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Article

A New Version of the Generalized Krätzel–Fox Integral Operators

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Abstract: This article deals with some variants of Krätzel integral operators involving Fox's H-function and their extension to classes of distributions and spaces of Boehmians. For real numbers a and $b > 0$, the Fréchet space $H_{a,b}$ of testing functions has been identified as a subspace of certain Boehmian spaces. To establish the Boehmian spaces, two convolution products and some related axioms are established. The generalized variant of the cited Krätzel-Fox integral operator is well defined and is the operator between the Boehmian spaces. A generalized convolution theorem has also been given.

Keywords: H-function; kernel method; Krätzel function; Krätzel operator; distribution space; Boehmian space

MSC: Primary 54C40, 14E20; Secondary 46E25, 20C20

1. Introduction

The notion of Fox's H-function is identified by various authors such as Mathai-Saxena [1], Srivastava et al. [2], Kilbas and Saigo [3], and some others. For $m, n, p, q \in N_0 = N \cup \{0\}$ where $mq, np, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R_+ = (0, \infty)$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$, Fox's H-functions, via the Mellin–Barnes-type integrals, are defined as [4]:

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{array}{l} (a_i, \alpha_i), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array}\right) := \frac{1}{2\pi i}\int_L T_{p,q}^{m,n}(s)\, z^{-s} ds,$$

with:

$$T_{p,q}^{m,n}(s) = \frac{\prod\limits_{j=1}^{m} \Gamma\left(b_j, \beta_j s\right) \prod\limits_{i=1}^{n} \Gamma\left(1 - a_i, \alpha_i s\right)}{\prod\limits_{i=n+1}^{p} \Gamma\left(a_i, \alpha_i s\right) \prod\limits_{j=m+1}^{q} \Gamma\left(1 - b_j, \beta_j s\right)} \tag{1}$$

where, for $z \neq 0$, $z^{-s} = \exp\left(-s\left(\log|z| + i\arg z\right)\right)$. An empty product of Equation (1) is to be considered one. The infinite contour L separates all possible poles of the gamma functions $\Gamma\left(b_j, \beta_j s\right)$ $(j = 1, 2, \ldots, m)$ to the left of the contour and all possible poles of the gamma functions $\Gamma\left(1 - a_i, \alpha_i s\right)$ $(i = 1, 2, \ldots, n)$ to the right of the said contour. Moreover, L can be formulated as in the following:

(i) $L = L_{-\infty}$ is the loop to the left that begins at $-\infty + i\varphi_1$ and ends at $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$.

(ii) $L = L_{+\infty}$ is the loop to the right that begins at $+\infty + i\varphi_1$ and ends at $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < \infty$.
(iii) $L = L_{i\gamma\infty}$ is the contour that begins at $\gamma - i\infty$ and ends at $\gamma + i\infty + i\varphi_2$ where $\gamma \in R$.

Let $x > 0, p \in R, v \in C$ and $Re\,(v) < 0$ when $p \leq 0$, then the function $Z_p^v\,(x)$ is defined by the improper integral:

$$Z_p^v\,(x) = \int_0^\infty t^{v-1} \exp\left(-t^p - \frac{x}{t}\right) dt \;\; (x > 0)\,. \tag{2}$$

A straightforward analytic investigation and evaluation of this integral representation by a statistical technique is given in the literature. Most investigations were devoted to $Z_p^v\,(x)$, $x > 0$, $p > 0$. However, Kilbas et al. [5] have considered this function for real and complex values of p and z, respectively. Their results substantially relied on a representation of $Z_p^v\,(x)$ in terms of Fox's H-functions as follows:

$$Z_p^v\,(x) = \frac{1}{p} H_{0,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \\ (0,1)\,, \left(\frac{v}{p}, \frac{1}{p}\right) \end{matrix}\right) (v \in C; z \in, z \neq 0)\,, \tag{3}$$

provided $p > 0$, while:

$$Z_p^v\,(x) = \frac{1}{|p|} H_{1,1}^{1,1}\left(z \,\middle|\, \begin{matrix} \left(1 - \frac{v}{p}, -\frac{1}{p}\right) \\ (0,1) \end{matrix}\right) (Re\,(v) < 0; z \in C; z \neq 0) \tag{4}$$

provided $p < 0$. The function Z_p^v plays an important role in the astrophysical thermonuclear functions that have been derived on the basis of Boltzmann–Gibbs statistical mechanics, and it, in particular, for $p = 1$ and $x = \frac{t^2}{4}$, gives:

$$Z_1^v\left(\frac{t^2}{4}\right) = 2\left(\frac{t}{2}\right)^2 K_v\,(t)\,,$$

where $K_v\,(t)$ is the McDonald function. The function $Z_p^v\,(x)$ was further introduced as the kernel function of the so-called Krätzel integral operator defined by [6]:

$$\left(K_v^p f\right)(x) = \int_0^\infty \left(Z_p^v\right)(xt)\, f\,(t)\, dt \;\; (x > 0) \tag{5}$$

to include the Meijer integral operator for $p = 1$ and the Laplace integral operator for $p = 1$ and $v = \pm\frac{1}{2}$. However, the investigations of the Krätzel integral operators are continued by obtaining Tauberian and Abelian theorems and some related inversion formulas in the classical theory. Later in [7], Rao–Debnath have discussed the Krätzel integral on a certain space of distributions based on the kernel method of extension. Here, we give a revised version of the generalized Krätzel integral discussed by Al-Omari and Kilicman [8,9] in terms of the generality and clearance of results. In view of (3), we introduce Krätzel–Fox's integral operator as:

$$\left(K_H^{v,p} f\right)(x) = \frac{1}{p} \int_0^\infty f\,(t)\, H_{0,2}^{2,0}\left(xt \,\middle|\, \begin{matrix} \\ (0,1)\,, \left(\frac{v}{p}, \frac{1}{p}\right) \end{matrix}\right) dt \;\; (x > 0)\,. \tag{6}$$

provided the integral exists. For a real number a, being fixed, satisfying $a \leq c + 1$ where $c = \min Re\,\{0, v\}$ and $b > 0$, the set $H_{a,b}$ is defined as the collection of those test functions that are C^∞-functions on $I, I =: (0, \infty)$ that possess the property:

$$\gamma_k\,(\phi) = \gamma_{a,b,k}\,(\phi) \overset{\Delta}{=} \sup_{t \in (0,\infty)} \left| e^{-bt} t^{1-a+k} D_t^k \phi\,(t) \right| \tag{7}$$

for each $k = 0, 1, 2, \ldots$ and $D_t = \frac{d}{dt}$.

The topology associated with $H_{a,b}$ can be generated by the semi-norms $\{\gamma_k\}_{k=0}^{\infty}$. The sequence $\{\phi_n\}$ of $H_{a,b}$ converges to ϕ in the topology of $H_{a,b}$ if:

$$e^{-bt}t^{1-a+k}\frac{d^k}{dt^k}\phi_k(t) \to e^{-bt}t^{1-a+k}\frac{d^k}{dt^k}\phi(t)$$

as $n \to \infty$, uniformly in t, for each $k = 0, 1, 2, \ldots$. It turns out that $H_{a,b}$ is a Fréchet space. $H'_{a,b}$ denotes the strong dual of $H_{a,b}$. If, in $H_{a,b}$, $\{\phi_n\}_1^{\infty}$ converges, then $\left\{D_t^k\phi_n(t)\right\}_1^{\infty}$ also converges on every compact subset. Therefore, if D_I denotes the Schwartz space of test functions of compact supports, then D_I is contained in $H_{a,b}$, and the topology of D_I induced on D_I by $H_{a,b}$ is weaker than that of D_I. Indeed, any restriction of $f \in \acute{H}_{a,b}$ to D_I is in the Schwartz space $\acute{D}_I$ of distributions.

We divide this paper into four sections. In Section 1, we have given the necessary definitions we need for our next investigation. In Section 2, we introduce convolution products and generate the Krätzel spaces of Boehmians. In Section 3, we give an estimation of the Krätzel integral operator and obtain some properties in the class of Boehmians. The Conclusion Section is given at the end of the article.

2. Boehmian Spaces

For the construction of the Boehmian spaces, readers are to be familiar with the abstract construction of the Boehmian space. Otherwise, they may refer to [8–16]. Here, we make use of the following convolution products. Throughout various works of the first author, the convolution product was often used:

$$(f \star g)(y) = \int_0^{\infty} x^{-1}f\left(x^{-1}y\right)g(x)\,dx \tag{8}$$

whereas we suggest another convolution product defined by:

$$(f \bullet g)(y) = \int_0^{\infty} f(xy)\,g(x)\,dx \tag{9}$$

provided the integrals exist. Our construction in this article begins by proving the following theorem.

Theorem 1. *Let $f \in H_{a,b}$ and $\phi \in D_I$. Then, we have $f \bullet \phi \in H_{a,b}$.*

Proof. Assume $f \in H_{a,b}$ and $\phi \in D_I$ are given. Then, by (9) and the concept of semi-norms of $H_{a,b}$, we, for some positive numbers b and fixed a satisfying $a \leq c+1$, write:

$$\sup_{0<t<\infty}\left|e^{-bt}t^{1-a+k}D_t^k(f \bullet \phi)(t)\right| \leq \int_{a_1}^{a_2}|\phi(x)|\left|e^{-bt}t^{1-a+k}D_t^k f(tx)\right|dx, \tag{10}$$

where a_1 and a_2 are the bounds of the support of $\phi(x)$. Hence, (10), for some positive constant A, gives:

$$\sup_{0<t<\infty}\left|e^{-bt}t^{1-a+k}D_t^k(f \bullet \phi)(t)\right| \leq A\int_{a_1}^{a_2}|\phi(x)|\,dx,$$

since $f \in H_{a,b}$.

Hence, the fact that $\phi \in D_I$ yields $\gamma_k(f \bullet \phi) < \infty$ for all $k = 0, 1, 2, \ldots$.

This finishes the proof of the theorem. $\square$

Theorem 2. *Let $f \in H_{a,b}$ and $\phi_1, \phi_2 \in D_I$, then we have:*

$$(f \bullet (\phi_1 \star \phi_2))(t) = ((f \bullet \phi_1) \bullet \phi_2)(t)$$

in $H_{a,b}$.

Proof. It is proven by Theorem 1 that $f \bullet \phi_1 \in H_{a,b}$ for any $f \in H_{a,b}$ and $\phi \in D_I$. To establish the equality in the previous equation, we employ (9) , (8) and Fubini's theorem to write:

$$(f \bullet (\phi_1 \star \phi_2))(t) = \int_0^\infty y^{-1}\phi_2(y)\left(\int_0^\infty \phi_1\left(y^{-1}x\right) f(tx)\,dx\right) dy.$$

Then, setting variables as $y^{-1}x = z$ gives:

$$\begin{aligned}(f \bullet (\phi_1 \star \phi_2))(t) &= \int_0^\infty \phi_2(y) \int_0^\infty \phi_1(z) f(tyz)\,dzdy \\ &= \int_0^\infty \phi_2(y)(f \bullet \phi_1)(ty)\,dy.\end{aligned}$$

Once again, (9) finishes our proof of this theorem. □

We introduce delta sequences as usual. A sequence $(\delta_n) \in D_I$ is a delta sequence if it can satisfy the following:

$$\left.\begin{array}{l}\text{(i) } \int_0^\infty \delta_n(x)\,dx = 1, \text{ where } n \text{ is a positive integer.}\\[2ex] \text{(ii) } |\delta_n(x)| < M,\ M \text{ is a positive real number.}\\[2ex] \text{(iii) } supp\delta_n \subseteq (a_n, b_n), a_n, b_n \to 0 \text{ as } n \to \infty.\end{array}\right\} \quad (11)$$

The set of all such sequences is denoted by Δ.

By using simple integral integration, we state without proof the following theorem.

Theorem 3. *Let $\alpha \in C, \{f_n\}, f \in H_{a,b}$ such that $f_n \to f$ as $n \to \infty$ and $\phi, \phi_1 \in D_I$, then the following hold.*

(i) $f \bullet (\phi + \phi_1) = f \bullet \phi + f \bullet \phi_1$ *in* $H_{a,b}$;
(ii) $\alpha(f \bullet \phi) = (\alpha f) \bullet \phi$ *in* $H_{a,b}$;
(iii) $f_n \bullet \phi \to f \bullet \phi$ *as* $n \to \infty$ *in* $H_{a,b}$.

Theorem 4. *Let $f \in H_{a,b}$ and $(\delta_n) \in \Delta$. Then, we have $f \bullet \delta_n \to f$ as $n \to \infty$ in $H_{a,b}$.*

Proof. Assume the hypothesis of the theorem is satisfied for f and $\{\delta_n\}$. Then, the use of (7), (11) (i) and (11) (iii) yields:

$$\sup_{t\in(0,\infty)} \left|e^{-bt}t^{1-a+k}D_t^k(f \bullet \delta_n - f)(t)\right| \le \int_{a_n}^{b_n} |\delta_n(x)| \sup_{t\in(0,\infty)}\left|e^{-bt}t^{1-a+k}D_t^k(f(xt) - f(t))\right| dx, \quad (12)$$

where $b > 0$ and $a \le c + 1$. The hypothesis that $f(xt) - f(t) \in H_{a,b}$ and (9) (ii) implies:

$$\sup_{t\in(0,\infty)} \left|e^{-bt}t^{1-a+k}D_t^k(f \bullet \delta_n - f)(t)\right| \le M_1 \int_{a_n}^{b_n} |\delta_n(x)| < M_1 M(b_n - a_n)$$

where M_1 and M are positive real numbers.

Hence, considering the limit as $n \to \infty$ and (11) (iii) finishes our proof of Theorem 4. □

The following Theorem 5 follows from the fact $f \star \phi = \phi \star f$ and (11) . We avoid the details.

Theorem 5. *Let $\{\delta_n\}, \{\epsilon_n\} \in \Delta$. Then $\{\delta_n \star \epsilon_n\} \in \Delta$.*

Hence, we have constructed the Boehmian space β_2 by the set $(H_{a,b}, \bullet)$, the subset $(D_I, \star)$ and the set Δ of delta sequences.

Another Boehmian space, say β_1, with $(H_{a,b}, \star), (D_I, \star)$ and Δ, follows from techniques similar to that of β_1 and the facts that [17]:

$$f \star \phi = \phi \star f; (f \star \phi_1) \star \phi_2 = f \star (\phi_1 \star \phi_2), (f + \phi_1) \star \phi_2 = f \star \phi_1 + f \star \phi_2.$$

For addition, convergence and multiplication in β_1 and β_2, we refer readers to [8,9] and the citations therein.

3. The Krätzel–Fox Integral Operator of Generalized Functions

By using the kernel method, we introduce the distributional Krätzel–Fox integral as a mapping acting on the space $H_{a,b}$ of testing functions. For certain appropriateness in our results, we throughout this paper set:

$$H_{0,2}^{2,0}(xt) = H_{0,2}^{2,0}\left(xt \middle| \begin{matrix} \\ (0,1), \left(\frac{v}{p}, \frac{1}{p}\right) \end{matrix}\right).$$

The following result is very necessary.

Theorem 6. *Let $H_{a,b}$ be the space determined by (7). Then, we have:*

$$H_{0,2}^{2,0}(xt) \in H_{a,b}, \tag{13}$$

where a and $b > 0$ are real numbers, but b is fixed.

Proof. By using differential properties of the H-function together with simple computations, we obtain
$$\sup_{t \in (0,\infty)} \left| e^{-bt} t^{1-a+k} D_t^k H_{0,2}^{2,0}(xt) \right| = \sup_{t \in (0,\infty)} \left| e^{-bt} t^{1-a} \frac{1}{p} H_{0,2}^{2,0}\left(xt \middle| \begin{matrix} (0,1) \\ (0,1), \left(\frac{v}{p}, \frac{1}{p}\right), (k,1) \end{matrix}\right) \right|.$$
Hence, for each $k = 0, 1, 2, \ldots$, the asymptotic behavior of H-functions reveals that:

$$\sup_{t \in (0,\infty)} \left| e^{-bt} t^{1-a+k} D_t^k \frac{1}{p} H_{0,2}^{2,0}\left(xt \middle| \begin{matrix} \\ (0,1), \left(\frac{v}{p}, \frac{1}{p}\right) \end{matrix}\right) \right| < \infty.$$

This finishes the proof of our Theorem. □

Therefore, for $f \in \acute{H}_{a,b}$ and a fixed real number a satisfying $a \leq c + 1$ where $c = \min Re\{0, v\}$ and $b > 0$, the generalized Krätzel integral operator applied to a distribution $f \in \acute{H}_{a,b}$ can be introduced as:

$$H(x) \triangleq \left\langle f(t), \frac{1}{p} H_{0,2}^{2,0}\left(xt \middle| \begin{matrix} \\ (0,1), \left(\frac{v}{p}, \frac{1}{p}\right) \end{matrix}\right) \right\rangle \tag{14}$$

which is the application of f to the kernel function.

The mandatory step of defining the Krätzel–Fox integral operator of a Boehmian is as follows.

Theorem 7. *Let $f \in H_{a,b}$ and $\phi \in D_I$. Then, we have:*

$$\left(K_v^p (f \star \phi)\right)(x) = \left(\left(K_v^p f\right) \bullet \phi\right)(x)\,(x > 0).$$

where a is a fixed real number satisfying $a \leq c + 1$, $c = \min Re\{0, v\}$ and $b > 0$.

Proof. Applying (6) to (9) gives:

$$\begin{aligned}\left(K_v^p\left(f\star\phi\right)\right)(x) &= \frac{1}{p}\int_0^\infty (f\star\phi)(t)\,H_{0,2}^{2,0}\left(xt\,\middle|\,\begin{matrix} \\ (0,1),\left(\frac{v}{p},\frac{1}{p}\right)\end{matrix}\right)dt\\ &= \int_0^\infty \phi(x)\,\frac{1}{p}\int_0^\infty f(yt)\,H_{0,2}^{2,0}\left(xt\,\middle|\,\begin{matrix} \\ (0,1),\left(\frac{v}{p},\frac{1}{p}\right)\end{matrix}\right)dtdy.\end{aligned}$$

Setting variables $yt=z$ gives $t=y^{-1}z$,

$$\begin{aligned}\left(K_v^p\left(f\star\phi\right)\right)(x) &= \int_0^\infty \phi(y)\,\frac{1}{p}\int_0^\infty f(z)\,H_{0,2}^{2,0}\left(xy^{-1}z\,\middle|\,\begin{matrix} \\ (0,1),\left(\frac{v}{p},\frac{1}{p}\right)\end{matrix}\right)y^{-1}dzdy\\ &= \int_0^\infty y^{-1}\left(K_v^p f\right)\left(xy^{-1}\right)\phi(y)\,dy.\end{aligned}$$

Hence the theorem.

In view of the above, we define the Krätzel–Fox integral operator of $\frac{f_n}{\phi_n}\in\beta$ and:

$$G_v^p\frac{f_n}{\phi_n}=\frac{K_v^p f_n}{\phi_n}\in\beta_2 \tag{15}$$

where a is a fixed real number satisfying $a\le c+1$ where $c=\min Re\{0,v\}$ and $b>0$. □

Theorem 8. *The Krätzel–Fox integral operator $G_v^p:\beta_1\to\beta_2$ is well defined.*

Proof. Let $\dfrac{f_n}{\mu_n}=\frac{\hat{f}_n}{\epsilon_n}\in\beta_1$, then $f_n\star\epsilon_m=\hat{f}_m\star\mu_n$. Taking into account Theorem 7, we have:

$$K_v^p f_n\bullet\epsilon_m=K_v^p\hat{f}_m\bullet\mu_n. \tag{16}$$

It follows from (16) that $\frac{K_v^p f_n}{\mu_n}\sim\frac{K_v^p\hat{f}}{\epsilon_n}$ in β_2. Therefore:

$$G_v^p\frac{f_n}{\mu_n}=G_v^p\frac{\hat{f}}{\epsilon_n}.$$

This finishes our proof of the proposition. □

Theorem 9. *The Krätzel–Fox integral operator $G_v^p:\beta_1\to\beta_2$ is linear.*

Proof. Let $\frac{f_n}{\mu_n},\frac{\hat{f}}{\epsilon_n}\in\beta_1$. Then, we write:

$$\begin{aligned}G_v^p\left(\frac{f_n}{\mu_n}+\frac{\hat{f}}{\epsilon_n}\right) &= G_v^p\left(\frac{f_n\star\epsilon_n+\hat{f}\star\mu_n}{\mu_n\star\epsilon_n}\right)\\ \text{i.e.,}\quad &= \frac{K_v^p\left(f_n\star\epsilon_n+\hat{f}\star\mu_n\right)}{\mu_n\star\epsilon_n}\\ \text{i.e.,}\quad &= \frac{K_v^p\left(f_n\star\epsilon_n\right)+K_v^p\left(\hat{f}\star\mu_n\right)}{\mu_n\star\epsilon_n}\\ \text{i.e.,}\quad &= \frac{K_v^p f_n\bullet\epsilon_n+K_v^p\hat{f}\bullet\mu_n}{\mu_n\star\epsilon_n}\\ \text{i.e.,}\quad &= \frac{K_v^p f_n}{\mu_n}+\frac{K_v^p\hat{f}}{\epsilon_n}.\end{aligned}$$

Hence,

$$G_v^p\left(\frac{f_n}{\mu_n}+\frac{\hat{f}}{\epsilon_n}\right)=G_v^p\frac{f_n}{\mu_n}+G_v^p\frac{\hat{f}}{\epsilon_n}.$$

Furthermore, for $\alpha \in C$, we have:

$$\alpha G_v^p\frac{f_n}{\mu_n}=\alpha\frac{K_v^p f_n}{\mu_n}=\frac{K_v^p(\alpha f_n)}{\mu_n}.$$

The proof is completed. □

Theorem 10. *Let* $\frac{f_n}{\mu_n}$ *and* $\frac{\hat{f}_n}{\epsilon_n}\in\beta_1$, *then* $G_v^p\left(\frac{f_n}{\mu_n}\star\frac{\hat{f}_n}{\epsilon_n}\right)=G_v^p\frac{f_n}{\mu_n}\bullet\frac{\hat{f}_n}{\epsilon_n}$.

Proof. Assume $\frac{f_n}{\mu_n}$ and $\frac{\hat{f}_n}{\epsilon_n}\in\beta_1$, then we get $G_v^p\left(\frac{f_n}{\mu_n}\star\frac{\hat{f}_n}{\epsilon_n}\right)=G_v^p\left(\frac{f_n\star\hat{f}_n}{\mu_n\star\epsilon_n}\right)$. (15) reveals:

$$G_v^p\left(\frac{f_n}{\mu_n}\star\frac{\hat{f}_n}{\epsilon_n}\right)=\frac{K_v^p\left(f_n\star\hat{f}_n\right)}{\mu_n\star\epsilon_n}$$

Now, Theorem 7 gives:

$$G_v^p\left(\frac{f_n}{\mu_n}\star\frac{\hat{f}_n}{\epsilon_n}\right)=\frac{K_v^p f_n\bullet\hat{f}_n}{\mu_n\star\epsilon_n}=\frac{K_v^p f_n}{\mu_n}\bullet\frac{\hat{f}_n}{\epsilon_n}.$$

This finishes the proof of the theorem. □

4. Conclusions

This paper has presented an extension of a Fréchet space of smooth functions to a Fréchet space of Boehmians. It has also presented a Krätzel–Fox convolution product and established the Krätzel–Fox convolution theorem as well. Consequently, the generalized Krätzel–Fox integral operator has been given as a well-defined linear mapping whose convolution theorem coincides with the convolution theorem of the classical integral.

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