.

Theorem 5. *If* $(X,\tilde{\mu})$ *is a length* 1*-fuzzy subalgebra of* $(X,*,0)$ *in which* $\tilde{\mu}_{\inf}$ *is constant on* X*, then it is a* $(k,1)$*-hyperfuzzy subalgebra of* $(X,*,0)$ *for* $k \in \{1,2,3,4\}$*.*

Proof. Assume that $(X,\tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X,*,0)$ in which $\tilde{\mu}_{\inf}$ is constant on X. It is clear that $(X,\tilde{\mu}_{\inf})$ is a k-fuzzy subalgebra of $(X,*,0)$ for $k \in \{1,2,3,4\}$. Let $\tilde{\mu}_{\inf}(x) = k$ for all $x \in X$. Then,

$$\begin{aligned}
\tilde{\mu}_{\sup}(x*y) &= \tilde{\mu}_{\ell}(x*y) + \tilde{\mu}_{\inf}(x*y)\\
&= \tilde{\mu}_{\ell}(x*y) + k\\
&\geq \min\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\} + k\\
&= \min\{\tilde{\mu}_{\ell}(x) + k, \tilde{\mu}_{\ell}(y) + k\}\\
&= \min\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\}
\end{aligned}$$

for all $x, y \in X$. Thus, $(X,\tilde{\mu}_{\sup})$ is a 1-fuzzy subalgebra of X. Therefore, $(X,\tilde{\mu})$ is a $(k,1)$-hyperfuzzy subalgebra of $(X,*,0)$ for $k \in \{1,2,3,4\}$. □

Corollary 5. *If* $(X,\tilde{\mu})$ *is a length* 3*-fuzzy subalgebra of* $(X,*,0)$ *in which* $\tilde{\mu}_{\inf}$ *is constant on* X*, then it is a* $(k,1)$*-hyperfuzzy subalgebra of* $(X,*,0)$ *for* $k \in \{1,2,3,4\}$*.*

Corollary 6. *Let* $(X,\tilde{\mu})$ *be a hyper structure over* $(X,*,0)$ *in which* $\tilde{\mu}_{\inf}$ *is constant on* X*. Then,* $(X,\tilde{\mu})$ *is a* $(k,1)$*-hyperfuzzy subalgebra of* $(X,*,0)$ *for* $k \in \{1,2,3,4\}$ *if and only if* $(X,\tilde{\mu})$ *is a length* 1*-fuzzy subalgebra of* $(X,*,0)$*.*

Theorem 6. *If* $(X,\tilde{\mu})$ *is a length* 1*-fuzzy subalgebra of* $(X,*,0)$ *in which* $\tilde{\mu}_{\sup}$ *is constant on* X*, then it is a* $(4,k)$*-hyperfuzzy subalgebra of* $(X,*,0)$ *for* $k \in \{1,2,3,4\}$*.*

Proof. Let $(X,\tilde{\mu})$ be a length 1-fuzzy subalgebra of $(X,*,0)$ in which $\tilde{\mu}_{\sup}$ is constant on X. Clearly, $(X,\tilde{\mu}_{\sup})$ is a k-fuzzy subalgebra of $(X,*,0)$ for $k \in \{1,2,3,4\}$. Let $\tilde{\mu}_{\sup}(x) = t$ for all $x \in X$. Then,

$$\begin{aligned}
\tilde{\mu}_{\inf}(x*y) &= \tilde{\mu}_{\sup}(x*y) - \tilde{\mu}_{\ell}(x*y)\\
&= t - \tilde{\mu}_{\ell}(x*y)\\
&\leq t - \min\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}\\
&= t + \max\{-\tilde{\mu}_{\ell}(x), -\tilde{\mu}_{\ell}(y)\}\\
&= \max\{t - \tilde{\mu}_{\ell}(x), t - \tilde{\mu}_{\ell}(y)\}\\
&= \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}
\end{aligned}$$

for all $x, y \in X$, and so $(X,\tilde{\mu}_{\inf})$ is a 4-fuzzy subalgebra of $(X,*,0)$. Therefore, $(X,\tilde{\mu})$ is a $(4,k)$-hyperfuzzy subalgebra of $(X,*,0)$ for $k \in \{1,2,3,4\}$. □

Theorem 7. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (5). For any $k \in \{1, 2, 3, 4\}$, if $(X, \tilde{\mu})$ is a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$, then it is a length 1-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Let $(X, \tilde{\mu})$ be a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (5). Then, $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(x)$ and $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(y)$ for all $x, y \in X$, and $(X, \tilde{\mu}_{\inf})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. It follows from (6) that

$$\begin{aligned}
\tilde{\mu}_{\ell}(x * y) &= \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \\
&\geq \tilde{\mu}_{\sup}(x * y) - \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \\
&= \min\{\tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(y)\} \\
&\geq \min\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y)\} \\
&= \min\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}
\end{aligned}$$

for all $x, y \in X$. Hence, $(X, \tilde{\mu})$ is a length 1-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 7. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (5). For any $k \in \{1, 2, 3, 4\}$, every $(2, k)$-hyperfuzzy subalgebra is a length 1-fuzzy subalgebra.*

Theorem 8. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (5). If $(X, \tilde{\mu})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is a length 4-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Assume that $(X, \tilde{\mu})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (5). Then, $\tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\inf}(x)$ and $\tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\inf}(y)$ for all $x, y \in X$, and $(X, \tilde{\mu}_{\sup})$ is a 4-fuzzy subalgebra of X. Hence,

$$\begin{aligned}
\tilde{\mu}_{\ell}(x * y) &= \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \\
&\leq \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} - \tilde{\mu}_{\inf}(x * y) \\
&= \max\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x * y),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(x * y)\} \\
&\leq \max\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y)\} \\
&= \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}
\end{aligned}$$

for all $x, y \in X$, and so $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 8. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\inf})$ satisfies the Condition (5). If $(X, \tilde{\mu})$ is a $(k, 2)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is a length 4-fuzzy subalgebra of $(X, *, 0)$.*

Corollary 9. *For $j \in \{2, 4\}$, every $(3, j)$-hyperfuzzy subalgebra is a length 4-fuzzy subalgebra.*

Theorem 9. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}_{\inf}$ is constant. Then, every length 4-fuzzy subalgebra is a $(k, 4)$-hyperfuzzy subalgebra for $k \in \{1, 2, 3, 4\}$.*

Proof. Let $(X, \tilde{\mu})$ be a length 4-fuzzy subalgebra of $(X, *, 0)$ in which $\tilde{\mu}_{\inf}$ is constant. It is obvious that $(X, \tilde{\mu}_{\inf})$ is a k-fuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Let $\tilde{\mu}_{\inf}(x) = t$ for all $x \in X$. Then,

$$
\begin{aligned}
\tilde{\mu}_{\sup}(x*y) = \tilde{\mu}_{\ell}(x*y) + \tilde{\mu}_{\inf}(x*y) &= \tilde{\mu}_{\ell}(x*y) + t \\
&\leq \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\} + t \\
&= \max\{\tilde{\mu}_{\ell}(x) + t, \tilde{\mu}_{\ell}(y) + t\} \\
&= \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}
\end{aligned}
$$

for all $x, y \in X$, and hence $(X, \tilde{\mu}_{\sup})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Therefore, $(X, \tilde{\mu})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

Corollary 10. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}_{\inf}$ is constant. Then, every length 2-fuzzy subalgebra is a $(k, 4)$-hyperfuzzy subalgebra for $k \in \{1, 2, 3, 4\}$.*

Theorem 10. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (4). For every $k \in \{1, 2, 3, 4\}$, every $(1, k)$-hyperfuzzy subalgebra is a length 4-fuzzy subalgebra.*

Proof. For every $k \in \{1, 2, 3, 4\}$, let $(X, \tilde{\mu})$ be a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (4). Then, $\tilde{\mu}_{\sup}(x*y) \leq \tilde{\mu}_{\sup}(x)$ and $\tilde{\mu}_{\sup}(x*y) \leq \tilde{\mu}_{\sup}(y)$ for all $x, y \in X$. Since $(X, \tilde{\mu}_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$, we have

$$
\begin{aligned}
\tilde{\mu}_{\ell}(x*y) &= \tilde{\mu}_{\sup}(x*y) - \tilde{\mu}_{\inf}(x*y) \\
&\leq \tilde{\mu}_{\sup}(x*y) - \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \\
&= \max\{\tilde{\mu}_{\sup}(x*y) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(x*y) - \tilde{\mu}_{\inf}(y)\} \\
&\leq \max\{\tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y)\} \\
&= \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}
\end{aligned}
$$

for all $x, y \in X$. Thus, $(X, \tilde{\mu})$ is a length 4-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 11. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\mu}_{\sup})$ satisfies the Condition (4). For every $k \in \{1, 2, 3, 4\}$, every $(3, k)$-hyperfuzzy subalgebra is a length 4-fuzzy subalgebra.*

Theorem 11. *Let $(X, \tilde{\mu})$ be a length 4-fuzzy subalgebra of $(X, *, 0)$. If $\tilde{\mu}_{\sup}$ is constant on X, then $(X, \tilde{\mu})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $\tilde{\mu}_{\sup}$ is constant on X in a length 4-fuzzy subalgebra $(X, \tilde{\mu})$ of $(X, *, 0)$. Obviously, $(X, \tilde{\mu}_{\sup})$ is a k-fuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Let $\tilde{\mu}_{\sup}(x) = t$ for all $x \in X$. Then,

$$
\begin{aligned}
\tilde{\mu}_{\inf}(x*y) &= \tilde{\mu}_{\sup}(x*y) - \tilde{\mu}_{\ell}(x*y) \\
&= t - \tilde{\mu}_{\ell}(x*y) \\
&\geq t - \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\} \\
&= \min\{t - \tilde{\mu}_{\ell}(x), t - \tilde{\mu}_{\ell}(y)\} \\
&= \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}
\end{aligned}
$$

for all $x, y \in X$, and so $(X, \tilde{\mu}_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$. Therefore, $(X, \tilde{\mu}_{\inf})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

Corollary 12. *Let $(X, \tilde{\mu})$ be a length 2-fuzzy subalgebra of $(X, *, 0)$. If $\tilde{\mu}_{\sup}$ is constant on X, then $(X, \tilde{\mu})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

4. Conclusions

In order to consider a generalization of fuzzy sets and interval-valued fuzzy sets, the notion of hyperfuzzy sets was introduced by Ghosh and Samanta (see [3]). Jun et al. [4] and Song et al. [7] have applied the hyperfuzzy sets to BCK/BCI-algebras. In this article, we have introduced the concept of length-fuzzy sets based on hyperfuzzy sets, and have presented an application in BCK/BCI-algebras. We have introduced the notion of length fuzzy subalgebras in BCK/BCI-algebras, and have investigated related properties. We have discussed characterizations of a length fuzzy subalgebra, and have established relations between length fuzzy subalgebras and hyperfuzzy subalgebras. Recently, many kinds of fuzzy sets have several applications to deal with uncertainties from our different kinds of daily life problems, in particular, for solving decision-making problems (see [8–12]). In the future, from a purely mathematical standpoint, we will apply the notions and results in this manuscript to related algebraic structures, for example, MV-algebras, BL-algebras, MTL-algebras, EQ-algebras, effect algebras and so on. From an applicable standpoint, we shall extend our proposed approach to some decision-making problems under the field of fuzzy cluster analysis, uncertain programming, mathematical programming, decision-making problems and so on.

Acknowledgments: The authors wish to thank the anonymous reviewers for their valuable suggestions. To the memory of Lotfi A. Zadeh.

Author Contributions: All authors contributed equally and significantly to the study and preparation of the article. They have read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
2. Marty, F. Sur une generalization de la notion de groupe. In Proceedings of the 8th Congress Math Scandenaves, Stockholm, Sweden, 1934; pp. 45–49.
3. Ghosh, J.; Samanta, T.K. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol.* **2012**, *41*, 27–37.
4. Jun, Y.B.; Hur, K.; Lee, K.J. Hyperfuzzy subalgebras of BCK/BCI-algebras. *Ann. Fuzzy Math. Inform.* **2017**, in press.
5. Huang, Y.S. *BCI-Algebra*; Science Press: Beijing, China, 2006.
6. Meng, J.; Jun, Y.B. *BCK-Algebras*; Kyungmoon Sa Co.: Seoul, Korea, 1994.
7. Song, S.Z.; Kim, S.J.; Jun, Y.B. Hyperfuzzy ideals in BCK/BCI-algebras. *Mathematics* **2017**, *5*, 81, doi:10.3390/math5040081.
8. Garg, H. A robust ranking method for intuitionistic multiplicative sets under crisp, interval environments and its applications. *IEEE Trans. Emerg. Top. Comput. Intell.* **2017**, *1*, 366–374.
9. Feng, F.; Jun, Y.B.; Liu, X.; Li, L. An adjustable approach to fuzzy soft set based decision making. *J. Comput. Appl. Math.* **2010**, *234*, 10–20.
10. Xia, M.; Xu, Z. Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **2011**, *52*, 395–407.
11. Tang, H. Decision making based on interval-valued intuitionistic fuzzy soft sets and its algorithm. *J. Comput. Anal. Appl.* **2017**, *23*, 119–131.
12. Wei, G.; Alsaadi, F.E.; Hayat, T.; Alsaedi, A. Hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2017**, *33*, 1119–1128.