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Some Results on the Deficiencies of Some Differential-Difference Polynomials of Meromorphic Function

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Abstract: For a transcendental meromorphic function f(z), the main aim of this paper is to investigate the properties on the zeros and deficiencies of some differential-difference polynomials. Some results about the deficiencies of some differential-difference polynomials concerning Nevanlinna deficiency and Valiron deficiency are obtained, which are a generalization of and improvement on previous theorems given by Liu, Lan and Zheng, etc.

Keywords: differential-difference polynomial; nevanlinna deficiency; valiron deficiency

MSC: 30D35, 39A10

1. Introduction and Main Results

In 1926, R. Nevanlinna established the famous Nevanlinna theory, which is an important tool in studying the value distribution of meromorphic functions in Complex Analysis. After several decades or even hundreds of years of development, a lot of interesting and important results exist on the value distribution of meromorphic functions (see Hayman [1], Gol'dberg-Ostrovskii [2], Yang [3] and Yi-Yang [4]). This article is devoted to the study of value distribution of some differential-difference polynomials of meromorphic function concerning the Nevanlinna and Valiron exceptional values. In addition, for meromorphic function *f*, we use *S*(*r*, *f*) to denote any quantity satisfying *S*(*r*, *f*) = o(T(r, f)) for all *r* outside a possible exceptional set *E* of finite logarithmic measure $\lim_{r\to\infty} \int_{[1,r)\cap E} \frac{dt}{t} < \infty$. Additionally, from Reference [3], the order $\rho(f)$ of *f* is defined as:

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r},$$

then f(z) is called a meromorphic function of finite order if $\rho(f) < +\infty$, and f(z) is called a meromorphic function of zero order if $\rho(f) = 0$.

As we all know, there were a number of important results focusing on the deficiencies of meromorphic functions in the study of value distributions (see [2,3]). In order to estimate the value distribution of meromorphic functions more accurately, many people often use the concept of deficiency, including Nevanlinna deficiency, Valiron deficiency, and so on (see [1,2]). For $a \in \mathbb{C} \cup \{\infty\}$, the following notations can be found in References [1,2]:

$$\delta(a,f) = \liminf_{r \to +\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r \to +\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

$$\Delta(a,f) = \limsup_{r \to +\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \liminf_{r \to +\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)}$$

where $m(r, f) = m(r, \frac{1}{f-a})$ and $N(r, f) = N(r, \frac{1}{f-a})$ if $a = +\infty$. Further, $\delta(a, f)$ is called the Nevanlinna deficiency of function f(z) at point a, and the quantity $\Delta(a, f)$ is called the Valiron deficiency. It is clear from Nevanlinna's first fundamental theorem that:

$$0 \le \delta(a, f) \le \Delta(a, f) \le 1.$$

Recently, many scholars have exhibited an increasing interest in studying some properties of meromorphic function and its difference by relying on the Nevanlinna theory, and they have produced a number of papers focusing on the value distribution and uniqueness of complex difference, differential-difference operators, differential-difference equations, and so on (see [5–11]).

However, there are seldom any papers concerning the Nevanlinna deficiency and Valiron deficiency of differential-difference polynomials. Thus, our purpose in this paper is to investigate the relation of the Nevanlinna deficiency and the Valiron deficiency among meromorphic functions, their derivations, and their differential-difference polynomials.

For a nonconstant meromorphic function f(z), let:

$$F_1(z) = f(z+c)^s [f'(z)]^k, \ F_2(z) = f(z)^n f(z+c)^s [f'(z)]^k,$$

where $c \in \mathbb{C}$ and s, n, k are positive integers. We first give one theorem below, which shows some relations on the Valiron and Nevanlinna deficiencies of some differential-difference monomials. It seems this problem has never been treated before.

Theorem 1. Let f(z) be a transcendental meromorphic function, such that $T(r, f) = O((\log r)^2)$ and $\delta(1 + \frac{1}{\Lambda}) \leq 1$, where $\delta := \delta(\infty, f) \in (0, 1)$ and $\Delta := \Delta(\infty, f)$. If s, k, δ, Δ satisfy one of the following cases:

1. k > 2s and

$$\Psi_1(s,k,\Delta,\delta) := (s+2k)\delta^2 - (3s+4k+\frac{k}{2\Delta})\delta + 2s+2k < 0;$$
(1)

2. s > 2k and $\delta > \frac{4k}{s+2k}$.

Then, $\delta(\infty, F_1) > 0$.

Theorem 2. Let f(z) be a transcendental meromorphic function, such that $T(r, f) = O((\log r)^2)$, $\delta(1 + \frac{1}{\Delta}) \le 1$ and $0 < \delta < 1$. If s, k, δ, Δ , and n satisfy one of the following cases:

1. k > 2(s - n) and $s \ge n$, and

$$\Psi_{2}(n, s, k, \Delta, \delta) := (n + s + 2k)\delta^{2} - (n + 3s + 4k + \frac{k}{2\Delta})\delta + 2s + 2k < 0;$$

2. n > s + 2k and $\delta > \frac{2s+4k}{n+s+2k}$; 3. s > n+2k and $\delta > \frac{2n+4k}{n+s+2k}$.

Then, $\delta(\infty, F_2) > 0$.

Remark 1. A meromorphic function is considered transcendental if it is not rational. From Reference [2], we have that f(z) is a transcendental meromorphic function if and only if f(z) satisfies:

$$\liminf_{r \to +\infty} \frac{T(r, f)}{\log r} = \infty.$$

In 1959, Hayman [12] discussed the Picard values of meromorphic functions concerning their derivatives, and got the following theorem:

Theorem 3 (see [12]). If f(z) is a transcendental entire function, $n \ge 3$ is an integer, and $a \ne 0$ is a constant, then $f'(z) - af(z)^n$ assumes all finite values infinitely often.

In 1970s, Yang [13,14] further investigated this problem and extended the results to some differential polynomial in f(z), when f(z) is a transcendental meromorphic function satisfying $N(r, f) + N(r, \frac{1}{f}) = S(r, f)$.

Recently, relying on some establishments of difference analogues of the classic Nevanlinna theory (including [5,6,15]), many mathematicians paid considerable attention to studying some properties of the value distribution of complex difference expressions of meromorphic functions (see [16–23]).

In 2013, Zheng-Chen [24] obtained a difference counterpart of Theorem A as follows:

Theorem 4 (see [24]). Let f(z) be a transcendental entire function of finite order, and let a, c be nonzero constants. Then, for any integer $n \ge 3$, $f(z + c) - af(z)^n$ assumes all finite values $b \in \mathbb{C}$ infinitely often.

A differential-difference polynomial is a polynomial in f(z), its shifts, its derivatives, and derivatives of its shifts (see [23]), that is, an expression of the form:

$$P(z,f) = \sum_{\lambda \in I} a_{\lambda}(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \cdots f^{(m)}(z)^{\lambda_{0,m}} \times f(z+c_1)^{\lambda_{1,0}} f'(z+c_1)^{\lambda_{1,1}} \cdots f^{(m)}(z+c_1)^{\lambda_{1,m}} \cdots f(z+c_n)^{\lambda_{n,0}} f'(z+c_n)^{\lambda_{n,1}} \cdots f^{(m)}(z+c_n)^{\lambda_{n,m}} = \sum_{\lambda \in I} a_{\lambda}(z) \prod_{i=0}^{n} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}},$$
(2)

where *I* is a finite set of multi-indices $\lambda = (\lambda_{0,0}, ..., \lambda_{0,m}, \lambda_{1,0}, ..., \lambda_{1,m}, ..., \lambda_{n,0}, ..., \lambda_{n,m})$, and $c_0(=0)$ and $c_1, ..., c_n$ are distinct complex constants. Additionally, let the meromorphic coefficients $a_{\lambda}(z), \lambda \in I$ of P(z, f) be of growth S(r, f). We denote the degree of P(z, f) by:

$$d(P) = \max_{\lambda \in I} \{ d(\lambda) \},$$

where $d(\lambda) = \sum_{i=0}^{n} \sum_{j=0}^{m} \lambda_{i,j}$ is the degree of the monomial $\prod_{i=0}^{n} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}$ of P(z, f). Further, set:

$$\Gamma(\lambda) = \sum_{i=0}^{n} \sum_{j=0}^{m} j \lambda_{i,j}$$

and:

$$\Gamma(P) = \max_{\lambda \in I} \{ \Gamma(\lambda) \}.$$

In 2014, Zheng-Xu [25] further considered the value distribution of some complex differential-difference polynomials by combining complex differentiates and complex differences and extended and improved the previously results given by Chen, Liu, etc. (see [9,26]).

Theorem 5 (see [25]). Suppose that f(z) is a transcendental meromorphic function satisfying $\rho_2(f) < 1$ and:

$$N(r,f) + N(r,\frac{1}{f}) = S(r,f),$$
 (3)

and P(z, f) is a differential-difference polynomial of the form (2). If $v \ge d(P) + 2 \ge 3$, then:

$$Q(z,f) = f^v + P(z,f) \tag{4}$$

satisfies $\delta(\alpha, Q(z, f)) < 1$, where $\alpha \neq 0$ is a meromorphic function of growth S(r, f) and:

$$\rho_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r}$$

Therefore, $Q(z, f) - \alpha(z)$ *has infinitely many zeros.*

In this paper, we will relax (3) by considering f(z) with deficient value ∞ . In fact, we have obtained a series of improvements of the previous results as follow:

Theorem 6. Let f(z) be a transcendental meromorphic function of finite order, and let $P(z, f) \neq 0$ be a differential-difference polynomial of the form (2), with n different shifts. Assume that $\delta(\infty, f) > 1 - \frac{1}{2n+7}$ and $v \geq d(P) + 2$. Then, the differential-difference polynomial Q(z, f) has infinitely many zeros and satisfies $\delta(0, Q(z, f)) < 1$.

Theorem 7. Let f(z) be a transcendental meromorphic function of finite order satisfying $\overline{N}(r, \frac{1}{f}) = S(r, f)$, and let $P(z, f) (\neq 0)$ be a differential-difference polynomial of the form (2), with n different shifts. Assume that $\delta(\infty, f) > 1 - \frac{1}{2n+5}$ and $v \ge d(P) + 1$. Then, the differential-difference polynomial Q(z, f) has infinitely many zeros and satisfies $\delta(0, Q(z, f)) < 1$.

Theorem 8. Let f(z) be a transcendental meromorphic function of finite order satisfying $\overline{N}(r, f) = S(r, f)$, and let $P(z, f) (\neq 0)$ be a differential-difference polynomial of the form (2), with n different shifts. Assume that $\delta(\infty, f) > \frac{2}{3}$ and $v \ge d(P) + 2$. Then, the differential-difference polynomial Q(z, f) has infinitely many zeros and satisfies $\delta(0, Q(z, f)) < 1$.

Theorem 9. Let f(z) be a transcendental meromorphic function of finite order satisfying $\delta(\infty, f) > 0$, and:

$$\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f),$$
(5)

and let $P(z, f) (\neq 0)$ be a differential-difference polynomial of the form (2), with n different shifts. Then, the differential-difference polynomial Q(z, f) has infinitely many zeros and satisfies $\delta(0, Q(z, f)) < 1$.

Remark 2. Let $Q_1(z, f) = Q(z, f) - b$, for $b \in \mathbb{C}$. By applying Theorems 6–9 to $Q_1(z, f)$, we then conclude that $Q_1(z, f)$ has infinitely many zeros and $\delta(b, Q_1(z, f)) < 1$.

Remark 3. Let $Q_2(z, f) = \alpha(z)f^v + P(z, f)$ and $Q_3(z, f) = \frac{Q_2(z, f)}{\alpha(z)}$, where $\alpha(z)$ is stated as in Theorem 1.5. By applying Theorems 6–9 to $Q_3(z, f)$, we then conclude that $Q_2(z, f)$ has infinitely many zeros and $\delta(0, Q_2(z, f)) < 1$.

2. Proofs of Theorems 1 and 2

To prove Theorems 1 and 2, we will require some lemmas as follow:

Lemma 1 (see [27], Valiron). Suppose that f is a meromorphic function, such that $T(r, f) = O((\log r)^2)$. Then, for any distinct $a, b \in \mathbb{C}$, we have:

$$T(r, f) = (1 + o(1)) \max\left\{ N(r, \frac{1}{f - a}), N(r, \frac{1}{f - b}) \right\}.$$

From [5] and ([2] p. 66), we immediately deduce:

Lemma 2. Let f(z) be a transcendental meromorphic function of finite order and $c \in \mathbb{C}$. Then:

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f), N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f),$$
$$N(r, f(z+c)) = N(r, f) + S(r, f), \overline{N}\left(r, \frac{1}{f(z+c)}\right) = \overline{N}\left(r, \frac{1}{f}\right) + S(r, f),$$
$$\overline{N}(r, f(z+c)) = \overline{N}(r, f) + S(r, f).$$

Lemma 3 (see [28] Hayman). Suppose that f is a meromorphic function, such that $T(r, f) = O((\log r)^2)$. Then:

$$T(r, f) \le (2 + o(1))T(r, f').$$

Lemma 4 (see [15] Theorem 2.1 or [6]). Let f(z) be a meromorphic function of finite order ρ and let c be a fixed non zero complex number. Then, for each $\varepsilon > 0$, we have:

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f).$$

Lemma 5 (see [4] p. 37). Let f(z) be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then:

$$T(r, f^{(l)}(z)) \le T(r, f) + l\overline{N}(r, f) + S(r, f), \ N(r, f^{(l)}(z)) = N(r, f) + l\overline{N}(r, f).$$

Lemma 6 (see [29] Toppila). *Let f be a transcendental meromorphic function, such that:*

$$\lim_{r \to +\infty} \frac{T(2r, f)}{T(r, f)} = 1$$

and $\delta = \delta(\infty, f) > 0$. Let $\Delta = \Delta(\infty, f)$. If $\delta(1 + \frac{1}{\Delta}) > 1$, then $\Delta(0, f') = 0$, and:

$$\delta(\infty, f') \geq rac{2\delta}{\Delta - \delta\Delta + \delta} - 1,$$

and if $\delta(1+\frac{1}{\Delta}) \leq 1$, then:

$$\Delta(0,f') \leq rac{1-\delta(1+rac{1}{\Delta})}{1-\delta}\limsup_{r
ightarrow+\infty}rac{N(r,f)}{N(r,f')}.$$

2.1. The Proof of Theorem 1

1. k > 2s. For any positive integers s, k, we first prove that $1 \ge \delta := \delta(\infty, f) > \eta > 0$ as $\Psi_1(s, k, \Delta, \delta) < 0$. From the condition of Theorem 1, we have $\Delta := \Delta(\infty, f) > 0$. Then, it follows from Equation (1) that:

$$(3s+4k+\frac{k}{2\Delta})^2 - 8(s+k)(s+2k) = s^2 + \frac{3sk+4k^2}{\Delta} + \frac{k^2}{4\Delta^2} > 0.$$

Therefore, we get that the equation $\Psi_1(s, k, \Delta, \delta) = 0$ has two distinct zeros and:

$$\delta_{1,2} = \frac{3s + 4k + \frac{k}{2\Delta} \pm \sqrt{s^2 + \frac{3sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2}}}{2s + 4k}.$$

Obviously,

$$\delta_1 := \frac{3s + 4k + \frac{k}{2\Delta} + \sqrt{s^2 + \frac{3sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2}}}{2s + 4k} > 1, \tag{6}$$

and since:

$$(s + \frac{k}{2\Delta})^2 = s^2 + \frac{sk}{\Delta} + \frac{k^2}{4\Delta^2} = s^2 + \frac{3sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2} - \frac{2sk}{\Delta} - \frac{4k^2}{\Delta},$$

then it follows that:

$$s + 2s + 4k + \frac{k}{2\Delta} - 2s - 4k < \sqrt{s^2 + \frac{3sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2}}$$

Thus, it yields:

$$\delta_2 := \frac{s + 2s + 4k + \frac{k}{2\Delta} - \sqrt{s^2 + \frac{sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2}}}{2s + 4k} < \frac{s + 2k}{s + 2k} = 1.$$
(7)

Hence, we have:

$$0 < \delta_2 := \frac{3s + 4k + \frac{k}{2\Delta} - \sqrt{s^2 + \frac{3sk + 4k^2}{\Delta} + \frac{k^2}{4\Delta^2}}}{2s + 4k} < 1$$

Thus, $\Psi_1(s, k, \Delta, \delta) < 0$ implies $\delta_2 < \delta < \delta_1$, and, by combining $0 < \delta \le 1$, it follows $\delta_2 < \delta \le 1$ as $\Psi_1(s, k, \Delta, \delta) < 0$.

Next, we will prove that $\delta > \eta > 0$ as $\Psi_1(s, k, \Delta, \delta) < 0$, since $\delta(1 + \frac{1}{\Delta}) \le 1$, that is, $\Delta \ge \frac{\delta}{1-\delta}$. Moreover, we see that $\Psi_1(s, k, \Delta, \delta)$ is an increasing function of Δ for $\delta > 0$. Therefore, the set of $\delta > 0$ as $\Psi_1(s, k, \Delta, \delta) < 0$ is contained in the corresponding set with the choice $\Delta = \frac{\delta}{1-\delta}$. Then, $\Psi_1(s, k, \Delta, \delta)$ is:

$$\Psi(s,k,\delta) = (s+2k)\delta^2 - (3s+4k + \frac{k(1-\delta)}{2\delta})\delta + 2s + 2k = (s+2k)\delta^2 - (3s+\frac{7k}{2})\delta + 2s + \frac{3k}{2}.$$

Thus, the equation $\Psi(s, k, \delta) = 0$ has two distinct roots, and one root between 0 and 1 is:

$$\eta = \frac{(3s + \frac{7k}{2}) - \sqrt{(3s + \frac{7k}{2})^2 - 4(2s + \frac{3k}{2})(s + 2k)}}{2s + 4k} = \frac{4s + 3k}{2s + 4k}.$$

Since k > 2s, then $0 < \eta < 1$. Hence, it follows that:

$$\delta > \eta := \frac{4s + 3k}{2s + 4k} > 0. \tag{8}$$

Thus, in view of Equations (6)–(8), we have $0 < \eta < \delta := \delta(\infty, f) \le 1$ as $\Psi_1(s, k, \Delta, \delta) < 0$. Hence, in view of the definition of $\delta(\infty, f)$, for any $\varepsilon \in (0, \eta)$, it follows that:

$$N(r,f) < (1 - \eta + \varepsilon)T(r,f) < T(r,f).$$
(9)

Since $T(r, f) = O((\log r)^2)$, and let a = 0 and $b = \infty$ in Lemma 1, then it follows from Equation (9) that:

$$T(r,f) = (1+o(1))N(r,\frac{1}{f}).$$
(10)

Thus, in view of Lemma 2 and from Equations (9) and (10), it yields:

$$T(r, f(z+c)) = (1+o(1))N(r, \frac{1}{f(z+c)}).$$
(11)

Hence, we can conclude from Equations (9)-(11) that:

$$N(r, \frac{1}{F_1}) \ge kN(r, \frac{1}{f'(z)}) - sN(r, f(z+c)) + S(r, f).$$
(12)

Thus, by Lemmas 3 and 4 and (12), for any small $\varepsilon > 0$ and $r \to +\infty$, it follows that:

$$T(r, F_{1}) \geq N(r, \frac{1}{F_{1}}) + O(1)$$

$$\geq k(1 - \Delta(0, f') - \varepsilon)T(r, f'(z)) - sT(r, f(z + c)) + S(r, f)$$

$$\geq \left[\frac{k}{2}(1 - \Delta(0, f') - \varepsilon + o(1)) - s\right]T(r, f).$$
(13)

In addition, we can conclude from Lemmas 2 and 5 that for the above ε :

$$N(r, F_1) \le N(r, f(z+c)^s) + kN(r, f'(z)) \le (s+2k)N(r, f) + S(r, f) \le (s+2k)(1-\delta+\varepsilon)T(r, f) + S(r, f).$$
(14)

Thus, it follows from Equations (13) and (14) that:

$$\frac{N(r, F_1)}{T(r, F_1)} \le \frac{(s+2k)(1-\delta+\varepsilon)}{\frac{k}{2}(1-\Delta(0, f')-\varepsilon)-s}(1+o(1)).$$
(15)

Since $T(r, f) = O((\log r)^2)$, let $T(r, f) = K(\log r)^2$, then:

$$\lim_{r \to +\infty} \frac{T(2r, f)}{T(r, f)} = \lim_{r \to +\infty} \frac{K(\log 2r)^2}{K(\log r)^2} = 1,$$

where *K* is a constant. Thus, by Lemmas 5 and 6, we have:

$$\Delta(0, f') \le \frac{1 - \delta(1 + \frac{1}{\Delta})}{1 - \delta}.$$
(16)

In view of $\Psi_1(s, k, \Delta, \delta) < 0$, and combining Equations (15) and (16), we have:

$$\frac{N(r,F_1)}{T(r,F_1)} \leq \frac{(s+2k)(1-\delta+\varepsilon)}{\frac{k}{2}(1-\Delta(0,f')-\varepsilon)-s}(1+o(1)) \leq \frac{(s+2k)(1-\delta+\varepsilon)}{\frac{k\delta}{2\Delta(1-\delta)}-\varepsilon-s}(1+o(1)).$$

Let $\varepsilon \to 0$ and $r \to +\infty$, we can conclude that F_1 has deficient poles, that is, $\delta(\infty, F_1) > 0$. 2. s > 2k. In view of Lemmas 2 and 5, we have:

$$T(r,F_1) \ge sT(r,f(z+c)) - kT(r,f'(z)) + S(r,f) \ge (s-2k+o(1))T(r,f).$$
(17)

Thus, from Equations (14) and (17), it follows that:

$$\frac{N(r, F_1)}{T(r, F_1)} \le \frac{(s+2k)(1-\delta+\varepsilon)}{s-2k}(1+o(1)).$$
(18)

Since s > 2k and $\delta > \frac{4k}{s+2k}$, then, in view of Equation (18), it is easy to prove that F_1 has deficient poles, that is, $\delta(\infty, F_1) > 0$. Therefore, this completes the proof of Theorem 1.

2.2. The Proof of Theorem 2

By using the same argument as in the proof of Theorem 1, and combining with (1), we can also easily prove $\delta(\infty, F_2) > 0$.

3. Proofs of Theorems 6-9

To prove Theorems 6–9, we will require some lemmas as follow:

Lemma 7 (see [4]). Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + \cdots + a_n f^n$, where a_0, a_1, \cdots, a_n are constants and $a_n \neq 0$. Then:

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 8 (see [23]). *Let* f *be a transcendental meromorphic solution of finite order* ρ *of a difference equation of the form:*

$$H(z,f)P(z,f) = Q(z,f),$$

where H(z, f), P(z, f), Q(z, f) are difference polynomials in f, such that the total degree of H(z, f) in f and its shifts is n, and the corresponding total degree of Q(z, f) is $\leq n$. If H(z, f) contains just one term of maximal total degree, then for any $\varepsilon > 0$,

$$m(r, P(z, f)) = S(r, f),$$

possibly outside of an exceptional set of finite logarithmic measures.

Yang-Laine in Reference [23] also pointed out that:

Remark 4. If, in the above lemma, $H(z, f) = f^n$, then a similar conclusion holds if P(z, f), Q(z, f) are differential-difference polynomials in f.

Lemma 9 (see [30]). Let f(z) be a transcendental meromorphic function with $\delta(\infty, f) > 0$, and let P(f) be an algebraic polynomial in f of the form $P(f) = a_n(z)f(z)^n + a_{n-1}(z)f(z)^{n-1} + \cdots + a_1(z)f(z) + a_0(z)$, where $a_n(z) \neq 0$, $a_i(z)$ satisfy $m(r, a_i) = S(r, f)$, $j = 0, \ldots, n$, then:

$$m(r, P(f)) \le nm(r, f) + S(r, f).$$

3.1. The Proof of Theorem 6

Let f(z) be a transcendental meromorphic function of finite order. We first prove that $Q(z, f) \neq A$, where *A* is a constant. Suppose that $Q(z, f) \equiv A$, then:

$$f(z)^v = -P(z, f) + A.$$
 (19)

In view of $v \ge d(P) + 2$, and by applying Lemma 8 to (19), we conclude that:

$$m(r,f) = S(r,f).$$
⁽²⁰⁾

Since $\delta := \delta(\infty, f) > 0$, then for any $\varepsilon_1 \in (0, \delta)$ and sufficiently large *r*, it follows that:

$$m(r,f) \ge (\delta - \varepsilon_1)T(r,f).$$
(21)

From Equations (20) and (21), it yields T(r, f) = S(r, f); this is in contradiction to the assumption of f(z) being transcendental. Hence, Q(z, f) is not constant.

In view of the expression of P(z, f), we can rewrite Q(z, f) as the following form:

$$Q(z,f) = f(z)^{v} + \sum_{k=0}^{d(P)} \tilde{a}_{k}(z)f(z)^{k},$$
(22)

where the coefficients $\tilde{a}_k(z)$, k = 0, 1, ..., d(P) are the sum of finitely many terms of the form:

$$a_{\lambda}(z) \prod_{i=0}^{n} \prod_{j=0}^{m} \left(\frac{f^{(j)}(z+c_i)}{f(z)} \right)^{\lambda_{i,j}}$$

From Lemma 4, it follows that:

$$m(r,\tilde{a}_{k}) \leq m(r,a_{\lambda}) + \sum_{i=0}^{n} \sum_{j=0}^{m} \left[m\left(r,\frac{f^{(j)}(z+c_{i})}{f(z+c_{i})}\right) + m\left(r,\frac{f(z+c_{i})}{f(z)}\right) \right] + O(1) = S(r,f),$$

for k = 0, 1, ..., d(P). Thus, it follows from Lemma 8 and Equations (21) and (22) that:

$$m(r,Q(z,f)) \le vm(r,f) + S(r,f).$$
(23)

In view of Lemmas 2 and 5, we have:

$$N(r,Q(z,f)) \le vN(r,f) + d(P)\sum_{j=1}^{n} N(r,f(z+c_j)) + \Gamma(P)\sum_{j=1}^{n} \overline{N}(r,f(z+c_j)) + S(r,f)$$

= $[v + nd(P) + n\Gamma(P)]N(r,f) + S(r,f),$ (24)

and:

$$\overline{N}(r,Q(z,f)) \leq \overline{N}(r,f) + \sum_{j=1}^{n} \overline{N}(r,f(z+c_j)) + S(r,f)$$
$$= (n+1)\overline{N}(r,f) + S(r,f).$$
(25)

Hence, from Equations (23) and (24), it follows that:

$$T(r, Q(z, f)) = m(r, Q(z, f)) + N(r, Q(z, f))$$

$$\leq [v + nd(P) + n\Gamma(P)]T(r, f) + S(r, f).$$
(26)

Thus, we can conclude S(r, Q(z, f)) = S(r, f). With the differential two sides of Equation (4), we have:

$$Q'(z,f) = vf^{v-1}f' + P'(z,f).$$
(27)

Multiplying two sides of Equation (4) by $\frac{Q'}{Q}$, it follows that:

$$Q'(z,f) = \frac{Q'(z,f)}{Q(z,f)} f^{v} + \frac{Q'(z,f)}{Q(z,f)} P(z,f).$$

and with a view of Equation (27), it yields:

$$vf^{v-1}f' + P'(z,f) = \frac{Q'(z,f)}{Q(z,f)}f^v + \frac{Q'(z,f)}{Q(z,f)}P(z,f),$$

that is,

$$f^{v-1}\left(vf' - \frac{Q'}{Q}f\right) = \frac{Q'}{Q}P(z,f) - P'(z,f).$$
(28)

Firstly, $vf' - \frac{Q'}{Q}f \neq 0$. Otherwise, $v\frac{f'}{f} = \frac{Q'}{Q}$, which leads to $Q = A_2 f^v$ for some constant A_2 . This means:

$$(A_2 - 1)f(z)^v = P(z, f).$$
(29)

In view of $P(z, f) \neq 0$, it follows that $A_2 \neq 1$. Since v > d(P), and in view of Lemma 8 and Equation (29), we have m(r, f) = S(r, f), which leads to T(r, f) = S(r, f), a contradiction.

Since $v \ge d(P) + 2$, in view of Lemma 8, it follows that:

$$m\left(r,vf'-\frac{Q'}{Q}f\right) = S(r,f)$$
(30)

and:

$$m\left(r, f\left(vf' - \frac{Q'}{Q}f\right)\right) = S(r, f).$$
(31)

Further, iin view of Lemma 2, it follows that:

$$N\left(r,vf'-\frac{Q'}{Q}f\right) \le N(r,f) + \overline{N}(r,f) + \overline{N}(r,Q) + \overline{N}(r,\frac{1}{Q}) + O(1)$$
$$\le N(r,f) + (n+2)\overline{N}(r,f) + \overline{N}(r,\frac{1}{Q}) + S(r,f),$$
(32)

and:

$$N\left(r, f\left(vf' - \frac{Q'}{Q}f\right)\right) \le 2N(r, f) + \overline{N}(r, f) + \overline{N}(r, Q) + \overline{N}(r, \frac{1}{Q}) + O(1)$$
$$\le 2N(r, f) + (n+2)\overline{N}(r, f) + \overline{N}(r, \frac{1}{Q}) + S(r, f).$$
(33)

Thus, it follows from Equations (30)–(33) that:

$$T\left(r,vf'-\frac{Q'}{Q}f\right) = m\left(r,vf'-\frac{Q'}{Q}f\right) + N\left(r,vf'-\frac{Q'}{Q}f\right)$$
$$\leq N(r,f) + (n+2)\overline{N}(r,f) + \overline{N}(r,\frac{1}{Q}) + S(r,f),$$
(34)

and:

$$T\left(r, f\left(vf' - \frac{Q'}{Q}f\right)\right) = m\left(r, f\left(vf' - \frac{Q'}{Q}f\right)\right) + N\left(r, f\left(vf' - \frac{Q'}{Q}f\right)\right)$$
$$\leq 2N(r, f) + (n+2)\overline{N}(r, f) + \overline{N}(r, \frac{1}{Q}) + S(r, f).$$
(35)

Hence, from Equations (34) and (35), we conclude that:

$$\begin{split} T(r,f) &\leq T\left(r,vf'-\frac{Q'}{Q}f\right) + T\left(r,f\left(vf'-\frac{Q'}{Q}f\right)\right) + O(1) \\ &\leq 3N(r,f) + 2(n+2)\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{Q}) + S(r,f) \\ &\leq (2n+7)N(r,f) + 2\overline{N}(r,\frac{1}{Q}) + S(r,f). \end{split}$$

Since $\delta > 1 - \frac{1}{2n+7}$, then for any given $\varepsilon \in (0, \delta - \frac{2n+6}{2n+7})$, and combining with Equation (26), we can deduce:

$$[1-(2n+7)(1-\delta+\varepsilon)+o(1)]T(r,Q) \leq 2[v+n(d(P)+\Gamma(P))]\overline{N}(r,\frac{1}{Q}),$$

which leads to:

$$\limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{Q})}{T(r,Q)} \geq \frac{1-(2n+7)(1-\delta+\varepsilon)}{2[v+n(d(P)+\Gamma(P))]} > 0.$$

Therefore, this means that Q(z, f) has infinitely many zeros and:

$$\delta(0,Q) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{Q})}{T(r,Q)} < 1.$$

This completes the proof of Theorem 6.

3.2. The Proof of Theorem 7

Similar to the proof of Theorem 6, it follows that Q(z, f) is not constant, and Equations (22)–(28), (30), (31), and (33) also hold. Thus, it yields from Equation (26) that:

$$m\left(r, v\frac{f'}{f} - \frac{Q'}{Q}\right) \le S(r, f) + S(r, Q) = S(r, f).$$
(36)

From Equation (25), and combining with $\overline{N}(r, \frac{1}{f}) = S(r, f)$, we obtain:

$$N\left(r, v\frac{f'}{f} - \frac{Q'}{Q}\right) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, Q) + \overline{N}(r, \frac{1}{Q}) + O(1)$$
$$\leq (n+2)\overline{N}(r, f) + \overline{N}(r, \frac{1}{Q}) + S(r, f).$$
(37)

Hence, we can deduce from Equations (36) and (37) that:

$$T\left(r, v\frac{f'}{f} - \frac{Q'}{Q}\right) \le (n+2)\overline{N}(r, f) + \overline{N}(r, \frac{1}{Q}) + S(r, f).$$
(38)

Thus, from Equations (34) and (38), it follows that:

$$\begin{split} T(r,f) &\leq T\left(r,v\frac{f'}{f} - \frac{Q'}{Q}\right) + T\left(r,vf' - \frac{Q'}{Q}f\right) + O(1) \\ &\leq N(r,f) + 2(n+2)\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{Q}) + S(r,f) \\ &\leq (2n+5)N(r,f) + 2\overline{N}(r,\frac{1}{Q}) + S(r,f). \end{split}$$

Since $\delta > 1 - \frac{1}{2n+5}$, then for any given $\varepsilon \in (0, \delta - \frac{2n+4}{2n+5})$, and combining with Equation (26), we can deduce that:

$$[1 - (2n+5)(1 - \delta + \varepsilon) + o(1)]T(r, Q) \le 2[v + n(d(P) + \Gamma(P))]\overline{N}(r, \frac{1}{Q}),$$

which leads to:

$$\limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{Q})}{T(r, Q)} \ge \frac{1 - (2n + 5)(1 - \delta + \varepsilon)}{2[v + n(d(P) + \Gamma(P))]} > 0$$

Therefore, this means that Q(z, f) has infinitely many zeros and:

$$\delta(0,Q) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{Q})}{T(r,Q)} < 1.$$

This completes the proof of Theorem 7.

3.3. The Proof of Theorem 8

Similar to the proof of Theorem 6, in view of $\overline{N}(r, f) = S(r, f)$, we have:

$$T(r, Q(z, f)) \le (v + nd(P))T(r, f) + S(r, f),$$
(39)

and:

$$T(r,f) \le 3N(r,f) + 2\overline{N}(r,\frac{1}{Q}) + S(r,f).$$

$$\tag{40}$$

Since $\delta > \frac{2}{3}$, then for any given $\varepsilon \in (0, \delta - \frac{2}{3})$, and combining with Equations (39) and (40), we can deduce that:

$$[1-3(1-\delta+\varepsilon)+o(1)]T(r,Q) \le 2[v+nd(P)]\overline{N}(r,\frac{1}{Q}),$$

which leads to:

$$\limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{Q})}{T(r, Q)} \geq \frac{1 - 3(1 - \delta + \varepsilon)}{2[v + nd(P)]} > 0.$$

Therefore, this means that Q(z, f) has infinitely many zeros and:

$$\delta(0,Q) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,\frac{1}{Q})}{T(r,Q)} < 1.$$

This completes the proof of Theorem 8.

3.4. The Proof of Theorem 9

By using the same method as in the proof of Theorem 7, we can easily get the conclusion of Theorem 9.

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