mathematics

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Approaches to Multiple Attribute Decision Making with Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Information

Jie Wang, Guiwu Wei * and Hui Gao *

School of Business, Sichuan Normal University, Chengdu 610101, China; JW970326@163.com
* Correspondence: weiguiwu@uestc.edu.cn (G.W.); gaohuisxy@sicnu.edu.cn (H.G.)

Received: 27 September 2018; Accepted: 8 October 2018; Published: 13 October 2018

Abstract: The Maclaurin symmetric mean (MSM) operator is a classical mean type aggregation operator used in modern information fusion theory, which is suitable to aggregate numerical values. The prominent characteristic of the MSM operator is that it can capture the interrelationship among multi-input arguments. Motivated by the ideal characteristic of the MSM operator, in this paper, we expand the MSM operator, generalized MSM (GMSM), and dual MSM (DMSM) operator with interval-valued 2-tuple linguistic Pythagorean fuzzy numbers (IV2TLPFNs) to propose the interval-valued 2-tuple linguistic Pythagorean fuzzy MSM (IV2TLPFMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy weighted MSM (IV2TLPFWMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy GMSM (IN2TLPFGMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy weighted GMSM (IV2TLPFWGMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy DMSM (IN2TLPFDMSM) operator, Interval-valued 2-tuple linguistic Pythagorean fuzzy weighted DMSM (IV2TLPFWDMSM) operator. Then the multiple attribute decision making (MADM) methods are developed with these three operators. Finally, an example of green supplier selection is used to show the proposed methods.

Keywords: multiple attribute decision making (MADM); Pythagorean fuzzy numbers; Interval-valued 2-tuple linguistic Pythagorean fuzzy set (IV2TLPFSs); interval-valued 2-tuple linguistic Pythagorean fuzzy MSM (IV2TLPFMSM) operator; Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Generalized MSM (IV2TLPFGMSM) operator; interval-valued 2-tuple linguistic Pythagorean fuzzy dual MSM (IV2TLPFDMSM) operator; green supplier selection

1. Introduction

Atanassov [1,2] introduced the concept of an intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set [3]. Each element in the IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to 1. More recently, the Pythagorean fuzzy set (PFS) [4,5] has emerged as an effective tool for depicting uncertainty of the multiple attribute decision making (MADM) problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. Zhang and Xu [6] developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the multiple criteria decision making (MCDM) problem within Pythagorean fuzzy numbers (PFNs). Peng and Yang [7] proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problems with PFNs. Ren et al. [8] developed the Pythagorean fuzzy

TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making) approach which considers the DMs' psychological behaviors. Garg [9] proposed the new generalized Pythagorean fuzzy information aggregation by using Einstein Operations. Zhang [10] proposed the hierarchical QUALIFLEX (qualitative flexible multiple criteria method) approach in PFS. Chen [11] developed the Pythagorean fuzzy VIKOR (VIseKriterijumska Optimizacija I KOmpromisno Resenje) models. Peng and Yang [12] proposed the MABAC (multi-attributive border approximation area comparison) method for multiple attribute group decision making (MAGDM) with PFNs. Zeng et al. [13] developed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg [14] studied a novel accuracy function under interval-valued PFSs for solving multicriteria decision making problem. Wei [15] utilized arithmetic and geometric operations [16–24] to develop some Pythagorean fuzzy interaction aggregation operators. Wei and Lu [25] utilize power aggregation operators [26–29] to develop some Pythagorean fuzzy power aggregation operators. Wei and Lu [30] we extend the Maclaurin symmetric mean (MSM) [31] to the Pythagorean fuzzy environment to propose the Pythagorean fuzzy Maclaurin symmetric mean (PFMSM) operator and Pythagorean fuzzy weighted Maclaurin symmetric mean (PFWMSM) operator. Wan et al. [32] proposed a Pythagorean fuzzy mathematical programming method for MAGDM with Pythagorean fuzzy truth degrees. Peng and Dai [33] studied the Pythagorean fuzzy stochastic MADM with prospect theory. Garg [34] proposed linear programming for MADM with interval-valued PFS (IVPFSs). Chen [35] proposed an interval-valued Pythagorean fuzzy outranking method with a closeness-based assignment model for MADM. Garg [36] developed a novel improved accuracy function for interval-valued Pythagorean fuzzy sets.

Although, IVPFSs theory has been broadly applied to many domains, all the above approaches are unsuitable to depict the membership degree and a non-membership degree of an element to a set by interval-valued 2-tuple linguistic sets (IV2TLSs) [37]. In order to overcome this issue, we propose the definition of interval-valued 2-tuple linguistic Pythagorean fuzzy sets (IV2TLPFSs) based on the IVPFSs [34] and 2-tuple linguistic sets [38,39]. The MSM operator [31] is a famous aggregation operator which can depict interrelationships among the multi-input arguments. Moreover, for a given collection of arguments, the MSM operator is monotonically decreasing with respect to the values of parameters, which can reflect the risk preferences of the decision makers in practical situations. In the past few years, the MSM has received more and more attentions, many important results both in theory and application are developed [40–43]. Therefore, the MSM operators can supply a robust and flexible mechanism to deal with the information fusion in MADM problems. Because IV2TLPFSs can easily describe the fuzzy information, and the MSM operator can capture interrelationships among the multi-input arguments, it is necessary to extend the MSM operators to deal with the 2TLPFNs. The aim of this paper is to propose some MSM operators with IV2TLPFNs, then to study some properties of these operators, and applied them to cope with the MADM with IV2TLPFNs. In order to do so, the rest of this paper is organized as follows. In Section 2, we develop the IV2TLPFSs. In Section 3, we develop MSM operators with IV2TLPFSs. In Section 4, we develop generalized MSM operators with IV2TLPFSs. In Section 5, we develop dual MSM operators with IV2TLPFSs. In Section 6, we present an example for green supplier selection. Conclusions are given in Section 7.

2. Preliminaries

In this section, we shall propose the concept of interval-valued 2-tuple linguistic Pythagorean fuzzy sets (IV2TLPFSs) based on the IVPFSs [34] and 2TLSs [38,39].

2.1. 2-Tuple Linguistic Sets

Definition 1. *[38,39] Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and S can be defined as:*

$$S = \left\{ \begin{array}{l} s_0 = extremely\ poor,\ s_1 = very\ poor,\ s_2 = poor,\ s_3 = medium, \\ s_4 = good,\ s_5 = very\ good,\ s_6 = extremely\ good. \end{array} \right\}$$

Herrera and Martinez [38,39] developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple (s_i, ρ_i), where s_i is a linguistic label for predefined linguistic term set S and ρ_i is the value of symbolic translation, and $\rho_i \in [-0.5, 0.5)$.

2.2. Pythagorean Fuzzy Sets

In this section, some basic concepts of the PFSs and IVPFSs have been reviewed over a fix set X.

Definition 2. *[4,5] A PFS P is defined as*

$$P = \{\langle x, (\mu_P(x), \nu_P(x))\rangle | x \in X\} \tag{1}$$

where the functions $\mu_P : X \to [0,1]$ and $\nu_P : X \to [0,1]$ defines the degrees of membership and non-membership of the element $x \in X$ to P, such that for each x, the condition $\left(\mu_p(x)\right)^2 + \left(\nu_p(x)\right)^2 \leq 1$, holds.

Definition 3. *[8] The $p = (\mu, \nu)$ be called as a Pythagorean fuzzy number (PFN) and define the score and accuracy functions as $S(p) = \mu^2 - \nu^2$ and $H(p) = \mu^2 + \nu^2$. In order to compare two or more PFNs p_1 and p_2, a comparison law is defined as*

(1) if $S(p_1) < S(p_2)$, then $p_1 < p_2$;
(2) if $S(p_1) = S(p_2)$, then

  (i) if $H(p_1) = H(p_2)$, then $p_1 = p_2$;
  (ii) if $H(p_1) < H(p_2)$, then $p_1 < p_2$.

2.3. Interval-Valued Pythagorean Fuzzy Sets

Zhang [10] extended the PFSs to the IVPFSs which is defined as followed over the fix set X.

Definition 4. *[10] An IVPFS $\widetilde{P}$ is defined as*

$$\widetilde{P} = \{\langle x, (\widetilde{\mu}_{\widetilde{P}}(x), \widetilde{\nu}_{\widetilde{P}}(x))\rangle | x \in X\} \tag{2}$$

where $\widetilde{\mu}_{\widetilde{P}}(x) = \left[\mu^L_{\widetilde{P}}(x), \mu^R_{\widetilde{P}}(x)\right], \widetilde{\nu}_{\widetilde{P}}(x) = \left[\nu^L_{\widetilde{P}}(x), \nu^R_{\widetilde{P}}(x)\right]$ are the interval numbers of [0,1] with the condition $0 \leq \left(\mu^R_{\widetilde{P}}(x)\right)^2 + \left(\nu^R_{\widetilde{P}}(x)\right)^2 \leq 1, \forall x \in X$. The pair $\widetilde{p} = \left(\left[u^L_{\widetilde{p}}, u^R_{\widetilde{p}}\right], \left[v^L_{\widetilde{p}}, v^R_{\widetilde{p}}\right]\right)$ is called as an IVPF number (IVPFN), where $u_{\widetilde{p}}, v_{\widetilde{p}} \subseteq [0,1]$ and $(u^R_{\widetilde{p}})^2 + (v^R_{\widetilde{p}})^2 \leq 1$.

Theorem 1. *[14] For three IVPFNs* $\widetilde{p}_1 = \left(\left[u^L_{\widetilde{p}_1}, u^R_{\widetilde{p}_1}\right], \left[v^L_{\widetilde{p}_1}, v^R_{\widetilde{p}_1}\right]\right)$, $\widetilde{p}_2 = \left(\left[u^L_{\widetilde{p}_2}, u^R_{\widetilde{p}_2}\right], \left[v^L_{\widetilde{p}_2}, v^R_{\widetilde{p}_2}\right]\right)$, *and* $\widetilde{p} = \left(\left[u^L_{\widetilde{p}}, u^R_{\widetilde{p}}\right], \left[v^L_{\widetilde{p}}, v^R_{\widetilde{p}}\right]\right)$, *the basic operational laws are defined as follows:*

$$(1)\ \widetilde{p}_1 \oplus \widetilde{p}_2 = \left(\left[\begin{array}{c}\sqrt{\left(u^L_{\widetilde{p}_1}\right)^2 + \left(u^L_{\widetilde{p}_2}\right)^2 - \left(u^L_{\widetilde{p}_1}\right)^2\left(u^L_{\widetilde{p}_2}\right)^2},\\ \sqrt{\left(u^R_{\widetilde{p}_1}\right)^2 + \left(u^R_{\widetilde{p}_2}\right)^2 - \left(u^R_{\widetilde{p}_1}\right)^2\left(u^R_{\widetilde{p}_2}\right)^2}\end{array}\right], \left[v^L_{\widetilde{p}_1} v^L_{\widetilde{p}_2}, v^R_{\widetilde{p}_1} v^R_{\widetilde{p}_2}\right]\right);$$

$$(2)\ \widetilde{p}_1 \otimes \widetilde{p}_2 = \left(\left[\mu^L_{\widetilde{p}_1} v^L_{\widetilde{p}_2}, \mu^R_{\widetilde{p}_1} v^R_{\widetilde{p}_2}\right], \left[\begin{array}{c}\sqrt{\left(v^L_{\widetilde{p}_1}\right)^2 + \left(v^L_{\widetilde{p}_2}\right)^2 - \left(v^L_{\widetilde{p}_1}\right)^2\left(v^L_{\widetilde{p}_2}\right)^2},\\ \sqrt{\left(v^R_{\widetilde{p}_1}\right)^2 + \left(v^R_{\widetilde{p}_2}\right)^2 - \left(v^R_{\widetilde{p}_1}\right)^2\left(v^R_{\widetilde{p}_2}\right)^2}\end{array}\right]\right);$$

$$(3)\ \lambda\widetilde{p} = \left(\left[\sqrt{1 - \left(1 - \left(u^L_{\widetilde{p}}\right)^2\right)^\lambda}, \sqrt{1 - \left(1 - \left(u^R_{\widetilde{p}}\right)^2\right)^\lambda}\right], \left[\left(v^L_{\widetilde{p}}\right)^\lambda, \left(v^R_{\widetilde{p}}\right)^\lambda\right]\right), \lambda > 0;$$

$$(4)\ (\widetilde{p})^\lambda = \left(\left[\left(\mu^L_{\widetilde{p}}\right)^\lambda, \left(\mu^R_{\widetilde{p}}\right)^\lambda\right], \left[\sqrt{1 - \left(1 - \left(v^L_{\widetilde{p}}\right)^2\right)^\lambda}, \sqrt{1 - \left(1 - \left(v^R_{\widetilde{p}}\right)^2\right)^\lambda}\right]\right), \lambda > 0;$$

$$(5)\ \widetilde{p}^c = \left(\left[v^L_{\widetilde{p}}, v^R_{\widetilde{p}}\right], \left[\mu^L_{\widetilde{p}}, \mu^R_{\widetilde{p}}\right]\right).$$

2.4. Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Sets

Then, we shall propose the concept of interval-valued 2-tuple linguistic Pythagorean fuzzy sets (IV2TLPFSs) based on the IVPFSs [34] and 2TLSs [38,39].

Definition 5. *Assume that* $P = \{p_0, p_1, \ldots, p_t\}$ *is a 2TLSs with odd cardinality* $t + 1$. *If* $\widetilde{p} = \left\langle\left[\left(s^L_\phi, \varphi^L\right), \left(s^R_\phi, \varphi^R\right)\right], \left[\left(s^L_\chi, \xi^L\right), \left(s^R_\chi, \xi^R\right)\right]\right\rangle$ *is defined for* $\left[\left(s^L_\phi, \varphi^L\right), \left(s^R_\phi, \varphi^R\right)\right], \left[\left(s^L_\chi, \xi^L\right), \left(s^R_\chi, \xi^R\right)\right] \in P$ *where* $\left[\left(s^L_\phi, \varphi^L\right), \left(s^R_\phi, \varphi^R\right)\right]$ *and* $\left[\left(s^L_\chi, \xi^L\right), \left(s^R_\chi, \xi^R\right)\right]$ *express independently the degree of membership and non-membership by 2TLSs, then IV2TLPFSs is defined as follows:*

$$\widetilde{p}_j = \left\langle\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\rangle \tag{3}$$

where $\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right), \Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right) \subseteq [0, t]$, *and* $0 \le \left(\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right)^2 + \left(\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right)^2 \le t^2$.

Definition 6. *Let* $\widetilde{p}_1 = \left\langle\left[\left(s^L_{\phi_1}, \varphi^L_1\right), \left(s^R_{\phi_1}, \varphi^R_1\right)\right], \left[\left(s^L_{\chi_1}, \xi^L_1\right), \left(s^R_{\chi_1}, \xi^R_1\right)\right]\right\rangle$ *be an IV2TLPFN in P. Then the score and accuracy functions of* $\widetilde{p}_1$ *are defined as follows:*

$$S(\widetilde{p}_1) = \Delta\left\{\frac{1}{2t}\sqrt{\left(\begin{array}{c}2t^2 + \left(\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right)\right)^2 + \left(\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right)^2\\ -\left(\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right)\right)^2 - \left(\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right)^2\end{array}\right)}\right\}, S(\widetilde{p}_1) \in [0, 1] \tag{4}$$

$$H(\widetilde{p}_1) = \Delta\left\{\frac{1}{2t}\sqrt{\left(\begin{array}{c}\left(\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right)\right)^2 + \left(\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right)^2\\ +\left(\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right)\right)^2 + \left(\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right)^2\end{array}\right)}\right\}, H(\widetilde{p}_1) \in [0, 1] \tag{5}$$

Definition 7. *Let* $\widetilde{p}_1 = \left\langle \left[\left(s^L_{\phi_1}, \varphi^L_1\right), \left(s^R_{\phi_1}, \varphi^R_1\right)\right], \left[\left(s^L_{\chi_1}, \xi^L_1\right), \left(s^R_{\chi_1}, \xi^R_1\right)\right] \right\rangle$ *and* $\widetilde{p}_2 = \left\langle \left[\left(s^L_{\phi_2}, \varphi^L_2\right), \left(s^R_{\phi_2}, \varphi^R_2\right)\right], \left[\left(s^L_{\chi_2}, \xi^L_2\right), \left(s^R_{\chi_2}, \xi^R_2\right)\right] \right\rangle$ *be two IV2TLPFNs, then*

(1) if $S(\widetilde{p}_1) \prec S(\widetilde{p}_2)$, then $\widetilde{p}_1 \prec \widetilde{p}_2$;
(2) if $S(\widetilde{p}_1) \succ S(\widetilde{p}_2)$, then $\widetilde{p}_1 \succ \widetilde{p}_2$;
(3) if $S(\widetilde{p}_1) = S(\widetilde{p}_2), H(\widetilde{p}_1) \prec H(\widetilde{p}_2)$, then $\widetilde{p}_1 \prec \widetilde{p}_2$;
(4) if $S(\widetilde{p}_1) = S(\widetilde{p}_2), H(\widetilde{p}_1) \succ H(\widetilde{p}_2)$, then $\widetilde{p}_1 \succ \widetilde{p}_2$;
(5) if $S(\widetilde{p}_1) = S(\widetilde{p}_2), H(\widetilde{p}_1) = H(\widetilde{p}_2)$, then $\widetilde{p}_1 = \widetilde{p}_2$.

Definition 8. *Let* $\widetilde{p}_1 = \left\langle \left[\left(s^L_{\phi_1}, \varphi^L_1\right), \left(s^R_{\phi_1}, \varphi^R_1\right)\right], \left[\left(s^L_{\chi_1}, \xi^L_1\right), \left(s^R_{\chi_1}, \xi^R_1\right)\right] \right\rangle$ *and* $\widetilde{p}_2 = \left\langle \left[\left(s^L_{\phi_2}, \varphi^L_2\right), \left(s^R_{\phi_2}, \varphi^R_2\right)\right], \left[\left(s^L_{\chi_2}, \xi^L_2\right), \left(s^R_{\chi_2}, \xi^R_2\right)\right] \right\rangle$ *be two IV2TLPFNs, then*

$$(1) \quad \widetilde{p}_1 \oplus \widetilde{p}_2 = \left\{ \begin{array}{c} \left[\begin{array}{l} \Delta\left(t\sqrt{\left(\frac{\Delta^{-1}\left(s^L_{\phi_1},\varphi^L_1\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^L_{\phi_2},\varphi^L_2\right)}{t}\right)^2 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_1},\varphi^L_1\right)}{t}\right)^2 \left(\frac{\Delta^{-1}\left(s^L_{\phi_2},\varphi^L_2\right)}{t}\right)^2}\right), \\ \Delta\left(t\sqrt{\left(\frac{\Delta^{-1}\left(s^R_{\phi_1},\varphi^R_1\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^R_{\phi_2},\varphi^R_2\right)}{t}\right)^2 - \left(\frac{\Delta^{-1}\left(s^R_{\phi_1},\varphi^R_1\right)}{t}\right)^2 \left(\frac{\Delta^{-1}\left(s^R_{\phi_2},\varphi^R_2\right)}{t}\right)^2}\right) \end{array} \right], \\ \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\chi_1},\xi^L_1\right)}{t}\right)\left(\frac{\Delta^{-1}\left(s^L_{\chi_2},\xi^L_2\right)}{t}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\chi_1},\xi^R_1\right)}{t}\right)\left(\frac{\Delta^{-1}\left(s^R_{\chi_2},\xi^R_2\right)}{t}\right)\right)\right] \end{array} \right\}$$

$$(2) \quad \widetilde{p}_1 \otimes \widetilde{p}_2 = \left\{ \begin{array}{l} \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\phi_1},\varphi^L_1\right)}{t}\right)\left(\frac{\Delta^{-1}\left(s^L_{\phi_2},\varphi^L_2\right)}{t}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\phi_1},\varphi^R_1\right)}{t}\right)\left(\frac{\Delta^{-1}\left(s^R_{\phi_2},\varphi^R_2\right)}{t}\right)\right)\right], \\ \left[\begin{array}{l} \Delta\left(t\sqrt{\left(\frac{\Delta^{-1}\left(s^L_{\chi_1},\xi^L_1\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^L_{\chi_2},\xi^L_2\right)}{t}\right)^2 - \left(\frac{\Delta^{-1}\left(s^L_{\chi_1},\xi^L_1\right)}{t}\right)^2 \left(\frac{\Delta^{-1}\left(s^L_{\chi_2},\xi^L_2\right)}{t}\right)^2}\right), \\ \Delta\left(t\sqrt{\left(\frac{\Delta^{-1}\left(s^R_{\chi_1},\xi^R_1\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^R_{\chi_2},\xi^R_2\right)}{t}\right)^2 - \left(\frac{\Delta^{-1}\left(s^R_{\chi_1},\xi^R_1\right)}{t}\right)^2 \left(\frac{\Delta^{-1}\left(s^R_{\chi_2},\xi^R_2\right)}{t}\right)^2}\right) \end{array} \right] \end{array} \right\}$$

$$(3) \quad \lambda\widetilde{p}_1 = \left\{ \begin{array}{l} \left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_1},\varphi^L_1\right)}{t}\right)^2\right)^\lambda}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_1},\varphi^R_1\right)}{t}\right)^2\right)^\lambda}\right)\right], \\ \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\chi_1},\xi^L_1\right)}{t}\right)^\lambda\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\chi_1},\xi^R_1\right)}{t}\right)^\lambda\right)\right] \end{array} \right\}, \lambda > 0;$$

$$(4) \quad (\widetilde{p}_1)^\lambda = \left\{ \begin{array}{l} \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\phi_1},\varphi^L_1\right)}{t}\right)^\lambda\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\phi_1},\varphi^R_1\right)}{t}\right)^\lambda\right)\right], \\ \left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_1},\xi^L_1\right)}{t}\right)^2\right)^\lambda}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_1},\xi^R_1\right)}{t}\right)^2\right)^\lambda}\right)\right] \end{array} \right\}, \lambda > 0.$$

2.5. Maclaurin Symmetric Mean Operators

Maclaurin [31] proposed the MSM operator.

Definition 9. *[31] Let* $x_i (i = 1, 2, \ldots, n)$ *be a set of nonnegative real numbers, and* $k = (1, 2, \ldots, n)$*. If*

$$\mathrm{MSM}^{(k)}(x_1, x_2, \ldots, x_n) = \left(\frac{\sum\limits_{1 \le i_1 < \ldots < i_k \le i_n} \prod\limits_{j=1}^{k} x_{i_j}}{C_n^k} \right)^{\frac{1}{k}} \tag{6}$$

Then we called $\mathrm{MSM}^{(k)}$ *the Maclaurin symmetric mean operator, where* $(i_1, i_2, \ldots, i_k)$ *traverses all the k-tuple combinations of* $(1, 2, \ldots, n)$, C_n^k *is the binomial coefficient.*

3. Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy MSM and Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Weighted MSM Operators

3.1. The IV2TLPFMSM Operator

This section extends MSM and to fuse the IV2TLPFNs and proposes the Interval-valued 2-tuple linguistic Pythagorean fuzzy MSM (IV2TLPFMSM) operator.

Definition 10. *Let* $\widetilde{p}_i = \left\langle \left[\left(s_{\phi_i}^L, \varphi_i^L \right), \left(s_{\phi_i}^R, \varphi_i^R \right) \right], \left[\left(s_{\chi_i}^L, \xi_i^L \right), \left(s_{\chi_i}^R, \xi_i^R \right) \right] \right\rangle$ *be a set of IV2TLPFNs. The Interval-valued 2-tuple linguistic Pythagorean fuzzy MSM (IV2TLPFMSM) operator:*

$$\mathrm{IV2TLPFMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_k \le i_n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} \widetilde{p}_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} \tag{7}$$

Theorem 2. *Let* $\widetilde{p}_i = \left\langle \left[\left(s_{\phi_i}^L, \varphi_i^L \right), \left(s_{\phi_i}^R, \varphi_i^R \right) \right], \left[\left(s_{\chi_i}^L, \xi_i^L \right), \left(s_{\chi_i}^R, \xi_i^R \right) \right] \right\rangle$ *be a set of IV2TLPFNs. The aggregated value by using IV2TLPFMSM operators is also an IV2TLPFN where*

$$\begin{aligned}
&\mathrm{IV2TLPFMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_k \le i_n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} \widetilde{p}_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} \\
&= \left\{ \begin{array}{l}
\left[\begin{array}{l}
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^L, \varphi_{i_j}^L \right)}{t} \right)^2 \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{k}} \right), \\
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^R, \varphi_{i_j}^R \right)}{t} \right)^2 \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{k}} \right)
\end{array} \right], \\
\left[\begin{array}{l}
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^L, \xi_{i_j}^L \right)}{t} \right)^2 \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}} \right), \\
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^R, \xi_{i_j}^R \right)}{t} \right)^2 \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}} \right)
\end{array} \right]
\end{array} \right\}
\end{aligned} \tag{8}$$

Proof. According to Definition 8, we can obtain

$$\underset{j=1}{\overset{k}{\otimes}}\widetilde{p}_{i_j}=\left\{\begin{array}{c}\left[\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)\right),\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)\right)\right],\\ \left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right),\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)\right]\end{array}\right\} \tag{9}$$

Thus,

$$\begin{array}{l}\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\underset{j=1}{\overset{k}{\otimes}}\widetilde{p}_{i_j}\right)\\ =\left\{\begin{array}{c}\left[\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right),\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)\right],\\ \left[\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right),\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)\right]\end{array}\right\}\end{array} \tag{10}$$

Thereafter,

$$\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\underset{j=1}{\overset{k}{\otimes}}\widetilde{p}_{i_j}\right)}{C_n^k}=\left\{\begin{array}{c}\left[\begin{array}{c}\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right),\\ \Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right)\end{array}\right],\\ \left[\begin{array}{c}\Delta\left(t\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right),\\ \Delta\left(t\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right)\end{array}\right]\end{array}\right\} \tag{11}$$

Therefore,

$$\mathrm{IV2TLPFMSM}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\widetilde{p}_{i_j}\right)}{C_n^k}\right)^{\frac{1}{k}}$$

$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right),\\ \Delta\left(t\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}\right)\end{array}\right]\end{array}\right\} \quad (12)$$

Hence, (8) is kept.

Then we need to prove that (8) is an IV2TLPFN. We need to prove two conditions as follows:

① $\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right),\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right),\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right),\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\subseteq[0,t]$

② $0\le\left(\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)\right)^2+\left(\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\right)^2\le t^2.$

Let

$$\frac{\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right)}{t}=\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right)}{t}=\sqrt{1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\sqrt{1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}$$

Proof. ① Since $0 \le \frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t} \le 1$, we get

$$0 \le \prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2 \le 1 \; and \; 0 \le \left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}} \le 1 \tag{13}$$

Then,

$$0 \le \prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}} \le 1 \tag{14}$$

$$0 \le \left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}} \le 1 \tag{15}$$

That means $0 \le \Delta^{-1}\left(s^L_{\phi}, \varphi^L\right) \le t$, so ① is maintained, similarly, we can get $0 \le \Delta^{-1}\left(s^R_{\phi}, \varphi^R\right) \le t$, $0 \le \Delta^{-1}\left(s^L_{\chi}, \xi^L\right) \le t, 0 \le \Delta^{-1}\left(s^R_{\chi}, \xi^R\right) \le t$.
②

$$\begin{aligned}
0 \le & \left(\frac{\Delta^{-1}\left(s^R_{\phi}, \varphi^R\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^R_{\chi}, \xi^R\right)}{t}\right)^2 = \left(\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)^2 \\
& + \left(\sqrt{1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}\right)^2 = \left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \\
& + 1-\left(1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}} \le 1
\end{aligned}$$

Example 1. *Let* $\widetilde{p}_1 = \langle[(s_2, 0.3), (s_4, 0.2)], [(s_1, 0.4), (s_2, -0.3)]\rangle$, $\widetilde{p}_2 = \langle[(s_1, 0.4), (s_3, 0.3)], [(s_2, -0.4), (s_2, 0.1)]\rangle$ *and* $\widetilde{p}_3 = \langle[(s_3, 0.2), (s_4, -0.4)], [(s_1, 0.1), (s_2, -0.2)]\rangle$ *be three IV2TLPFNs, and suppose* $k = 2$ *then according to (8), we have*

$$\mathrm{IV2TLPFMSM}^{(2)}(\tilde{p}_1,\tilde{p}_2,\ldots,\tilde{p}_n)=\left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\tilde{p}_{i_j}\right)}{C_n^k}\right)^{\frac{1}{k}}$$

$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(6\times\left(\sqrt{1-\left(1-\left(\frac{2.3}{6}\right)^2\times\left(\frac{1.4}{6}\right)^2\right)^{\frac{1}{3}}\times\left(1-\left(\frac{2.3}{6}\right)^2\times\left(\frac{3.2}{6}\right)^2\right)^{\frac{1}{3}}\times\left(1-\left(\frac{1.4}{6}\right)^2\times\left(\frac{3.2}{6}\right)^2\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right),\\ \Delta\left(6\times\left(\sqrt{1-\left(1-\left(\frac{4.2}{6}\right)^2\times\left(\frac{3.3}{6}\right)^2\right)^{\frac{1}{3}}\times\left(1-\left(\frac{4.2}{6}\right)^2\times\left(\frac{3.6}{6}\right)^2\right)^{\frac{1}{3}}\times\left(1-\left(\frac{3.3}{6}\right)^2\times\left(\frac{3.6}{6}\right)^2\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(6\times\sqrt{1-\left(\begin{array}{c}1-\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)\times\left(1-\left(\frac{1.6}{6}\right)^2\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)\times\left(1-\left(\frac{1.1}{6}\right)^2\right)\right)^{\frac{1}{3}}\\ \times\left(1-\left(1-\left(\frac{1.6}{6}\right)^2\right)\times\left(1-\left(\frac{1.1}{6}\right)^2\right)\right)^{\frac{1}{3}}\end{array}\right)^{\frac{1}{2}}}\right),\\ \Delta\left(6\times\sqrt{1-\left(\begin{array}{c}1-\left(1-\left(1-\left(\frac{1.7}{6}\right)^2\right)\times\left(1-\left(\frac{2.1}{6}\right)^2\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^2\right)\times\left(1-\left(\frac{1.8}{6}\right)^2\right)\right)^{\frac{1}{3}}\\ \times\left(1-\left(1-\left(\frac{2.1}{6}\right)^2\right)\times\left(1-\left(\frac{1.8}{6}\right)^2\right)\right)^{\frac{1}{3}}\end{array}\right)^{\frac{1}{2}}}\right)\end{array}\right]\end{array}\right\}$$

$$=\{[(s_2,0.3073),(s_4,-0.2978)],[(s_1,0.3758),(s_2,-0.1288)]\}$$

Then we will discuss some properties of IV2TLPFMSM operator.

Property 1. *(Idempotency) If* $\tilde{p}_j=\left\langle\left[\left(s^L_{\phi_j},\varphi^L_j\right),\left(s^R_{\phi_j},\varphi^R_j\right)\right],\left[\left(s^L_{\chi_j},\xi^L_j\right),\left(s^R_{\chi_j},\xi^R_j\right)\right]\right\rangle(j=1,2,\ldots,n)$ *are equal, that's to say,* $\tilde{p}_j=\tilde{p}=\left\langle\left[\left(s^L,\varphi^L\right),\left(s^R,\varphi^R\right)\right],\left[\left(s^L,\xi^L\right),\left(s^R,\xi^R\right)\right]\right\rangle(j=1,2,\ldots,n)$*, then*

$$\mathrm{IV2TLPFMSM}^{(k)}(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\tilde{p} \tag{16}$$

Property 2. *(Monotonicity) Let* $\tilde{p}_{x_i}=\left\langle\left[\left(s^L_{\phi_{x_i}},\varphi^L_{x_i}\right),\left(s^R_{\phi_{x_i}},\varphi^R_{x_i}\right)\right],\left[\left(s^L_{\chi_{x_i}},\xi^L_{x_i}\right),\left(s^R_{\chi_{x_i}},\xi^R_{x_i}\right)\right]\right\rangle$ $(i=1,2,\ldots,n)$ *and* $\tilde{p}_{y_i}=\left\langle\left[\left(s^L_{\phi_{y_i}},\varphi^L_{y_i}\right),\left(s^R_{\phi_{y_i}},\varphi^R_{y_i}\right)\right],\left[\left(s^L_{\chi_{y_i}},\xi^L_{y_i}\right),\left(s^R_{\chi_{y_i}},\xi^R_{y_i}\right)\right]\right\rangle(i=1,2,\ldots,n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s^L_{\phi_{x_i}},\varphi^L_{x_i}\right)\le\Delta^{-1}\left(s^L_{\phi_{y_i}},\varphi^L_{y_i}\right)$, $\Delta^{-1}\left(s^R_{\phi_{x_i}},\varphi^R_{x_i}\right)\ge\Delta^{-1}\left(s^R_{\phi_{y_i}},\varphi^R_{y_i}\right)$, $\Delta^{-1}\left(s^L_{\chi_{x_i}},\xi^L_{x_i}\right)\ge\Delta^{-1}\left(s^L_{\chi_{y_i}},\xi^L_{y_i}\right)$ *and* $\Delta^{-1}\left(s^R_{\chi_{x_i}},\xi^R_{x_i}\right)\ge\Delta^{-1}\left(s^R_{\chi_{y_i}},\xi^R_{y_i}\right)$ *hold for all i, then*

$$\mathrm{IV2TLPFMSM}^{(k)}(\tilde{p}_{x_1},\tilde{p}_{x_2},\cdots,\tilde{p}_{x_n})\le\mathrm{IV2TLPFMSM}^{(k)}(\tilde{p}_{y_1},\tilde{p}_{y_2},\cdots,\tilde{p}_{y_n}) \tag{17}$$

Property 3. *(Boundedness) Let* $\tilde{p}_j=\left\{\left[\left(s^L_{\phi_j},\varphi^L_j\right),\left(s^R_{\phi_j},\varphi^R_j\right)\right],\left[\left(s^L_{\chi_j},\xi^L_j\right),\left(s^R_{\chi_j},\xi^R_j\right)\right]\right\}(j=1,2,\ldots,n)$ *be a set of IV2TLPFNs. If* $\tilde{p}^+=\left\{\begin{array}{l}\left[\max\Delta^{-1}\left(s^L_{\phi_j},\varphi^L_j\right),\max\Delta^{-1}\left(s^R_{\phi_j},\varphi^R_j\right)\right],\\ \left[\min\Delta^{-1}\left(s^L_{\chi_j},\xi^L_j\right),\min\Delta^{-1}\left(s^R_{\chi_j},\xi^R_j\right)\right]\end{array}\right\}(j=1,2,\ldots,n)$ *and* $\tilde{p}^-=\left\{\begin{array}{l}\left[\min\Delta^{-1}\left(s^L_{\phi_j},\varphi^L_j\right),\min\Delta^{-1}\left(s^R_{\phi_j},\varphi^R_j\right)\right],\\ \left[\max\Delta^{-1}\left(s^L_{\chi_j},\xi^L_j\right),\max\Delta^{-1}\left(s^R_{\chi_j},\xi^R_j\right)\right]\end{array}\right\}(j=1,2,\ldots,n)$ *then*

$$\tilde{p}^-\le\mathrm{IV2TLPFMSM}^{(k)}(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)\le\tilde{p}^+ \tag{18}$$

Property 4. *(Commutativity) If Let* $\tilde{p}_j(j=1,2,\cdots,n)$ *be a collection of IV2TLPFNs, and* $\tilde{p}'_j(j=1,2,\cdots,n)$ *is any permutation of* $\tilde{p}_j(j=1,2,\cdots,n)$*, then*

$$\mathrm{IV2TLPFMSM}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_n)=\mathrm{IV2TLPFMSM}^{(k)}(\widetilde{p}'_1,\widetilde{p}'_2,\cdots,\widetilde{p}'_n) \tag{19}$$

3.2. The IV2TLPFWMSM Operator

In actual MADM, it is important to consider attribute weights. This section will propose interval-valued 2-tuple linguistic Pythagorean fuzzy weighted MSM (IV2TLPFWMSM) operator as follows.

Definition 11. *Let* $\widetilde{p}_j=\left\langle\left[\left(s^L_{\phi_j},\varphi^L_j\right),\left(s^R_{\phi_j},\varphi^R_j\right)\right],\left[\left(s^L_{\chi_j},\xi^L_j\right),\left(s^R_{\chi_j},\xi^R_j\right)\right]\right\rangle(j=1,2,\ldots,n)$ *be a set of IV2TLPFNs with their weight vector be* $w_i=(w_1,w_2,\ldots,w_n)^T$*, thereby satisfying* $w_i\in[0,1]$ *and* $\sum_{i=1}^n w_i=1$*. If*

$$\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_n)=\left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)}{C^k_n}\right)^{\frac{1}{k}} \tag{20}$$

Then we called $\mathrm{IV2TLPFWMSM}^{(k)}_{nw}$ *the interval-valued 2-tuple linguistic Pythagorean fuzzy weighted MSM (IV2TLPFWMSM) operator.*

Theorem 3. *Let* $\widetilde{p}_j=\left\langle\left[\left(s^L_{\phi_j},\varphi^L_j\right),\left(s^R_{\phi_j},\varphi^R_j\right)\right],\left[\left(s^L_{\chi_j},\xi^L_j\right),\left(s^R_{\chi_j},\xi^R_j\right)\right]\right\rangle(j=1,2,\ldots,n)$ *be a set of IV2TLPFNs. The aggregated value by using IV2TLPFWMSM operators is also an IV2TLPFN where*

$$\begin{aligned}&\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)}{C^k_n}\right)^{\frac{1}{k}}\\ &=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\left(\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C^k_n}}}\right)^{\frac{1}{k}}\right),\\ \Delta\left(t\left(\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C^k_n}}}\right)^{\frac{1}{k}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C^k_n}}\right)^{\frac{1}{k}}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C^k_n}}\right)^{\frac{1}{k}}}\right)\end{array}\right]\end{array}\right\}\end{aligned} \tag{21}$$

Proof. According to Definition 8, we can obtain

$$\begin{aligned} & nw_{i_j} \otimes \widetilde{p}_{i_j} \\ & = \left\{ \begin{array}{l} \left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)\right], \\ \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j}\right)}{t}\right)^{nw_{i_j}}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^{nw_{i_j}}\right)\right] \end{array} \right\} \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} & \overset{k}{\underset{j=1}{\otimes}} \left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right) \\ & = \left\{ \begin{array}{l} \left[\Delta\left(t\prod_{j=1}^{k}\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right), \Delta\left(t\prod_{j=1}^{k}\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)\right], \\ \left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right), \Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^r_{\chi_{i_j}}, \xi^r_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right)\right] \end{array} \right\} \end{aligned} \tag{23}$$

Thereafter,

$$\begin{aligned} & \underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}} \left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)\right) \\ & = \left\{ \begin{array}{l} \left[\begin{array}{l} \Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)}\right), \\ \Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)}\right) \end{array} \right], \\ \left[\begin{array}{l} \Delta\left(t\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right)\right), \\ \Delta\left(t\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^r_{\chi_{i_j}}, \xi^r_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right)\right) \end{array} \right] \end{array} \right\} \end{aligned} \tag{24}$$

Furthermore,

$$\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)}{C_n^k}$$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \Delta\left(t\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}} \right), \\ \Delta\left(t\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}} \right) \end{array} \right], \\ \left[\begin{array}{l} \Delta\left(t\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right)\right)^{\frac{1}{C_n^k}} \right), \\ \Delta\left(t\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^r_{\chi_{i_j}},\xi^r_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)}\right)\right)^{\frac{1}{C_n^k}} \right) \end{array} \right] \end{array} \right\} \quad (25)$$

Therefore,

$$\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n) = \left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)}{C_n^k}\right)^{\frac{1}{k}}$$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \Delta\left(t\left(\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}} \right), \\ \Delta\left(t\left(\sqrt{1-\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}} \right) \end{array} \right], \\ \left[\begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \right), \\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \right) \end{array} \right] \end{array} \right\} \quad (26)$$

Hence, (21) is kept.

Then we need to prove that (21) is an IV2TLPFN. We need to prove two conditions as follows

① $\Delta^{-1}\left(s^L_{\phi},\varphi^L\right),\Delta^{-1}\left(s^R_{\phi},\varphi^R\right),\Delta^{-1}\left(s^L_{\chi},\xi^L\right),\Delta^{-1}\left(s^R_{\chi},\xi^R\right)\subseteq[0,t]$

② $0 \leq \left(\Delta^{-1}\left(s_{\phi}^{R}, \varphi^{R}\right)\right)^{2} + \left(\Delta^{-1}\left(s_{\chi}^{R}, \xi^{R}\right)\right)^{2} \leq t^{2}$.

Let

$$\frac{\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right)}{t} = \left(\sqrt{1-\left(\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t} = \left(\sqrt{1-\left(\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right)}{t} = \sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t} = \sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

① Since $0 \leq \frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t} \leq 1$, we get

$$0 \leq \left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}} \leq 1 \textit{ and } 0 \leq \prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right) \leq 1 \tag{27}$$

Then

$$0 \leq \prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)\right) \leq 1 \tag{28}$$

Furthermore,

$$0 \leq 1-\left(\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}} \leq 1 \tag{29}$$

Therefore,

$$0 \leq \left(\sqrt{1-\left(\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}} \leq 1 \tag{30}$$

That means $0 \le \Delta^{-1}\left(s_{\phi}^{L}, \varphi^{L}\right) \le t$, so ① is maintained, similarly, we can get $0 \le \Delta^{-1}\left(s_{\phi}^{R}, \varphi^{R}\right) \le t$, $0 \le \Delta^{-1}\left(s_{\chi}^{L}, \xi^{L}\right) \le t, 0 \le \Delta^{-1}\left(s_{\chi}^{R}, \xi^{R}\right) \le t$.

②

$$
\begin{aligned}
0 &\le \left(\frac{\Delta^{-1}\left(s_{\phi}^{R}, \varphi^{R}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\chi}^{R}, \xi^{R}\right)}{t}\right)^2 \\
&= \left(\left(\sqrt{1-\left(\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R}, \varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)^2 \\
&+ \left(\sqrt{1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R}, \xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)^2 \\
&= \left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R}, \varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \\
&+ 1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R}, \xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \le 1
\end{aligned}
$$

Example 2. *Let* $\widetilde{p}_1 = \langle[(s_2, 0.3), (s_4, 0.2)], [(s_1, 0.4), (s_2, -0.3)]\rangle$, $\widetilde{p}_2 = \langle[(s_1, 0.4), (s_3, 0.3)], [(s_2, -0.4), (s_2, 0.1)]\rangle$ *and* $\widetilde{p}_3 = \langle[(s_3, 0.2), (s_4, -0.4)], [(s_1, 0.1), (s_2, -0.2)]\rangle$ *be three IV2TLPFNs, and suppose* $k = 2$, $w = (0.2, 0.3, 0.5)$ *then according to (21), we have*

$$
\mathrm{IV2TLPFWMSM}_{nw}^{(2)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\bigoplus_{1\le i_1<\ldots<i_k\le i_n}\left(\bigotimes_{j=1}^{k}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)}{C_n^k}\right)^{\frac{1}{k}}
$$

$$
= \left\{\left[\begin{array}{l}
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)\times\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)^{0.9}\right)\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)\times\left(1-\left(1-\left(\frac{3.2}{6}\right)^2\right)^{1.5}\right)\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)^{0.9}\right)\times\left(1-\left(1-\left(\frac{3.2}{6}\right)^2\right)^{1.5}\right)\right)
\end{array}\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right), \\
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)^{0.6}\right)\times\left(1-\left(1-\left(\frac{3.3}{6}\right)^2\right)^{0.9}\right)\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)^{0.6}\right)\times\left(1-\left(1-\left(\frac{3.6}{6}\right)^2\right)^{1.5}\right)\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{3.3}{6}\right)^2\right)^{0.9}\right)\times\left(1-\left(1-\left(\frac{3.6}{6}\right)^2\right)^{1.5}\right)\right)
\end{array}\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)
\end{array}\right],\right\}
$$

$$= \left\{ \begin{bmatrix} \Delta\left(6 \times \sqrt{1-\left(1-\left(\begin{matrix}\left(1-\left(1-\left(\frac{1.4}{6}\right)^{1.2}\right)\times\left(1-\left(\frac{1.6}{6}\right)^{1.8}\right)\right)\times\left(1-\left(1-\left(\frac{1.4}{6}\right)^{1.2}\right)\times\left(1-\left(\frac{1.1}{6}\right)^{3}\right)\right) \\ \times\left(1-\left(1-\left(\frac{1.6}{6}\right)^{1.8}\right)\times\left(1-\left(\frac{1.1}{6}\right)^{3}\right)\right)\end{matrix}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right), \\ \Delta\left(6 \times \sqrt{1-\left(1-\left(\begin{matrix}\left(1-\left(1-\left(\frac{1.7}{6}\right)^{1.2}\right)\times\left(1-\left(\frac{2.1}{6}\right)^{1.8}\right)\right)\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{1.2}\right)\times\left(1-\left(\frac{1.8}{6}\right)^{3}\right)\right) \\ \times\left(1-\left(1-\left(\frac{2.1}{6}\right)^{1.8}\right)\times\left(1-\left(\frac{1.8}{6}\right)^{3}\right)\right)\end{matrix}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right) \end{bmatrix} \right\}$$

$$= \{[(s_2, 0.2565), (s_4, -0.3954)], [(s_2, -0.2430), (s_2, 0.1586)]\}$$

Then we will discuss some properties of IV2TLPFWMSM operator.

Property 5. *(Idempotency) If* $\tilde{p}_j = \left\langle \left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right] \right\rangle (j = 1, 2, \ldots, n)$ *are equal, then*

$$\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \tilde{p} \tag{31}$$

Property 6. *(Monotonicity) Let* $\tilde{p}_{x_i} = \left\langle \left[\left(s^L_{\phi_{x_i}}, \varphi^L_{x_i}\right), \left(s^R_{\phi_{x_i}}, \varphi^R_{x_i}\right)\right], \left[\left(s^L_{\chi_{x_i}}, \xi^L_{x_i}\right), \left(s^R_{\chi_{x_i}}, \xi^R_{x_i}\right)\right] \right\rangle$ $(i = 1, 2, \ldots, n)$ *and* $\tilde{p}_{y_i} = \left\langle \left[\left(s^L_{\phi_{y_i}}, \varphi^L_{y_i}\right), \left(s^R_{\phi_{y_i}}, \varphi^R_{y_i}\right)\right], \left[\left(s^L_{\chi_{y_i}}, \xi^L_{y_i}\right), \left(s^R_{\chi_{y_i}}, \xi^R_{y_i}\right)\right] \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s^L_{\phi_{x_i}}, \varphi^L_{x_i}\right) \le \Delta^{-1}\left(s^L_{\phi_{y_i}}, \varphi^L_{y_i}\right), \Delta^{-1}\left(s^R_{\phi_{x_i}}, \varphi^R_{x_i}\right) \ge \Delta^{-1}\left(s^R_{\phi_{y_i}}, \varphi^R_{y_i}\right)$, $\Delta^{-1}\left(s^L_{\chi_{x_i}}, \xi^L_{x_i}\right) \ge \Delta^{-1}\left(s^L_{\chi_{y_i}}, \xi^L_{y_i}\right)$ *and* $\Delta^{-1}\left(s^R_{\chi_{x_i}}, \xi^R_{x_i}\right) \ge \Delta^{-1}\left(s^R_{\chi_{y_i}}, \xi^R_{y_i}\right)$ *hold for all i, then*

$$\begin{aligned} &\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}_{x_1}, \tilde{p}_{x_2}, \cdots, \tilde{p}_{x_n}) \\ &\le \mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}_{y_1}, \tilde{p}_{y_2}, \cdots, \tilde{p}_{y_n}) \end{aligned} \tag{32}$$

Property 7. *(Boundedness) Let* $\tilde{p}_j = \left\{\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\} (j = 1, 2, \ldots, n)$ *be a set of IV2TLPFNs. If* $\tilde{p}^+ = \left\{ \begin{array}{l} \left[\max\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \max\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\min\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \min\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *and* $\tilde{p}^- = \left\{ \begin{array}{l} \left[\min\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \min\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\max\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \max\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *then*

$$\tilde{p}^- \le \mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \le \tilde{p}^+ \tag{33}$$

Property 8. *(Commutativity) If Let* $\tilde{p}_j (j = 1, 2, \cdots, n)$ *be a collection of IV2TLPFNs, and* $\tilde{p}'_j (j = 1, 2, \cdots, n)$ *is any permutation of* $\tilde{p}_j (j = 1, 2, \cdots, n)$*, then*

$$\mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \mathrm{IV2TLPFWMSM}^{(k)}_{nw}(\tilde{p}'_1, \tilde{p}'_2, \cdots, \tilde{p}'_n) \tag{34}$$

4. The Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Generalized MSM (IV2TLPFGMSM) and Interval-Valued 2-Tuple Linguistic Pythagorean Fuzzy Weighted GMSM (IV2TLPFWGMSM) Operators

4.1. IV2TLPFGMSM Operator

Detemple and Robertson [40] proposed the generalized MSM (GMSM) considering the individual differences.

Definition 12. *[40] Let $a_i (i = 1, 2, \ldots, n)$ be a set of nonnegative real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$. A generalized MSM operator of dimension n is a mapping $\mathrm{GMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)} : (R^+)^n \to R$, and it can be defined as follows:*

$$\mathrm{GMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(x_1, x_2, \ldots, x_n) = \left(\frac{\sum\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \prod\limits_{j=1}^{k} x_{i_j}^{\lambda_j}}{C_n^k} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \tag{35}$$

Then we called $\mathrm{GMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}$ the generalized Maclaurin symmetric mean (GMSM) operator, where $(i_1, i_2, \ldots, i_k)$ traverses all the k-tuple combinations of $(1, 2, \ldots, n)$, C_n^k is the binomial coefficient.

In this section, we will propose the GMSM operator for IV2TLPFNs as follows.

Definition 13. *Let $\widetilde{p}_j = \left\langle \left[\left(s_{\phi_j}^L, \varphi_j^L \right), \left(s_{\phi_j}^R, \varphi_j^R \right) \right], \left[\left(s_{\chi_j}^L, \xi_j^L \right), \left(s_{\chi_j}^R, \xi_j^R \right) \right] \right\rangle (j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ be a vector of parameters. If*

$$\mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\bigoplus\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(\bigotimes\limits_{j=1}^{n} \widetilde{p}_{i_j}^{\lambda_j} \right)}{C_n^k} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \tag{36}$$

Then we called $\mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}$ the interval-valued 2-tuple linguistic Pythagorean fuzzy generalized MSM (IV2TLPFGMSM) operator.

Theorem 4. *Let $\widetilde{p}_j = \left\langle \left[\left(s_{\phi_j}^L, \varphi_j^L \right), \left(s_{\phi_j}^R, \varphi_j^R \right) \right], \left[\left(s_{\chi_j}^L, \xi_j^L \right), \left(s_{\chi_j}^R, \xi_j^R \right) \right] \right\rangle (j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs. The aggregated value by using IV2TLPFGMSM operator is also an IV2TLPFN where*

$$\begin{aligned}
&\mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\bigoplus\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(\bigotimes\limits_{j=1}^{n} \widetilde{p}_{i_j}^{\lambda_j} \right)}{C_n^k} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \\
&= \left\{ \begin{array}{l}
\left[\begin{array}{l}
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^L, \varphi_{i_j}^L \right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \right), \\
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^R, \varphi_{i_j}^R \right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \right)
\end{array} \right], \\
\left[\begin{array}{l}
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^L, \xi_{i_j}^L \right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}}} \right), \\
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^R, \xi_{i_j}^R \right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}}} \right)
\end{array} \right]
\end{array} \right\}
\end{aligned} \tag{37}$$

Proof. According to Definition 8, we can obtain

$$\widetilde{p}_{i_j}^{\lambda_j}=\left\{\begin{array}{c}\left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{\lambda_j}\right),\Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^{\lambda_j}\right)\right],\\ \left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right),\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right)\right]\end{array}\right\} \tag{38}$$

Thus,

$$\begin{aligned}&\overset{k}{\underset{j=1}{\otimes}}\widetilde{p}_{i_j}^{\lambda_j}\\ &=\left\{\begin{array}{c}\left[\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{\lambda_j}\right),\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^{\lambda_j}\right)\right],\\ \left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right),\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right)\right]\end{array}\right\}\end{aligned} \tag{39}$$

Thereafter,

$$\begin{aligned}&\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\widetilde{p}_{i_j}^{\lambda_j}\right)\\ &=\left\{\begin{array}{c}\left[\begin{array}{c}\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{kn}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{2\lambda_j}\right)}\right),\\ \Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^{2\lambda_j}\right)}\right)\end{array}\right],\\ \left[\begin{array}{c}\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right),\\ \Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right)\end{array}\right]\end{array}\right\}\end{aligned} \tag{40}$$

Furthermore,

$$\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\widetilde{p}_{i_j}^{\lambda_j}\right)}{C_n^k} = \left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right), \\ \Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)\end{array}\right], \\ \left[\begin{array}{l}\Delta\left(t\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right)^{\frac{1}{C_n^k}}\right), \\ \Delta\left(t\left(\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}}\right)^{\frac{1}{C_n^k}}\right)\end{array}\right]\end{array}\right\} \tag{41}$$

Therefore,

$$\mathrm{IV2TLPFGMSM}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n) = \left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\widetilde{p}_{i_j}^{\lambda_j}\right)}{C_n^k}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$

$$= \left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}\right), \\ \Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}\right)\end{array}\right], \\ \left[\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}\right), \\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}\right)\end{array}\right]\end{array}\right\} \tag{42}$$

Hence, (37) is kept.

Then we need to prove that (37) is an IV2TLPFN. We need to prove two conditions as follows:

① $\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right),\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right),\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right),\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\subseteq[0,t]$

② $0\leq\left(\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)\right)^{2}+\left(\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\right)^{2}\leq t^{2}$.

Let

$$\frac{\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right)}{t}=\left(\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\left(\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right)}{t}=\sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}$$

① Since $0\leq\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\leq 1$, we get

$$0\leq\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2\lambda_j}\leq 1\ \textit{and}\ 0\leq\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2\lambda_j}\right)\leq 1 \tag{43}$$

Then

$$0\leq\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}\leq 1 \tag{44}$$

Furthermore,

$$0\leq\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{C_n^k}}}\leq 1 \tag{45}$$

Therefore,

$$0 \leq \left(\sqrt{1 - \prod_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \leq 1 \quad (46)$$

That means $0 \leq \Delta^{-1}\left(s^L_\phi, \varphi^L\right) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}\left(s^R_\phi, \varphi^R\right) \leq t$, $0 \leq \Delta^{-1}\left(s^L_\chi, \xi^L\right) \leq t, 0 \leq \Delta^{-1}\left(s^R_\chi, \xi^R\right) \leq t$.

②

$$\begin{aligned}
0 &\leq \left(\frac{\Delta^{-1}\left(s^R_\phi, \varphi^R\right)}{t} \right)^2 + \left(\frac{\Delta^{-1}\left(s^R_\chi, \xi^R\right)}{t} \right)^2 \\
&= \left(\left(\sqrt{1 - \prod_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{C_n^k}}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \right)^2 \\
&+ \left(\sqrt{1 - \left(1 - \prod_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}}} \right)^2 \\
&= \left(1 - \prod_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \\
&+ 1 - \left(1 - \prod_{1 \leq i_1 < \ldots < i_k \leq i_n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}} \leq 1
\end{aligned}$$

Example 3. *Let* $\widetilde{p}_1 = \langle [(s_2, 0.3), (s_4, 0.2)], [(s_1, 0.4), (s_2, -0.3)] \rangle$, $\widetilde{p}_2 = \langle [(s_1, 0.4), (s_3, 0.3)], [(s_2, -0.4), (s_2, 0.1)] \rangle$ *and* $\widetilde{p}_3 = \langle [(s_3, 0.2), (s_4, -0.4)], [(s_1, 0.1), (s_2, -0.2)] \rangle$ *be three IV2TLPFNs, and suppose* $k = 2$, $\lambda = (1, 2)$ *then according to (37), we have*

$$\mathrm{IV2TLPFGMSM}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1 \le i_1 < \ldots < i_k \le i_n}{\oplus} \left(\overset{n}{\underset{j=1}{\otimes}} \widetilde{p}_{i_j}^{\lambda_j} \right)}{C_n^k} \right)^{\frac{1}{\sum_{j=1}^{k} \lambda_j}}$$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \Delta\left(6 \times \left(\sqrt{1 - \left(\begin{array}{l} \left(1 - \left(\frac{2.3}{6}\right)^{2\times1} \times \left(\frac{1.4}{6}\right)^{2\times2}\right) \times \left(1 - \left(\frac{2.3}{6}\right)^{2\times1} \times \left(\frac{3.2}{6}\right)^{2\times2}\right) \\ \times\left(1 - \left(\frac{1.4}{6}\right)^{2\times1} \times \left(\frac{3.2}{6}\right)^{2\times2}\right) \end{array} \right)^{\frac{1}{3}}} \right)^{\frac{1}{1+2}} \right), \\ \Delta\left(6 \times \left(\sqrt{1 - \left(\begin{array}{l} \left(1 - \left(\frac{4.2}{6}\right)^{2\times1} \times \left(\frac{3.3}{6}\right)^{2\times2}\right) \times \left(1 - \left(\frac{4.2}{6}\right)^{2\times1} \times \left(\frac{3.6}{6}\right)^{2\times2}\right) \\ \times\left(1 - \left(\frac{3.3}{6}\right)^{2\times1} \times \left(\frac{3.6}{6}\right)^{2\times2}\right) \end{array} \right)^{\frac{1}{3}}} \right)^{\frac{1}{1+2}} \right) \end{array} \right], \\ \left[\begin{array}{l} \Delta\left(6 \times \sqrt{1 - \left(1 - \left(\begin{array}{l} \left(1 - \left(1 - \left(\frac{1.4}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{1.6}{6}\right)^2\right)^2\right) \times \left(1 - \left(1 - \left(\frac{1.4}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{1.1}{6}\right)^2\right)^2\right) \\ \times\left(1 - \left(1 - \left(\frac{1.6}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{1.1}{6}\right)^2\right)^2\right) \end{array} \right)^{\frac{1}{3}} \right)^{\frac{1}{1+2}}} \right), \\ \Delta\left(6 \times \sqrt{1 - \left(1 - \left(\begin{array}{l} \left(1 - \left(1 - \left(\frac{1.7}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{2.1}{6}\right)^2\right)^2\right) \times \left(1 - \left(1 - \left(\frac{1.7}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{1.8}{6}\right)^2\right)^2\right) \\ \times\left(1 - \left(1 - \left(\frac{2.1}{6}\right)^2\right)^1 \times \left(1 - \left(\frac{1.8}{6}\right)^2\right)^2\right) \end{array} \right)^{\frac{1}{3}} \right)^{\frac{1}{1+2}}} \right) \end{array} \right] \end{array} \right\}$$

$$= \{[(s_3, -0.4725), (s_4, -0.3675)], [(s_2, -0.4634), (s_2, -0.0223)]\}$$

Then we will discuss some properties of IV2TLPFGMSM operator.

Property 9. *(Idempotency) If* $\widetilde{p}_j = \left\langle \left[\left(s_{\phi_j}^L, \varphi_j^L\right), \left(s_{\phi_j}^R, \varphi_j^R\right) \right], \left[\left(s_{\chi_j}^L, \xi_j^L\right), \left(s_{\chi_j}^R, \xi_j^R\right) \right] \right\rangle (j = 1, 2, \ldots, n)$ *are equal, then*

$$\mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \widetilde{p} \tag{47}$$

Property 10. *(Monotonicity) Let* $\widetilde{p}_{x_i} = \left\langle \left[\left(s_{\phi_{x_i}}^L, \varphi_{x_i}^L\right), \left(s_{\phi_{x_i}}^R, \varphi_{x_i}^R\right) \right], \left[\left(s_{\chi_{x_i}}^L, \xi_{x_i}^L\right), \left(s_{\chi_{x_i}}^R, \xi_{x_i}^R\right) \right] \right\rangle$ $(i = 1, 2, \ldots, n)$ *and* $\widetilde{p}_{y_i} = \left\langle \left[\left(s_{\phi_{y_i}}^L, \varphi_{y_i}^L\right), \left(s_{\phi_{y_i}}^R, \varphi_{y_i}^R\right) \right], \left[\left(s_{\chi_{y_i}}^L, \xi_{y_i}^L\right), \left(s_{\chi_{y_i}}^R, \xi_{y_i}^R\right) \right] \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s_{\phi_{x_i}}^L, \varphi_{x_i}^L\right) \le \Delta^{-1}\left(s_{\phi_{y_i}}^L, \varphi_{y_i}^L\right), \Delta^{-1}\left(s_{\phi_{x_i}}^R, \varphi_{x_i}^R\right) \ge \Delta^{-1}\left(s_{\phi_{y_i}}^R, \varphi_{y_i}^R\right)$, $\Delta^{-1}\left(s_{\chi_{x_i}}^L, \xi_{x_i}^L\right) \ge \Delta^{-1}\left(s_{\chi_{y_i}}^L, \xi_{y_i}^L\right)$ *and* $\Delta^{-1}\left(s_{\chi_{x_i}}^R, \xi_{x_i}^R\right) \ge \Delta^{-1}\left(s_{\chi_{y_i}}^R, \xi_{y_i}^R\right)$ *hold for all i, then*

$$\begin{array}{c} \mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_{x_1}, \widetilde{p}_{x_2}, \cdots, \widetilde{p}_{x_n}) \\ \le \mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_{y_1}, \widetilde{p}_{y_2}, \cdots, \widetilde{p}_{y_n}) \end{array} \tag{48}$$

Property 11. *(Boundedness) Let* $\widetilde{p}_j = \left\{ \left[\left(s_{\phi_j}^L, \varphi_j^L\right), \left(s_{\phi_j}^R, \varphi_j^R\right) \right], \left[\left(s_{\chi_j}^L, \xi_j^L\right), \left(s_{\chi_j}^R, \xi_j^R\right) \right] \right\} (j = 1, 2, \ldots, n)$ *be a set of IV2TLPFNs. If* $\widetilde{p}^+ = \left\{ \begin{array}{l} \left[\max\Delta^{-1}\left(s_{\phi_j}^L, \varphi_j^L\right), \max\Delta^{-1}\left(s_{\phi_j}^R, \varphi_j^R\right) \right], \\ \left[\min\Delta^{-1}\left(s_{\chi_j}^L, \xi_j^L\right), \min\Delta^{-1}\left(s_{\chi_j}^R, \xi_j^R\right) \right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *and* $\widetilde{p}^- = \left\{ \begin{array}{l} \left[\min\Delta^{-1}\left(s_{\phi_j}^L, \varphi_j^L\right), \min\Delta^{-1}\left(s_{\phi_j}^R, \varphi_j^R\right) \right], \\ \left[\max\Delta^{-1}\left(s_{\chi_j}^L, \xi_j^L\right), \max\Delta^{-1}\left(s_{\chi_j}^R, \xi_j^R\right) \right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *then*

$$\widetilde{p}^- \le \mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) \le \widetilde{p}^+ \tag{49}$$

Property 12. *(Commutativity) If Let $\widetilde{p}_j(j = 1, 2, \cdots, n)$ be a collection of IV2TLPFNs, and $\widetilde{p}'_j(j = 1, 2, \cdots, n)$ is any permutation of $\widetilde{p}_j(j = 1, 2, \cdots, n)$, then*

$$\begin{aligned} &\mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) \\ &= \mathrm{IV2TLPFGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}'_1, \widetilde{p}'_2, \cdots, \widetilde{p}'_n) \end{aligned} \tag{50}$$

4.2. The IV2TLPFWGMSM Operator

In actual MADM, it is important to consider attribute weights; this section will propose an interval-valued 2-tuple linguistic Pythagorean fuzzy weighted generalized MSM (IV2TLPFWGMSM) operator as follows.

Definition 14. *Let $\widetilde{p}_j = \left\{\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\}(j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs with their weight vector be $w_i = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. If*

$$\mathrm{IV2TLPFWGMSM}_{nw}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)^{\lambda_j}\right)}{C_n^k}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}} \tag{51}$$

Then we called $\mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}$ *the interval-valued 2-tuple linguistic Pythagorean fuzzy weighted generalized MSM (IV2TLPFWGMSM) operator.*

Theorem 5. *Let $\widetilde{p}_j = \left\langle\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\rangle(j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs. The aggregated value by using IV2TLPFWGMSM operator is also an IV2TLPFN where*

$$\begin{aligned} &\mathrm{IV2TLPFWGMSM}_{nw}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)^{\lambda_j}\right)}{C_n^k}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}} \\ &= \left\{\begin{array}{l} \left[\begin{array}{l} \Delta\left(t\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^k\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}}\right), \\ \Delta\left(t\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^k\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}}\right) \end{array}\right], \\ \left[\begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^k\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}}}\right), \\ \Delta\left(t\sqrt{1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^k\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^k \lambda_j}}}\right) \end{array}\right] \end{array}\right\} \end{aligned} \tag{52}$$

Proof. According to Definition 8, we can obtain

$$
\begin{aligned}
& nw_{i_j} \otimes \widetilde{p}_{i_j} \\
& = \left\{ \begin{array}{l}
\left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)\right], \\
\left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{nw_{i_j}}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{nw_{i_j}}\right)\right]
\end{array} \right\}
\end{aligned}
\tag{53}
$$

Thus,

$$
\begin{aligned}
& \left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)^{\lambda_j} \\
& = \left\{ \begin{array}{l}
\left[\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\lambda_j}\right), \Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\lambda_j}\right)\right], \\
\left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)\right]
\end{array} \right\}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
& \mathop{\otimes}_{j=1}^{k} \left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)^{\lambda_j} \\
& = \left\{ \begin{array}{l}
\left[\Delta\left(t\prod_{j=1}^{k}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\lambda_j}\right), \Delta\left(t\prod_{j=1}^{k}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\lambda_j}\right)\right], \\
\left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right), \Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)\right]
\end{array} \right\}
\end{aligned}
\tag{55}
$$

Thereafter,

$$
\begin{aligned}
& \mathop{\oplus}_{1\le i_1<\ldots<i_k\le i_n} \left(\mathop{\otimes}_{j=1}^{k} \left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)^{\lambda_j} \right) \\
& = \left\{ \begin{array}{l}
\left[\begin{array}{l}
\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)}\right), \\
\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)}\right)
\end{array} \right], \\
\left[\begin{array}{l}
\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)\right), \\
\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)\right)
\end{array} \right]
\end{array} \right\}
\end{aligned}
\tag{56}
$$

Furthermore,

$$\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)^{\lambda_j}\right)}{C_n^k}$$
$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right),\\ \Delta\left(t\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)^{\frac{1}{C_n^k}}\right),\\ \Delta\left(t\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}}\right)^{\frac{1}{C_n^k}}\right)\end{array}\right]\end{array}\right\} \quad (57)$$

Therefore,

$$\mathrm{IV2TLPFWGMSM}_{nw}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}\left(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n\right)=\left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)^{\lambda_j}\right)}{C_n^k}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$
$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}\right),\\ \Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}\right)\end{array}\right]\end{array}\right\} \quad (58)$$

Hence, (52) is kept.

Then we need to prove that (52) is an IV2TLPFN. We need to prove two conditions as follows:

① $\Delta^{-1}\left(s^L_\phi, \varphi^L\right), \Delta^{-1}\left(s^R_\phi, \varphi^R\right), \Delta^{-1}\left(s^L_\chi, \xi^L\right), \Delta^{-1}\left(s^R_\chi, \xi^R\right) \subseteq [0, t]$

② $0 \leq \left(\Delta^{-1}\left(s^R_\phi, \varphi^R\right)\right)^2 + \left(\Delta^{-1}\left(s^R_\chi, \xi^R\right)\right)^2 \leq t^2.$

Let

$$\frac{\Delta^{-1}\left(s^L_\phi,\varphi^L\right)}{t} = \left(\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s^R_\phi,\varphi^R\right)}{t} = \left(\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s^L_\chi,\xi^L\right)}{t} = \sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s^R_\phi,\varphi^R\right)}{t} = \sqrt{1-\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}$$

① Since $0 \leq \frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t} \leq 1$, we get

$$0 \leq \left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}} \leq 1 \text{ and } 0 \leq \prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j} \leq 1 \tag{59}$$

Then

$$0 \leq \prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}} \leq 1 \tag{60}$$

Furthermore,

$$0 \leq \sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}} \leq 1 \tag{61}$$

Therefore,

$$0 \leq \left(\sqrt{1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}} \leq 1 \tag{62}$$

That means $0 \le \Delta^{-1}\left(s_{\phi}^{L}, \varphi^{L}\right) \le t$, so ① is maintained, similarly, we can get $0 \le \Delta^{-1}\left(s_{\phi}^{R}, \varphi^{R}\right) \le t$, $0 \le \Delta^{-1}\left(s_{\chi}^{L}, \xi^{L}\right) \le t, 0 \le \Delta^{-1}\left(s_{\chi}^{R}, \xi^{R}\right) \le t$.

②

$$
\begin{aligned}
0 &\le \left(\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}\right)^{2} + \left(\frac{\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)}{t}\right)^{2} \\
&= \left(\left(\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}\right)^{2} \\
&+ \left(\sqrt{1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}}\right)^{2} \\
&= \left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}} \\
&+1-\left(1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)^{\lambda_j}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}} \le 1
\end{aligned}
$$

Example 4. *Let* $\widetilde{p}_1 = \langle[(s_2, 0.3), (s_4, 0.2)], [(s_1, 0.4), (s_2, -0.3)]\rangle$, $\widetilde{p}_2 = \langle[(s_1, 0.4), (s_3, 0.3)], [(s_2, -0.4), (s_2, 0.1)]\rangle$ *and* $\widetilde{p}_3 = \langle[(s_3, 0.2), (s_4, -0.4)], [(s_1, 0.1), (s_2, -0.2)]\rangle$ *be three IV2TLPFNs, and suppose* $k = 2$, $\lambda = (1, 2), w = (0.2, 0.3, 0.5)$ *then according to (52), we have*

$$
\mathrm{IV2TLPFWGMSM}_{nw}^{(2,\lambda_1,\lambda_2)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \left(\frac{\underset{1\le i_1<\ldots<i_k\le i_n}{\oplus}\left(\underset{j=1}{\overset{k}{\otimes}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)^{\lambda_j}\right)}{C_n^k}\right)^{\frac{1}{\sum_{j=1}^{k}\lambda_j}}
$$

$$
= \left\{\left[\begin{array}{l}
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{1}\times\left(1-\left(1-\left(\frac{1.4}{6}\right)^{2}\right)^{0.9}\right)^{2}\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{1}\times\left(1-\left(1-\left(\frac{3.2}{6}\right)^{2}\right)^{1.5}\right)^{2}\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{1.4}{6}\right)^{2}\right)^{0.9}\right)^{1}\times\left(1-\left(1-\left(\frac{3.2}{6}\right)^{2}\right)^{1.5}\right)^{2}\right)
\end{array}\right)^{\frac{1}{3}}}\right)^{\frac{1}{1+2}}\right), \\
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{4.2}{6}\right)^{2}\right)^{0.6}\right)^{1}\times\left(1-\left(1-\left(\frac{3.3}{6}\right)^{2}\right)^{0.9}\right)^{2}\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{4.2}{6}\right)^{2}\right)^{0.6}\right)^{1}\times\left(1-\left(1-\left(\frac{3.6}{6}\right)^{2}\right)^{1.5}\right)^{2}\right) \\
\times\left(1-\left(1-\left(1-\left(\frac{3.3}{6}\right)^{2}\right)^{0.9}\right)^{1}\times\left(1-\left(1-\left(\frac{3.6}{6}\right)^{2}\right)^{1.5}\right)^{2}\right)
\end{array}\right)^{\frac{1}{3}}}\right)^{\frac{1}{1+2}}\right)
\end{array}\right],\right.
$$

$$= \left\{ \begin{bmatrix} \Delta\left(6 \times \sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.4}{6}\right)^{1.2}\right)^{1} \times \left(1-\left(\frac{1.6}{6}\right)^{1.8}\right)^{2}\right) \times \left(1-\left(1-\left(\frac{1.4}{6}\right)^{1.2}\right)^{1} \times \left(1-\left(\frac{1.1}{6}\right)^{3}\right)^{2}\right) \\ \times \left(1-\left(1-\left(\frac{1.6}{6}\right)^{1.8}\right)^{1} \times \left(1-\left(\frac{1.1}{6}\right)^{3}\right)^{2}\right)\end{array}\right)^{\frac{1}{3}}\right)}^{\frac{1}{1+2}}\right), \\ \Delta\left(6 \times \sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.7}{6}\right)^{1.2}\right)^{1} \times \left(1-\left(\frac{2.1}{6}\right)^{1.8}\right)^{2}\right) \times \left(1-\left(1-\left(\frac{1.7}{6}\right)^{1.2}\right)^{1} \times \left(1-\left(\frac{1.8}{6}\right)^{3}\right)^{2}\right) \\ \times \left(1-\left(1-\left(\frac{2.1}{6}\right)^{1.8}\right)^{1} \times \left(1-\left(\frac{1.8}{6}\right)^{3}\right)^{2}\right)\end{array}\right)^{\frac{1}{3}}\right)}^{\frac{1}{1+2}}\right) \end{bmatrix} \right\}$$

$$= \{[(s_3, -0.3535), (s_4, -0.2764)], [(s_1, 0.4452), (s_2, -0.2175)]\}$$

Then we will discuss some properties of IV2TLPFWGMSM operator.

Property 13. *(Idempotency) If* $\widetilde{p}_j = \left\langle \left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right] \right\rangle (j = 1, 2, \ldots, n)$ *are equal, then*

$$\mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) = \widetilde{p} \tag{63}$$

Property 14. *(Monotonicity) Let* $\widetilde{p}_{x_i} = \left\langle \left[\left(s^L_{\phi_{x_i}}, \varphi^L_{x_i}\right), \left(s^R_{\phi_{x_i}}, \varphi^R_{x_i}\right)\right], \left[\left(s^L_{\chi_{x_i}}, \xi^L_{x_i}\right), \left(s^R_{\chi_{x_i}}, \xi^R_{x_i}\right)\right] \right\rangle$ $(i = 1, 2, \ldots, n)$ *and* $\widetilde{p}_{y_i} = \left\langle \left[\left(s^L_{\phi_{y_i}}, \varphi^L_{y_i}\right), \left(s^R_{\phi_{y_i}}, \varphi^R_{y_i}\right)\right], \left[\left(s^L_{\chi_{y_i}}, \xi^L_{y_i}\right), \left(s^R_{\chi_{y_i}}, \xi^R_{y_i}\right)\right] \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s^L_{\phi_{x_i}}, \varphi^L_{x_i}\right) \le \Delta^{-1}\left(s^L_{\phi_{y_i}}, \varphi^L_{y_i}\right), \Delta^{-1}\left(s^R_{\phi_{x_i}}, \varphi^R_{x_i}\right) \ge \Delta^{-1}\left(s^R_{\phi_{y_i}}, \varphi^R_{y_i}\right), \Delta^{-1}\left(s^L_{\chi_{x_i}}, \xi^L_{x_i}\right) \ge \Delta^{-1}\left(s^L_{\chi_{y_i}}, \xi^L_{y_i}\right)$ *and* $\Delta^{-1}\left(s^R_{\chi_{x_i}}, \xi^R_{x_i}\right) \ge \Delta^{-1}\left(s^R_{\chi_{y_i}}, \xi^R_{y_i}\right)$ *hold for all i, then*

$$\begin{array}{l} \mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}_{nw}(\widetilde{p}_{x_1}, \widetilde{p}_{x_2}, \cdots, \widetilde{p}_{x_n}) \\ \le \mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}_{nw}(\widetilde{p}_{y_1}, \widetilde{p}_{y_2}, \cdots, \widetilde{p}_{y_n}) \end{array} \tag{64}$$

Property 15. *(Boundedness) Let* $\widetilde{p}_j = \left\{ \left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right] \right\} (j = 1, 2, \ldots, n)$ *be a set of IV2TLPFNs. If* $\widetilde{p}^+ = \left\{ \begin{array}{c} \left[\max\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \max\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\min\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \min\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *and* $\widetilde{p}^- = \left\{ \begin{array}{c} \left[\min\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \min\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\max\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \max\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right] \end{array} \right\} (j = 1, 2, \ldots, n)$ *then*

$$\widetilde{p}^- \le \mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) \le \widetilde{p}^+ \tag{65}$$

Property 16. *(Commutativity) If Let* $\widetilde{p}_j (j = 1, 2, \cdots, n)$ *be a collection of IV2TLPFNs, and* $\widetilde{p}'_j (j = 1, 2, \cdots, n)$ *is any permutation of* $\widetilde{p}_j (j = 1, 2, \cdots, n)$*, then*

$$\begin{array}{l} \mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) \\ = \mathrm{IV2TLPFWGMSM}^{(k,\lambda_1,\lambda_2,\ldots,\lambda_k)}(\widetilde{p}'_1, \widetilde{p}'_2, \cdots, \widetilde{p}'_n) \end{array} \tag{66}$$

5. The IV2TLPFDMSM and IV2TLPFWDMSM Operators

5.1. IV2TLPFDMSM Operator

Qin and Liu [44] proposed the dual MSM (DMSM) operator considering both the MSM and the dual operation.

Definition 15. *[44] Let $a_i (i = 1, 2, \ldots, n)$ be a set of nonnegative real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$. If*

$$\mathrm{DMSM}^{(k)}(x_1, x_2, \ldots, x_n) = \frac{1}{k}\left(\prod_{1 \leq i_1 < \ldots < i_k \leq i_n}\left(\sum_{j=1}^{k} x_{i_j}\right)^{\frac{1}{C_n^k}}\right) \tag{67}$$

Then we called $\mathrm{DMSM}^{(k)}$ the dual Maclaurin symmetric mean (DMSM) operator, where $(i_1, i_2, \ldots, i_k)$ traverses all the k-tuple combinations of $(1, 2, \ldots, n)$, C_n^k is the binomial coefficient.

In this section, we will propose the DMSM operator for IV2TLPFNs as follows.

Definition 16. *Let $\widetilde{p}_j = \left\langle\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\rangle (j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs. If*

$$\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \frac{1}{k}\left(\underset{1 \leq i_1 < \ldots < i_k \leq i_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}} \widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\right) \tag{68}$$

Then we called $\mathrm{IV2TLPFDMSM}^{(k)}$ the interval-valued 2-tuple linguistic Pythagorean fuzzy dual MSM (IV2TLPFDMSM) operator.

Theorem 6. *Let $\widetilde{p}_j = \left\langle\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\rangle (j = 1, 2, \ldots, n)$ be a set of IV2TLPFNs. The aggregated value by using IV2TLPFDMSM operator is also an IV2TLPFN where*

$$\begin{aligned}
&\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \tfrac{1}{k}\left(\underset{1 \leq i_1 < \ldots < i_k \leq i_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}} \widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\right) \\
&= \left\{\begin{array}{l}
\left[\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right), \\
\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)
\end{array}\right], \\
\left[\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right), \\
\Delta\left(t\left(\sqrt{1-\prod\limits_{1 \leq i_1 < \ldots < i_k \leq i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)
\end{array}\right]
\end{array}\right\}
\end{aligned} \tag{69}$$

Proof. According to Definition 8, we can obtain

$$\underset{j=1}{\overset{n}{\oplus}}\widetilde{p}_{i_j}=\left\{\begin{array}{c}\left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right),\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)\right],\\ \left[\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)\right),\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)\right)\right]\end{array}\right\} \tag{70}$$

Thus,

$$\begin{aligned}&\left(\underset{j=1}{\overset{n}{\oplus}}\widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\\&=\left\{\begin{array}{c}\left[\Delta\left(t\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right),\Delta\left(t\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right)\right],\\ \left[\Delta\left(t\sqrt{1-\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right),\Delta\left(t\sqrt{1-\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right)\right]\end{array}\right\}\end{aligned} \tag{71}$$

Thereafter,

$$\begin{aligned}&\underset{1\le i_1<\ldots<i_k\le i_n}{\otimes}\left(\underset{j=1}{\overset{n}{\oplus}}\widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\\&=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right),\\ \Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^{2}\right)}\right)^{\frac{1}{C_n^k}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right),\\ \Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2}\right)^{\frac{1}{C_n^k}}}\right)\end{array}\right]\end{array}\right\}\end{aligned} \tag{72}$$

Furthermore,

$$
\begin{aligned}
&\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \frac{1}{k}\left(\underset{1\le i_1<\ldots<i_k\le i_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\right) \\
&= \left\{\begin{array}{l}
\left[\begin{array}{l}
\Delta\left(t\sqrt{\left(1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right), \\
\Delta\left(t\sqrt{\left(1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)
\end{array}\right], \\
\left[\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right), \\
\Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)
\end{array}\right]
\end{array}\right\}
\end{aligned}
\tag{73}
$$

Hence, (69) is kept.

Then we need to prove that (69) is an IV2TLPFN. We need to prove two conditions as follows:

① $\Delta^{-1}\left(s^L_\phi, \varphi^L\right), \Delta^{-1}\left(s^R_\phi, \varphi^R\right), \Delta^{-1}\left(s^L_\chi, \xi^L\right), \Delta^{-1}\left(s^R_\chi, \xi^R\right) \subseteq [0, t]$

② $0 \le \left(\Delta^{-1}\left(s^R_\phi, \varphi^R\right)\right)^2 + \left(\Delta^{-1}\left(s^R_\chi, \xi^R\right)\right)^2 \le t^2.$

Let

$$\frac{\Delta^{-1}\left(s^L_\phi,\varphi^L\right)}{t} = \sqrt{\left(1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s^R_\phi,\varphi^R\right)}{t} = \sqrt{\left(1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s^L_\chi,\xi^L\right)}{t} = \left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s^R_\phi,\varphi^R\right)}{t} = \left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

① Since $0 \le \frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t} \le 1$, we get

$$0 \le \prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right) \le 1 \; and \; 0 \le \left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)\right) \le 1 \quad (74)$$

Then

$$0 \le \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}} \le 1 \quad (75)$$

Furthermore,

$$0 \le \left(1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \le 1 \quad (76)$$

Therefore,

$$0 \le \sqrt{1-\left(1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \le 1 \quad (77)$$

That means $0 \le \Delta^{-1}\left(s^L_{\phi}, \varphi^L\right) \le t$, so ① is maintained, similarly, we can get $0 \le \Delta^{-1}\left(s^R_{\phi}, \varphi^R\right) \le t$, $0 \le \Delta^{-1}\left(s^L_{\chi}, \xi^L\right) \le t, 0 \le \Delta^{-1}\left(s^R_{\chi}, \xi^R\right) \le t$.

②

$$\begin{aligned}
&0 \le \left(\frac{\Delta^{-1}\left(s^R_{\phi}, \varphi^R\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^R_{\chi}, \xi^R\right)}{t}\right)^2 \\
&= \left(\sqrt{1-\left(1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)^2 \\
&+ \left(\left(\sqrt{1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)^2 \\
&= 1-\left(1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \\
&+ \left(1-\prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^2\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \le 1
\end{aligned}$$

Example 5. *Let* $\widetilde{p}_1 = \langle[(s_2,0.3),(s_4,0.2)],[(s_1,0.4),(s_2,-0.3)]\rangle$, $\widetilde{p}_2 = \langle[(s_1,0.4),(s_3,0.3)],[(s_2,-0.4),(s_2,0.1)]\rangle$ *and* $\widetilde{p}_3 = \langle[(s_3,0.2),(s_4,-0.4)],[(s_1,0.1),(s_2,-0.2)]\rangle$ *be three IV2TLPFNs, and suppose* $k=2$*, then according to (69), we have*

$$\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\frac{1}{k}\left(\underset{1\le i_1<\ldots<i_k\le i_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\widetilde{p}_{i_j}\right)^{\frac{1}{C_n^k}}\right)$$

$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)\times\left(1-\left(\frac{1.4}{6}\right)^2\right)\right)\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)\times\left(1-\left(\frac{3.2}{6}\right)^2\right)\right)\\ \times\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)\times\left(1-\left(\frac{3.2}{6}\right)^2\right)\right)\end{array}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right),\\ \Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)\times\left(1-\left(\frac{3.3}{6}\right)^2\right)\right)\times\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)\times\left(1-\left(\frac{3.6}{6}\right)^2\right)\right)\\ \times\left(1-\left(1-\left(\frac{3.3}{6}\right)^2\right)\times\left(1-\left(\frac{3.6}{6}\right)^2\right)\right)\end{array}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{1.4}{6}\right)^2\times\left(\frac{1.6}{6}\right)^2\right)\times\left(1-\left(\frac{1.4}{6}\right)^2\times\left(\frac{1.1}{6}\right)^2\right)\times\left(1-\left(\frac{1.6}{6}\right)^2\times\left(\frac{1.1}{6}\right)^2\right)\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right),\\ \Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{1.7}{6}\right)^2\times\left(\frac{2.1}{6}\right)^2\right)\times\left(1-\left(\frac{1.7}{6}\right)^2\times\left(\frac{1.8}{6}\right)^2\right)\times\left(1-\left(\frac{2.1}{6}\right)^2\times\left(\frac{1.8}{6}\right)^2\right)\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)\end{array}\right]\end{array}\right\}$$

$$=\{[(s_2,0.3799),(s_4,-0.2775)],[(s_1,0.3674),(s_2,-0.1336)]\}$$

Then we will discuss some properties of IV2TLPFDMSM operator.

Property 17. *(Idempotency) If* $\widetilde{p}_j=\left\langle\left[\left(s_{\phi_j}^L,\varphi_j^L\right),\left(s_{\phi_j}^R,\varphi_j^R\right)\right],\left[\left(s_{\chi_j}^L,\xi_j^L\right),\left(s_{\chi_j}^R,\xi_j^R\right)\right]\right\rangle(j=1,2,\ldots,n)$ *are equal, then*

$$\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\widetilde{p} \tag{78}$$

Property 18. *(Monotonicity) Let* $\widetilde{p}_{x_i}=\left\langle\left[\left(s_{\phi_{x_i}}^L,\varphi_{x_i}^L\right),\left(s_{\phi_{x_i}}^R,\varphi_{x_i}^R\right)\right],\left[\left(s_{\chi_{x_i}}^L,\xi_{x_i}^L\right),\left(s_{\chi_{x_i}}^R,\xi_{x_i}^R\right)\right]\right\rangle$ $(i=1,2,\ldots,n)$ *and* $\widetilde{p}_{y_i}=\left\langle\left[\left(s_{\phi_{y_i}}^L,\varphi_{y_i}^L\right),\left(s_{\phi_{y_i}}^R,\varphi_{y_i}^R\right)\right],\left[\left(s_{\chi_{y_i}}^L,\xi_{y_i}^L\right),\left(s_{\chi_{y_i}}^R,\xi_{y_i}^R\right)\right]\right\rangle(i=1,2,\ldots,n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s_{\phi_{x_i}}^L,\varphi_{x_i}^L\right)\le\Delta^{-1}\left(s_{\phi_{y_i}}^L,\varphi_{y_i}^L\right)$, $\Delta^{-1}\left(s_{\phi_{x_i}}^R,\varphi_{x_i}^R\right)\ge\Delta^{-1}\left(s_{\phi_{y_i}}^R,\varphi_{y_i}^R\right)$, $\Delta^{-1}\left(s_{\chi_{x_i}}^L,\xi_{x_i}^L\right)\ge\Delta^{-1}\left(s_{\chi_{y_i}}^L,\xi_{y_i}^L\right)$ *and* $\Delta^{-1}\left(s_{\chi_{x_i}}^R,\xi_{x_i}^R\right)\ge\Delta^{-1}\left(s_{\chi_{y_i}}^R,\xi_{y_i}^R\right)$ *hold for all i, then*

$$\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_{x_1},\widetilde{p}_{x_2},\cdots,\widetilde{p}_{x_n})\le\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_{y_1},\widetilde{p}_{y_2},\cdots,\widetilde{p}_{y_n}) \tag{79}$$

Property 19. *(Boundedness) Let* $\widetilde{p}_j=\left\{\left[\left(s_{\phi_j}^L,\varphi_j^L\right),\left(s_{\phi_j}^R,\varphi_j^R\right)\right],\left[\left(s_{\chi_j}^L,\xi_j^L\right),\left(s_{\chi_j}^R,\xi_j^R\right)\right]\right\}(j=1,2,\ldots,n)$ *be a set of IV2TLPFNs. If* $\widetilde{p}^+=\left\{\begin{array}{l}\left[\max\Delta^{-1}\left(s_{\phi_j}^L,\varphi_j^L\right),\max\Delta^{-1}\left(s_{\phi_j}^R,\varphi_j^R\right)\right],\\ \left[\min\Delta^{-1}\left(s_{\chi_j}^L,\xi_j^L\right),\min\Delta^{-1}\left(s_{\chi_j}^R,\xi_j^R\right)\right]\end{array}\right\}(j=1,2,\ldots,n)$ *and* $\widetilde{p}^-=\left\{\begin{array}{l}\left[\min\Delta^{-1}\left(s_{\phi_j}^L,\varphi_j^L\right),\min\Delta^{-1}\left(s_{\phi_j}^R,\varphi_j^R\right)\right],\\ \left[\max\Delta^{-1}\left(s_{\chi_j}^L,\xi_j^L\right),\max\Delta^{-1}\left(s_{\chi_j}^R,\xi_j^R\right)\right]\end{array}\right\}(j=1,2,\ldots,n)$ *then*

$$\widetilde{p}^-\le\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_n)\le\widetilde{p}^+ \tag{80}$$

Property 20. *(Commutativity) If Let* $\widetilde{p}_j (j = 1, 2, \cdots, n)$ *be a collection of IV2TLPFNs, and* $\widetilde{p}'_j (j = 1, 2, \cdots, n)$ *is any permutation of* $\widetilde{p}_j (j = 1, 2, \cdots, n)$*, then*

$$\mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) = \mathrm{IV2TLPFDMSM}^{(k)}(\widetilde{p}'_1, \widetilde{p}'_2, \cdots, \widetilde{p}'_n) \tag{81}$$

5.2. The IV2TLPFWDMSM Operator

In actual MADM, it is important to consider attribute weights; this section will propose an interval-valued 2-tuple linguistic Pythagorean fuzzy weighted dual MSM (IV2TLPFWDMSM) operator as follows.

Definition 17. *Let* $\widetilde{p}_j = \left\{ \left[\left(s^L_{\phi_j}, \varphi^L_j \right), \left(s^R_{\phi_j}, \varphi^R_j \right) \right], \left[\left(s^L_{\chi_j}, \xi^L_j \right), \left(s^R_{\chi_j}, \xi^R_j \right) \right] \right\} (j = 1, 2, \ldots, n)$ *be a set of IV2TLPFNs with their weight vector be* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^n w_i = 1$*. If*

$$\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \frac{1}{k} \left(\underset{1 \le i_1 < \ldots < i_k \le i_n}{\otimes} \left(\overset{k}{\underset{j=1}{\oplus}} \left(nw_{i_j} \otimes \widetilde{p}_{i_j} \right) \right)^{\frac{1}{C^k_n}} \right) \tag{82}$$

Then we called $\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}$ the interval-valued 2-tuple linguistic Pythagorean fuzzy weighted dual MSM (IV2TLPFWDMSM) operator.

Theorem 7. *Let* $\widetilde{p}_j = \left\langle \left[\left(s^L_{\phi_j}, \varphi^L_j \right), \left(s^R_{\phi_j}, \varphi^R_j \right) \right], \left[\left(s^L_{\chi_j}, \xi^L_j \right), \left(s^R_{\chi_j}, \xi^R_j \right) \right] \right\rangle (j = 1, 2, \ldots, n)$ *be a set of IV2TLPFNs. The aggregated value by using 2TLPFWDMSM operator is also an IV2TLPFN where*

$$\begin{aligned}
&\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \tfrac{1}{k} \left(\underset{1 \le i_1 < \ldots < i_k \le i_n}{\otimes} \left(\overset{k}{\underset{j=1}{\oplus}} \left(nw_{i_j} \otimes \widetilde{p}_{i_j} \right) \right)^{\frac{1}{C^k_n}} \right) \\
&= \left\{ \begin{array}{l}
\left[\begin{array}{l}
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j} \right)}{t} \right)^2 \right)^{nw_{i_j}} \right)^{\frac{1}{C^k_n}} \right)^{\frac{1}{k}}} \right), \\
\Delta \left(t \sqrt{1 - \left(1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j} \right)}{t} \right)^2 \right)^{nw_{i_j}} \right)^{\frac{1}{C^k_n}} \right)^{\frac{1}{k}}} \right)
\end{array} \right], \\
\left[\begin{array}{l}
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}}, \xi^L_{i_j} \right)}{t} \right)^{2nw_{i_j}} \right)^{\frac{1}{C^k_n}}} \right)^{\frac{1}{k}} \right), \\
\Delta \left(t \left(\sqrt{1 - \prod\limits_{1 \le i_1 < \ldots < i_k \le i_n} \left(1 - \prod\limits_{j=1}^{k} \left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j} \right)}{t} \right)^{2nw_{i_j}} \right)^{\frac{1}{C^k_n}}} \right)^{\frac{1}{k}} \right)
\end{array} \right]
\end{array} \right\}
\end{aligned} \tag{83}$$

Proof. According to Definition 8, we can obtain

$$nw_{i_j} \otimes \widetilde{p}_{i_j} = \left\{ \begin{array}{l} \left[\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)\right], \\ \left[\Delta\left(t\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{nw_{i_j}}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{nw_{i_j}}\right)\right] \end{array} \right\} \tag{84}$$

Thus,

$$\mathop{\oplus}_{j=1}^{k}\left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right) = \left\{ \begin{array}{l} \left[\Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right), \Delta\left(t\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)\right], \\ \left[\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{nw_{i_j}}\right), \Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{nw_{i_j}}\right)\right] \end{array} \right\} \tag{85}$$

Thereafter,

$$\left(\mathop{\oplus}_{j=1}^{k}\left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)\right)^{\frac{1}{C_n^k}} = \left\{ \begin{array}{l} \left[\Delta\left(t\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\frac{1}{C_n^k}}\right), \Delta\left(t\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\frac{1}{C_n^k}}\right)\right], \\ \left[\Delta\left(t\sqrt{1-\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right), \Delta\left(t\sqrt{1-\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)\right] \end{array} \right\} \tag{86}$$

Furthermore,

$$\mathop{\otimes}_{1\le i_1<\ldots<i_k\le i_n}\left(\mathop{\oplus}_{j=1}^{k}\left(nw_{i_j} \otimes \widetilde{p}_{i_j}\right)\right)^{\frac{1}{C_n^k}} = \left\{ \begin{array}{l} \left[\begin{array}{l} \Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}},\varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\frac{1}{C_n^k}}\right), \\ \Delta\left(t\prod_{1\le i_1<\ldots<i_k\le i_n}\left(\sqrt{1-\prod_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}},\varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}}\right)^{\frac{1}{C_n^k}}\right) \end{array} \right], \\ \left[\begin{array}{l} \Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^L_{\chi_{i_j}},\xi^L_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right), \\ \Delta\left(t\sqrt{1-\prod_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right) \end{array} \right] \end{array} \right\} \tag{87}$$

Therefore,

$$\mathrm{IV2TLPFWDMSM}_{nw}^{(k)}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\frac{1}{k}\left(\underset{1\le i_1<\ldots<i_k\le i_n}{\otimes}\left(\overset{k}{\underset{j=1}{\oplus}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)^{\frac{1}{C_n^k}}\right)$$

$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right),\\ \Delta\left(t\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)\end{array}\right]\end{array}\right\} \tag{88}$$

Hence, (83) is kept.

Then we need to prove that (83) is an IV2TLPFN. We need to prove two conditions as follows:

① $\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right),\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right),\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right),\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\subseteq[0,t]$

② $0\le\left(\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)\right)^2+\left(\Delta^{-1}\left(s_{\chi}^{R},\xi^{R}\right)\right)^2\le t^2.$

Let

$$\frac{\Delta^{-1}\left(s_{\phi}^{L},\varphi^{L}\right)}{t}=\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{L},\varphi_{i_j}^{L}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\sqrt{1-\left(1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}^{R},\varphi_{i_j}^{R}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}$$

$$\frac{\Delta^{-1}\left(s_{\chi}^{L},\xi^{L}\right)}{t}=\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{L},\xi_{i_j}^{L}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

$$\frac{\Delta^{-1}\left(s_{\phi}^{R},\varphi^{R}\right)}{t}=\left(\sqrt{1-\prod\limits_{1\le i_1<\ldots<i_k\le i_n}\left(1-\prod\limits_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s_{\chi_{i_j}}^{R},\xi_{i_j}^{R}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}$$

① Since $0 \le \frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t} \le 1$, we get

$$0 \le \left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}} \le 1 \text{ and } 0 \le \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}} \le 1 \tag{89}$$

Then

$$0 \le \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}} \le 1 \tag{90}$$

Furthermore,

$$0 \le \left(1 - \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \le 1 \tag{91}$$

Therefore,

$$0 \le \sqrt{1 - \left(1 - \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^L_{\phi_{i_j}}, \varphi^L_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \le 1 \tag{92}$$

That means $0 \le \Delta^{-1}\left(s^L_{\phi}, \varphi^L\right) \le t$, so ① is maintained, similarly, we can get $0 \le \Delta^{-1}\left(s^R_{\phi}, \varphi^R\right) \le t$, $0 \le \Delta^{-1}\left(s^L_{\chi}, \xi^L\right) \le t, 0 \le \Delta^{-1}\left(s^R_{\chi}, \xi^R\right) \le t$.

②

$$\begin{aligned}
0 &\le \left(\frac{\Delta^{-1}\left(s^R_{\phi}, \varphi^R\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s^R_{\chi}, \xi^R\right)}{t}\right)^2 \\
&= \left(\sqrt{1 - \left(1 - \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}\right)^2 \\
&+ \left(\left(\sqrt{1 - \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}}, \xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}}\right)^{\frac{1}{k}}\right)^2 \\
&= 1 - \left(1 - \prod_{1 \le i_1 < \ldots < i_k \le i_n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\frac{\Delta^{-1}\left(s^R_{\phi_{i_j}}, \varphi^R_{i_j}\right)}{t}\right)^2\right)^{nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}
\end{aligned}$$

$$+\left(1-\prod_{1\leq i_1<\ldots<i_k\leq i_n}\left(1-\prod_{j=1}^{k}\left(\frac{\Delta^{-1}\left(s^R_{\chi_{i_j}},\xi^R_{i_j}\right)}{t}\right)^{2nw_{i_j}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\leq 1$$

Example 6. *Let* $\widetilde{p}_1 = \langle[(s_2,0.3),(s_4,0.2)],[(s_1,0.4),(s_2,-0.3)]\rangle$, $\widetilde{p}_2 = \langle[(s_1,0.4),(s_3,0.3)],[(s_2,-0.4),(s_2,0.1)]\rangle$ *and* $\widetilde{p}_3 = \langle[(s_3,0.2),(s_4,-0.4)],[(s_1,0.1),(s_2,-0.2)]\rangle$ *be three IV2TLPFNs, and suppose* $k = 2$, $\mathrm{w} = (0.2,0.3,0.5)$ *then according to (83), we have*

$$\mathrm{IV2TLPFWDMSM}^{(2)}_{nw}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\frac{1}{k}\left(\underset{1\leq i_1<\ldots<i_k\leq i_n}{\otimes}\left(\overset{k}{\underset{j=1}{\oplus}}\left(nw_{i_j}\otimes\widetilde{p}_{i_j}\right)\right)^{\frac{1}{C_n^k}}\right)$$

$$=\left\{\begin{array}{l}\left[\begin{array}{l}\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-(\frac{2.3}{6})^2\right)^{0.6}\times\left(1-\left(\frac{1.4}{6}\right)^2\right)^{0.9}\right)\times\left(1-\left(1-(\frac{2.3}{6})^2\right)^{0.6}\times\left(1-(\frac{3.2}{6})^2\right)^{1.2}\right)\\ \times\left(1-\left(1-\left(\frac{1.4}{6}\right)^2\right)^{0.9}\times\left(1-(\frac{3.2}{6})^2\right)^{1.2}\right)\end{array}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right),\\ \Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)^{0.6}\times\left(1-(\frac{3.3}{6})^2\right)^{0.9}\right)\times\left(1-\left(1-\left(\frac{4.2}{6}\right)^2\right)^{0.6}\times\left(1-(\frac{3.6}{6})^2\right)^{1.2}\right)\\ \times\left(1-\left(1-(\frac{3.3}{6})^2\right)^{0.9}\times\left(1-(\frac{3.6}{6})^2\right)^{1.2}\right)\end{array}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\right)\end{array}\right],\\ \left[\begin{array}{l}\Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{1.4}{6}\right)^{1.2}\times\left(\frac{1.6}{6}\right)^{1.8}\right)\times\left(1-\left(\frac{1.4}{6}\right)^{1.2}\times\left(\frac{1.1}{6}\right)^{3}\right)\times\left(1-\left(\frac{1.6}{6}\right)^{1.8}\times\left(\frac{1.1}{6}\right)^{3}\right)\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right),\\ \Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{1.7}{6}\right)^{1.2}\times\left(\frac{2.1}{6}\right)^{1.8}\right)\times\left(1-\left(\frac{1.7}{6}\right)^{1.2}\times\left(\frac{1.8}{6}\right)^{3}\right)\times\left(1-\left(\frac{2.1}{6}\right)^{1.8}\times\left(\frac{1.8}{6}\right)^{3}\right)\right)^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right)\end{array}\right]\end{array}\right\}$$

$$=\{[(s_2,0.2817),(s_4,-0.4935)],[(s_2,-0.3329),(s_2,0.0829)]\}$$

Then we will discuss some properties of IV2TLPFWDMSM operator.

Property 21. *(Idempotency) If* $\widetilde{p}_j = \left\langle\left[\left(s^L_{\phi_j},\varphi^L_j\right),\left(s^R_{\phi_j},\varphi^R_j\right)\right],\left[\left(s^L_{\chi_j},\xi^L_j\right),\left(s^R_{\chi_j},\xi^R_j\right)\right]\right\rangle (j = 1,2,\ldots,n)$ *are equal, then*

$$\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_n)=\widetilde{p} \tag{93}$$

Property 22. *(Monotonicity) Let* $\widetilde{p}_{x_i} = \left\langle\left[\left(s^L_{\phi_{x_i}},\varphi^L_{x_i}\right),\left(s^R_{\phi_{x_i}},\varphi^R_{x_i}\right)\right],\left[\left(s^L_{\chi_{x_i}},\xi^L_{x_i}\right),\left(s^R_{\chi_{x_i}},\xi^R_{x_i}\right)\right]\right\rangle$ $(i = 1,2,\ldots,n)$ *and* $\widetilde{p}_{y_i} = \left\langle\left[\left(s^L_{\phi_{y_i}},\varphi^L_{y_i}\right),\left(s^R_{\phi_{y_i}},\varphi^R_{y_i}\right)\right],\left[\left(s^L_{\chi_{y_i}},\xi^L_{y_i}\right),\left(s^R_{\chi_{y_i}},\xi^R_{y_i}\right)\right]\right\rangle (i = 1,2,\ldots,n)$ *be two sets of IV2TLPFNs. If* $\Delta^{-1}\left(s^L_{\phi_{x_i}},\varphi^L_{x_i}\right)\leq\Delta^{-1}\left(s^L_{\phi_{y_i}},\varphi^L_{y_i}\right)$, $\Delta^{-1}\left(s^R_{\phi_{x_i}},\varphi^R_{x_i}\right)\geq\Delta^{-1}\left(s^R_{\phi_{y_i}},\varphi^R_{y_i}\right)$, $\Delta^{-1}\left(s^L_{\chi_{x_i}},\xi^L_{x_i}\right)\geq\Delta^{-1}\left(s^L_{\chi_{y_i}},\xi^L_{y_i}\right)$ *and* $\Delta^{-1}\left(s^R_{\chi_{x_i}},\xi^R_{x_i}\right)\geq\Delta^{-1}\left(s^R_{\chi_{y_i}},\xi^R_{y_i}\right)$ *hold for all i, then*

$$\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_{x_1},\widetilde{p}_{x_2},\cdots,\widetilde{p}_{x_n})\leq\mathrm{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_{y_1},\widetilde{p}_{y_2},\cdots,\widetilde{p}_{y_n}) \tag{94}$$

Property 23. *(Boundedness) Let* $\widetilde{p}_j = \left\{\left[\left(s^L_{\phi_j}, \varphi^L_j\right), \left(s^R_{\phi_j}, \varphi^R_j\right)\right], \left[\left(s^L_{\chi_j}, \xi^L_j\right), \left(s^R_{\chi_j}, \xi^R_j\right)\right]\right\} (j = 1, 2, \ldots, n)$ *be a set of 2TLPFNs. If* $\widetilde{p}^+ = \left\{\begin{array}{l}\left[\max\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \max\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\min\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \min\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right]\end{array}\right\} (j = 1, 2, \ldots, n)$ *and* $\widetilde{p}^- = \left\{\begin{array}{l}\left[\min\Delta^{-1}\left(s^L_{\phi_j}, \varphi^L_j\right), \min\Delta^{-1}\left(s^R_{\phi_j}, \varphi^R_j\right)\right], \\ \left[\max\Delta^{-1}\left(s^L_{\chi_j}, \xi^L_j\right), \max\Delta^{-1}\left(s^R_{\chi_j}, \xi^R_j\right)\right]\end{array}\right\} (j = 1, 2, \ldots, n)$ *then*

$$\widetilde{p}^- \leq \text{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) \leq \widetilde{p}^+ \tag{95}$$

Property 24. *(Commutativity) If Let* $\widetilde{p}_j (j = 1, 2, \cdots, n)$ *be a collection of IV2TLPFNs, and* $\widetilde{p}'_j (j = 1, 2, \cdots, n)$ *is any permutation of* $\widetilde{p}_j (j = 1, 2, \cdots, n)$*, then*

$$\text{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) = \text{IV2TLPFWDMSM}^{(k)}_{nw}(\widetilde{p}'_1, \widetilde{p}'_2, \cdots, \widetilde{p}'_n) \tag{96}$$

6. Numerical Example and Comparative Analysis

6.1. Numerical Example

With the rapid development of economic globalization and the growing enterprise competition environment, the competition between modern enterprises has become the competition between supply chain and supply chain. The diversity of the people consumption concept, shorter new product life cycles, the volatility of demand market, and other external factors drive enterprises for effective supply chain integration and management, and strategic alliance with other enterprises in order to enhance core competitiveness and resist external risk. The key measure to achieving this goal is supplier selection. Therefore, the supplier selection problem has gained great attention in supply chain management theory and in actual production management problems. In practical supplier selection problems, there are a large number of uncertainties, such as the fuzziness of producers and consumers demand, fuzziness and dynamic of supplier supply capacity, risk from natural and man-made factors, visibility of the whole supply chain, etc. These uncertain factors have a great influence on actual assessment and selection. In order to properly deal with those uncertainty factors, thus, in this section we give an example to select green suppliers with IV2TLPFNs in order to show the method proposed. There are five possible green suppliers in the green supply chain management $A_i (i = 1, 2, 3, 4, 5)$ to select. The experts select four attributes to evaluate these green suppliers: ① G_1 is the price factor; ② G_2 is the delivery factor; ③ G_3 is the environmental factors; ④ G_4 is the product quality factor. The five possible green suppliers $A_i (i = 1, 2, 3, 4, 5)$ are to be assessed with 2TLPFNs by the decision maker according to these four attributes (whose weighting vector $\omega = (0.14, 0.37, 0.29, 0.20)$), the IV2TLPFNs decision matrix as listed in Table 1.

Step 1. According to Table 1, we can aggregate all IV2TLPFNs r_{ij} by using the IV2TLPFWMSM, IV2TLPFWGMSM (Suppose that $P = (1, 2, 3, 4)$), and IV2TLPFWDMSM operator to get the overall IV2TLPFNs $A_i (i = 1, 2, 3, 4, 5)$ of the green suppliers A_i, then the aggregating results are shown in Table 2. $(k = 2)$.

Step 2. According to the aggregating results listed in Table 2 and the score functions of the green suppliers are listed in Table 3.

Table 1. The interval-valued 2-tuple linguistic Pythagorean fuzzy number decision matrix.

	G_1	G_2
A_1	$\{[(s_2,0),(s_3,0)],[(s_1,0),(s_2,0)]\}$	$\{[(s_2,0),(s_4,0)],[(s_2,0),(s_3,0)]\}$
A_2	$\{[(s_1,0),(s_2,0)],[(s_2,0),(s_3,0)]\}$	$\{[(s_2,0),(s_3,0)],[(s_1,0),(s_2,0)]\}$
A_3	$\{[(s_3,0),(s_4,0)],[(s_1,0),(s_2,0)]\}$	$\{[(s_3,0),(s_5,0)],[(s_1,0),(s_2,0)]\}$
A_4	$\{[(s_2,0),(s_3,0)],[(s_1,0),(s_2,0)]\}$	$\{[(s_2,0),(s_3,0)],[(s_2,0),(s_3,0)]\}$
A_5	$\{[(s_1,0),(s_2,0)],[(s_2,0),(s_3,0)]\}$	$\{[(s_2,0),(s_4,0)],[(s_2,0),(s_3,0)]\}$
	G_3	G_4
A_1	$\{[(s_1,0),(s_2,0)],[(s_1,0),(s_3,0)]\}$	$\{[(s_2,0),(s_3,0)],[(s_1,0),(s_2,0)]\}$
A_2	$\{[(s_2,0),(s_3,0)],[(s_1,0),(s_4,0)]\}$	$\{[(s_1,0),(s_2,0)],[(s_2,0),(s_3,0)]\}$
A_3	$\{[(s_3,0),(s_4,0)],[(s_2,0),(s_4,0)]\}$	$\{[(s_1,0),(s_2,0)],[(s_2,0),(s_3,0)]\}$
A_4	$\{[(s_1,0),(s_2,0)],[(s_1,0),(s_3,0)]\}$	$\{[(s_1,0),(s_2,0)],[(s_2,0),(s_3,0)]\}$
A_5	$\{[(s_1,0),(s_2,0)],[(s_1,0),(s_4,0)]\}$	$\{[(s_1,0),(s_2,0)],[(s_1,0),(s_3,0)]\}$

Table 2. The operation results by the IV2TLPFWMSM, IV2TLPFWGMSM, and IV2TLPFWDMSM operators.

Aggregation Operator	**Green Supplier**	**Operation Results**
IV2TLPFWMSM	A_1	$\{[(s_1,0.4990),(s_3,-0.3639)],[(s_2,0.0079),(s_3,0.2511)]\}$
	A_2	$\{[(s_1,0.4130),(s_2,0.2682)],[(s_2,0.3463),(s_4,-0.2297)]\}$
	A_3	$\{[(s_2,0.2948),(s_3,0.4641)],[(s_2,0.3800),(s_4,-0.4372)]\}$
	A_4	$\{[(s_1,0.3026),(s_2,0.1673)],[(s_2,0.0079),(s_3,0.4758)]\}$
	A_5	$\{[(s_1,0.1226),(s_2,0.2467)],[(s_2,0.2644),(s_4,-0.0822)]\}$
IV2TLPFWGMSM	A_1	$\{[(s_4,0.2143),(s_5,-0.2477)],[(s_1,0.0561),(s_2,-0.1623)]\}$
	A_2	$\{[(s_4,0.1090),(s_5,-0.4737)],[(s_1,0.1838),(s_2,0.0989)]\}$
	A_3	$\{[(s_5,-0.4698),(s_5,0.1698)],[(s_1,0.2394),(s_2,0.-0307)]\}$
	A_4	$\{[(s_4,0.0031),(s_5,0.-4490)],[(s_1,0.0561),(s_2,-0.0399)]\}$
	A_5	$\{[(s_4,-0.1988),(s_5,-0.3714)],[(s_1,0.1872),(s_2,0.2531)]\}$
IV2TLPFWDMSM	A_1	$\{[(s_2,-0.4993),(s_3,-0.3427)],[(s_2,0.0023),(s_3,0.2460)]\}$
	A_2	$\{[(s_1,0.3906),(s_2,0.2675)],[(s_2,0.3993),(s_4,-0.2651)]\}$
	A_3	$\{[(s_2,0.3027),(s_3,0.4920)],[(s_2,0.3663),(s_4,-0.4670)]\}$
	A_4	$\{[(s_1,0.3020),(s_2,0.1752)],[(s_2,0.0023),(s_3,0.4677)]\}$
	A_5	$\{[(s_1,0.1240),(s_2,0.2680)],[(s_2,0.2492),(s_4,-0.0989)]\}$

Table 3. The score functions of the green suppliers.

	IV2TLPFWMSM	**IV2TLPFWGMSM**	**IV2TLPFWDMSM**
A_1	0.6800	0.8693	0.6809
A_2	0.6424	0.8409	0.6421
A_3	0.7017	0.8891	0.7042
A_4	0.6577	0.8489	0.6582
A_5	0.6337	0.8391	0.6354

Step 3. According to the score functions shown in Table 3 and the comparison formula of score functions, the ordering of the green suppliers is shown in Table 4. Note that ">" means "preferred to". As we can see, depending on the aggregation operators used, the best green supplier is A_3.

Table 4. Ordering of the green suppliers.

Methods	**Ordering**
IV2TLPFWMSM	$A_3>A_1>A_4>A_2>A_5$
IV2TLPFWGMSM	$A_3>A_1>A_4>A_2>A_5$
IV2TLPFWDMSM	$A_3>A_1>A_4>A_2>A_5$

6.2. Influence of the Parameter on the Final Result

In order to show the effects on the ranking results by changing parameters of k in the IV2TLPFWMSM, IV2TLPFWGMSM, and IV2TLPFWDMSM operators, all the results are shown in Tables 5–7.

Table 5. Ranking results for different operational parameters of the IV2TLPFWMSM operator.

	$\mathbf{s(A_1)}$	$\mathbf{s(A_2)}$	$\mathbf{s(A_3)}$	$\mathbf{s(A_4)}$	$\mathbf{s(A_5)}$	**Ordering**
K = 1	0.6909	0.6717	0.7366	0.6655	0.6567	A3>A1>A2>A4>A5
K = 2	0.6800	0.6424	0.7017	0.6577	0.6337	A3>A1>A4>A2>A5
K = 3	0.6755	0.6286	0.6884	0.6539	0.6243	A3>A1>A4>A2>A5
K = 4	0.6724	0.6210	0.6748	0.6514	0.6176	A3>A1>A4>A2>A5

Table 6. Ranking results for different operational parameters of the IV2TLPFWGMSM operator.

	$\mathbf{s(A_1)}$	$\mathbf{s(A_2)}$	$\mathbf{s(A_3)}$	$\mathbf{s(A_4)}$	$\mathbf{s(A_5)}$	**Ordering**
K = 1	0.4948	0.4729	0.5184	0.4723	0.4537	A3>A1>A2>A4>A5
K = 2	0.8693	0.8409	0.8891	0.8489	0.8391	A3>A1>A4>A2>A5
K = 3	0.8140	0.7821	0.8383	0.7914	0.7766	A3>A1>A4>A2>A5
K = 4	0.7573	0.7337	0.7906	0.7391	0.7205	A3>A1>A4>A2>A5

Table 7. Ranking results for different operational parameters of the IV2TLPFWDMSM operator.

	$\mathbf{s(A_1)}$	$\mathbf{s(A_2)}$	$\mathbf{s(A_3)}$	$\mathbf{s(A_4)}$	$\mathbf{s(A_5)}$	**Ordering**
K = 1	0.6724	0.6210	0.6748	0.6514	0.6176	A3>A1>A4>A2>A5
K = 2	0.6809	0.6421	0.7042	0.6582	0.6354	A3>A1>A4>A2>A5
K = 3	0.6863	0.6580	0.7195	0.6622	0.6465	A3>A1>A4>A2>A5
K = 4	0.6909	0.6717	0.7366	0.6655	0.6567	A3>A1>A2>A4>A5

6.3. Comparative Analysis

Qin & Liu [44] developed the uncertain linguistic dual Maclaurin symmetric mean (ULDMSM) operator, the uncertain linguistic weighted dual Maclaurin symmetric mean (ULWDMSM) operator and the uncertain linguistic Choquet dual Maclaurin symmetric mean (ULCDMSM) operator, but these operators can only deal with uncertain linguistic membership. However, the IV2TLPFWMSM, IV2TLPFWGMSM, and IV2TLPFWDMSM operators which are proposed in this paper can deal with both uncertain linguistic membership and uncertain linguistic non-membership.

Garg [14] proposed the IVPFWA operator and IVPFWG operator to fuse interval-valued Pythagorean fuzzy numbers (IVPFNs), but these operators for IVPFNs can only deal with both membership and non-membership of non-negative interval-valued numbers and cannot deal with both membership and non-membership of interval-valued 2-tuple linguistic information. However, the IV2TLPFWMSM, IV2TLPFWGMSM, and IV2TLPFWDMSM operators which are proposed in this paper can deal with both membership and non-membership of interval-valued 2-tuple linguistic information.

7. Conclusions

In this paper, we propose a new class of fuzzy sets (FSs), which can be seen as an extension of PFSs, named IV2TLPFSs. They can satisfactorily reflect imprecise, incomplete, and inconsistent information in order to address decision-making situations that involve qualitative information rather than numerical information. At the same time, in recent years, a number of researchers have developed aggregation operators under various fuzzy environments and applied these to solve different decision-making problems. However, these aggregation operators have some drawbacks in actual applications, such as being unable to reflect the correlation of all input arguments.

The MSM operator has an apparent advantage in that it can deal with the relationships between all the input arguments according to the parameter vector. Motivated by the ideal characteristic of the MSM operator, we utilize the MSM operator, GMSM operator, and DMSM operator to develop some MSM aggregation operators with IV2TLPFNs: Interval-valued 2-tuple linguistic Pythagorean fuzzy MSM (IV2TLPFMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy weighted MSM (IV2TLPFWMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy GMSM (IV2TLPFGMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy weighted GMSM (IV2TLPFWGMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy DMSM (IV2TLPFDMSM) operator, interval-valued 2-tuple linguistic Pythagorean fuzzy weighted DMSM (IV2TLPFWDMSM) operator. The characteristics of these operators are also investigated. Then, we have used these operators to solve the MADM problems with IV2TLPFNs. At the same time, some comparative analysis is given. Finally, a practical example for green supplier selection is used to show the developed approach. In the future, the application of the proposed operators of IV2TLPFNs needs to be explored in the other uncertain [45–66] and fuzzy MADM [67–84].

Author Contributions: J.W., G.W.W. and H.G. conceived and worked together to achieve this work; J.W. compiled the computing program by Matlab and analyzed the data; J.W. and G.W. Wei wrote the paper. Finally, all the authors have read and approved the final manuscript.

Funding: The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (17XJA630003) and the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province (15TD0004).

Conflicts of Interest: The authors declare no conflict of interest.

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