*mathematics*

MDPI

*Article*

# Inextensible Flows of Curves on Lightlike Surfaces

**Zühal Küçükarslan Yüzbaşı [1] and Dae Won Yoon [2,*]**

1 Department of Mathematics, Fırat University, Elazig 23119, Turkey; zuhal2387@yahoo.com.tr
2 Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 52828, Korea
* Correspondence: dwyoon@gnu.ac.kr; Tel.: +82-55-772-2256

Received: 28 September 2018; Accepted: 26 October 2018; Published: 29 October 2018

check for updates

**Abstract:** In this paper, we study inextensible flows of a curve on a lightlike surface in Minkowski three-space and give a necessary and sufficient condition for inextensible flows of the curve as a partial differential equation involving the curvatures of the curve on a lightlike surface. Finally, we classify lightlike ruled surfaces in Minkowski three-space and characterize an inextensible evolution of a lightlike curve on a lightlike tangent developable surface.

**Keywords:** inextensible flow; lightlike surface; ruled surface; Darboux frame

## 1. Introduction

It is well known that many nonlinear phenomena in physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces, and the time evolution of a curve and a surface has significance in computer vision and image processing. The time evolution of a curve and a surface is described by flows, in particular inextensible flows of a curve and a surface. Physically, inextensible flows give rise to motion, for which no strain energy is induced. The swinging motion of a cord of fixed length or of a piece of paper carried by the wind can be described by inextensible flows of a curve and a surface. Furthermore, the flows arise in the context of many problems in computer vision and computer animation [1–4].

Chirikjian and Burdick [1] studied applications of inextensible flows of a curve. In [5], the authors derived the time evolution equations for an inextensible flow of a space curve and also studied inextensible flows of a developable ruled surface. In [6], the author investigated the general description of the binormal motion of a spacelike and a timelike curve in a three-dimensional de Sitter space and gave some explicit examples of a binormal motion of the curves. Schief and Rogers [4] studied the binormal motions of curves with constant curvature and torsion. Many authors have studied geometric flow problems [7–11].

The outline of the paper is organized as follows: In Section 2, we give some geometric concepts in Minkowski space and present the pseudo-Darboux frames of a spacelike curve and a lightlike curve on a lightlike surface. In Sections 3 and 4, we study inextensible flows of a spacelike curve and a lightlike curve on a lightlike surface. In the last section, we classify lightlike ruled surfaces and study inextensible flows of lightlike tangent developable surfaces.

## 2. Preliminaries

The Minkowski three-space $\mathbb{R}^3_1$ is a real space $\mathbb{R}^3$ with the indefinite inner product $\langle \cdot\, , \cdot \rangle$ defined on each tangent space by:

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2,$$

where $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ are vectors in $\mathbb{R}^3_1$.

A nonzero vector $\mathbf{x}$ in $\mathbb{R}^3_1$ is said to be spacelike, timelike or lightlike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, respectively. Similarly, an arbitrary curve $\gamma = \gamma(s)$ is spacelike, timelike or lightlike if all of its tangent vectors $\gamma'(s)$ are spacelike, timelike or lightlike, respectively. Here "prime" denotes the derivative with respect to the parameter $s$.

Let $M$ be a lightlike surface in Minkowski three-space $\mathbb{R}^3_1$, that is the induced metric of $M$ is degenerate. Then, a curve $\gamma$ on $M$ is spacelike or lightlike.

Case 1: If $\gamma$ is a spacelike curve, we can reparametrize it by the arc length $s$. Therefore, we have the unit tangent vector $\mathbf{t}(s) = \gamma'(s)$ of $\gamma(s)$. Since $M$ is a lightlike surface, we have a lightlike normal vector $\mathbf{n}$ along $\gamma$. Therefore, we can choose a vector $\mathbf{g}$ satisfying:

$$\langle \mathbf{n}, \mathbf{g} \rangle = 1, \quad \langle \mathbf{t}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{g} \rangle = 0.$$

Then, we have pseudo-orthonormal frames $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$, which are called the Darboux frames along $\gamma(s)$. By standard arguments, we have the following Frenet formulae:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{g}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_n(s) & \tau_g(s) & 0 \\ -\kappa_g(s) & 0 & -\tau_g(s) \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{g}(s) \end{pmatrix}, \tag{1}$$

where $\kappa_n = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle$, $\kappa_g = \langle \mathbf{t}'(s), \mathbf{g}(s) \rangle$ and $\tau_g = -\langle \mathbf{n}(s), \mathbf{g}'(s) \rangle$.

Case 2: Let $\gamma$ be a lightlike curve parametrized by a pseudo arc length parameter $s$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. Since a normal vector $\mathbf{n}$ of a lightlike surface $M$ is lightlike, we can choose a vector $\mathbf{g}$ such that:

$$\langle \mathbf{g}, \mathbf{g} \rangle = 1, \quad \langle \mathbf{t}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{n} \rangle = 0.$$

Furthermore, we consider:

$$\langle \mathbf{t}, \mathbf{n} \rangle = 1.$$

Then, we have pseudo-orthonormal Darboux frames $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$ along a nongeodesic lightlike curve $\gamma(s)$ on $M$ and get the following Frenet formulae:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{g}(s) \end{pmatrix} = \begin{pmatrix} \kappa_n(s) & 0 & \kappa_g(s) \\ 0 & -\kappa_n(s) & \tau_g(s) \\ -\tau_g(s) & -\kappa_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{g}(s) \end{pmatrix}, \tag{2}$$

where $\kappa_n = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle$, $\kappa_g = \langle \mathbf{t}'(s), \mathbf{g}(s) \rangle$ and $\tau_g = -\langle \mathbf{n}(s), \mathbf{g}'(s) \rangle$.

## 3. Inextensible Flows of a Spacelike Curve

We assume that $\gamma : [0, l] \times [0, w] \to M \subset \mathbb{R}^3_1$ is a one-parameter family of the smooth spacelike curve on a lightlike surface in $\mathbb{R}^3_1$, where $l$ is the arc length of the initial curve. Let $u$ be the curve parametrization variable, $0 \leq u \leq l$. We put $v = ||\frac{\partial \gamma}{\partial u}||$, from which the arc length of $\gamma$ is defined by $s(u) = \int_0^u v du$. Furthermore, the operator $\frac{\partial}{\partial s}$ is given in terms of $u$ by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$, and the arc length parameter is given by $ds = vdu$.

On the Darboux frames $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$ of the spacelike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$, any flow of $\gamma$ can be given by:

$$\frac{\partial \gamma}{\partial t} = f_1 \mathbf{t} + f_2 \mathbf{n} + f_3 \mathbf{g}, \tag{3}$$

where $f_1, f_2, f_3$ are scalar speeds of the spacelike curve $\gamma$ on a lightlike surface $M$, respectively. We put $s(u, t) = \int_0^u v du$; it is called the arc length variation of $\gamma$. From this, the requirement that the curve is not subject to any elongation or compression can be expressed by the condition:

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0 \tag{4}$$

for all $u \in [0, l]$.

**Definition 1.** *A curve evolution $\gamma(u,t)$ and its flow $\frac{\partial \gamma}{\partial t}$ of a spacelike curve in $\mathbb{R}^3_1$ are said to be inextensible if:*

$$\frac{\partial}{\partial t}\left|\left|\frac{\partial \gamma}{\partial u}\right|\right| = 0.$$

Now, we give the arc length preserving condition for curve flows.

**Theorem 1.** *Let $M$ be a lightlike surface in Minkowski three-space $\mathbb{R}^3_1$ and $\{\mathbf{t},\mathbf{n},\mathbf{g}\}$ be the Darboux frames of a spacelike curve $\gamma$ on $M$. If $\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ is a flow of $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$, then we have the following equation:*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - vf_2\kappa_n - vf_3\kappa_g. \tag{5}$$

**Proof.** From the definition of a spacelike curve $\gamma$, we have $v^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle$. Since $u$ and $t$ are independent coordinates, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute. Therefore, by differentiating $v^2$, we have:

$$\begin{aligned}
2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle \\
&= 2\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \gamma}{\partial t})\right\rangle \\
&= 2\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u}(f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g})\right\rangle \\
&= 2v\left\langle \mathbf{t}, (\frac{\partial f_1}{\partial u} - vf_2\kappa_n - vf_3\kappa_g)\mathbf{t} + (\frac{\partial f_2}{\partial u} + vf_1\kappa_g + vf_2\tau_g)\mathbf{n} + (\frac{\partial f_3}{\partial u} + vf_1\kappa_n - vf_3\tau_g)\mathbf{g}\right\rangle \\
&= 2v\left(\frac{\partial f_1}{\partial u} - vf_2\kappa_n - vf_3\kappa_g\right).
\end{aligned}$$

This completes the proof. □

**Corollary 1.** *Let $\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ be a flow of a spacelike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. If the curve $\gamma$ is a geodesic curve or an asymptotic curve, then the following equation holds, respectively:*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - vf_2\kappa_n$$

*or:*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - vf_3\kappa_g.$$

**Theorem 2.** (Necessary and sufficient condition for an inextensible flow)

*Let $\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ be a flow of a spacelike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. Then, the flow is inextensible if and only if:*

$$\frac{\partial f_1}{\partial s} = f_2\kappa_n + f_3\kappa_g. \tag{6}$$

**Proof.** Suppose that the flow of a spacelike curve $\gamma$ on $M$ is inextensible. From (4) and (5), we have:

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}du = \int_0^u \left(\frac{\partial f_1}{\partial u} - vf_2\kappa_n - vf_3\kappa_g\right)du = 0.$$

It follows that:

$$\frac{\partial f_1}{\partial u} = vf_2\kappa_n + vf_3\kappa_g.$$

Since $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$, we can obtain (6).

Conversely, by following a similar way as above, the proof is completed. □

**Theorem 3.** *Let* $\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ *be a flow of a spacelike curve* $\gamma$ *on a lightlike surface* $M$ *in* $\mathbb{R}^3_1$*. If the flow is inextensible, then a time evolution of the Darboux frame* $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$ *along a curve* $\gamma$ *on a lightlike surface* $M$ *is given by:*

$$\frac{d}{dt}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ -\varphi_2 & \varphi_3 & 0 \\ -\varphi_1 & 0 & -\varphi_3 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{g} \end{pmatrix}, \tag{7}$$

*where:*

$$\begin{aligned} \varphi_1 &= \frac{\partial f_2}{\partial s} + f_1\kappa_g + f_2\tau_g, \\ \varphi_2 &= \frac{\partial f_3}{\partial s} + f_1\kappa_n - f_3\tau_g, \\ \varphi_3 &= \langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{g}\rangle. \end{aligned} \tag{8}$$

**Proof.** Noting that:

$$\begin{aligned} \frac{\partial \mathbf{t}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial \gamma}{\partial s}\right) = \frac{\partial}{\partial s}(f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}) \\ &= \left(\frac{\partial f_2}{\partial s} + f_1\kappa_g + f_2\tau_g\right)\mathbf{n} + \left(\frac{\partial f_3}{\partial s} + f_1\kappa_n - f_3\tau_g\right)\mathbf{g}. \end{aligned} \tag{9}$$

On the other hand,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\langle \mathbf{t}, \mathbf{n}\rangle = \langle \frac{\partial \mathbf{t}}{\partial t}, \mathbf{n}\rangle + \langle \mathbf{t}, \frac{\partial \mathbf{n}}{\partial t}\rangle = \frac{\partial f_3}{\partial s} + f_1\kappa_n - f_3\tau_g + \langle \mathbf{t}, \frac{\partial \mathbf{n}}{\partial t}\rangle \\ 0 &= \frac{\partial}{\partial t}\langle \mathbf{t}, \mathbf{g}\rangle = \langle \frac{\partial \mathbf{t}}{\partial t}, \mathbf{g}\rangle + \langle \mathbf{t}, \frac{\partial \mathbf{g}}{\partial t}\rangle = \frac{\partial f_2}{\partial s} + f_1\kappa_g + f_2\tau_g + \langle \mathbf{t}, \frac{\partial \mathbf{n}}{\partial t}\rangle \end{aligned}$$

because of $\langle \mathbf{n}, \mathbf{n}\rangle = \langle \mathbf{g}, \mathbf{g}\rangle = 0$ and $\langle \mathbf{n}, \mathbf{g}\rangle = 1$.

Thus, we have:

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(\frac{\partial f_3}{\partial s} + f_1\kappa_n - f_3\tau_g\right)\mathbf{t} + \varphi_3\mathbf{n}, \tag{10}$$

$$\frac{\partial \mathbf{g}}{\partial t} = -\left(\frac{\partial f_2}{\partial s} + f_1\kappa_g + f_2\tau_g\right)\mathbf{t} - \varphi_3\mathbf{g}, \tag{11}$$

where $\varphi_3 = \langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{g}\rangle$. This completes the proof. □

Now, by using Theorem 3, we give the time evolution equations of the geodesic curvature, the normal curvature and the geodesic torsion of a spacelike curve on a lightlike surface.

**Theorem 4.** *Let* $\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ *be a flow of a spacelike curve* $\gamma$ *on a lightlike surface* $M$ *in* $\mathbb{R}^3_1$*. Then, the time evolution equations of the functions* $\kappa_g$*,* $\kappa_n$ *and* $\tau_g$ *for the inextensible spacelike curve* $\gamma$ *are given by:*

$$\begin{aligned} \frac{\partial \kappa_g}{\partial t} &= \frac{\partial \varphi_1}{\partial s} + \varphi_1\tau_g - \varphi_3\kappa_g, \\ \frac{\partial \kappa_n}{\partial t} &= \frac{\partial \varphi_2}{\partial s} - \varphi_2\tau_g + \varphi_3\kappa_n \\ \frac{\partial \tau_g}{\partial t} &= \frac{\partial \varphi_3}{\partial s} + \varphi_1\kappa_n - \varphi_2\kappa_g + 2\varphi_3\tau_g. \end{aligned} \tag{12}$$

**Proof.** It is well known that the arc length and time derivatives commute. This implies the inextensibility of $\gamma$. Accordingly, the compatibility conditions are $\frac{\partial}{\partial s}\left(\frac{\partial \mathbf{t}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{t}}{\partial s}\right)$, etc. On the other hand,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial \mathbf{t}}{\partial t}\right) &= \frac{\partial}{\partial s}(\varphi_1\mathbf{n} + \varphi_2\mathbf{g}) \\ &= (-\varphi_1\kappa_n - \varphi_2\kappa_g)\mathbf{t} + (\frac{\partial \varphi_1}{\partial s} + \varphi_1\tau_g)\mathbf{n} + (\frac{\partial \varphi_2}{\partial s} - \varphi_2\tau_g)\mathbf{g},\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{t}}{\partial s}\right) &= \frac{\partial}{\partial t}(\kappa_g\mathbf{n} + \kappa_n\mathbf{g}) \\ &= (-\varphi_1\kappa_n - \varphi_2\kappa_g)\mathbf{t} + (\frac{\partial \kappa_g}{\partial t} + \varphi_3\kappa_g)\mathbf{n} + (\frac{\partial \kappa_n}{\partial t} - \varphi_3\kappa_n)\mathbf{g}.\end{aligned}$$

Comparing the two equations, we find:

$$\begin{aligned}\frac{\partial \kappa_g}{\partial t} &= \frac{\partial \varphi_1}{\partial s} + \varphi_1\tau_g - \varphi_3\kappa_g, \\ \frac{\partial \kappa_n}{\partial t} &= \frac{\partial \varphi_2}{\partial s} - \varphi_2\tau_g + \varphi_3\kappa_n.\end{aligned}$$

It follows from (8) that we can obtain the first and the second equation of (12).

Furthermore by using $\frac{\partial}{\partial s}\left(\frac{\partial \mathbf{n}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{n}}{\partial s}\right)$ and following a similar way as above, we can obtain the third equation of (12). The proof is completed. □

**Remark 1.** *As applications of inextensible flows of a spacelike curve on a lightlike surface, we can consider geometric phases of the repulsive-type nonlinear Schödinger equation ($NLS^-$) (cf. [12]).*

## 4. Inextensible Flows of a Lightlike Curve

Let $\gamma$ be a lightlike curve on a lightlike surface $M$ in $\mathbb{R}^3_1$. We note that a lightlike curve $\gamma(u)$ satisfies $\langle \gamma''(u), \gamma''(u)\rangle \geq 0$. We say that a lightlike curve $\gamma(u)$ is parametrized by the pseudo arc length if $\langle \gamma''(u), \gamma''(u)\rangle = 1$. If a lightlike curve $\gamma(u)$ satisfies $\langle \gamma''(u), \gamma''(u)\rangle \neq 0$, then $\langle \gamma''(u), \gamma''(u)\rangle > 0$, and:

$$s(u) = \int_0^u \langle \gamma''(u), \gamma''(u)\rangle^{\frac{1}{4}} du$$

becomes the pseudo arc length parameter. Let us consider a lightlike curve $\gamma(u)$ on a lightlike surface $M$ in $\mathbb{R}^3_1$ with $\langle \gamma''(u), \gamma''(u)\rangle \neq 0$.

Let $\gamma : [0, l] \times [0, w] \to M \subset \mathbb{R}^3_1$ be a one-parameter family of smooth lightlike curves on a lightlike surface in $\mathbb{R}^3_1$, where $l$ is the arc length of the initial curve. We put $v^4 = \langle \gamma''(u), \gamma''(u)\rangle$, from which the pseudo arc length of $\gamma$ is defined by $s(u) = \int_0^u v du$. Furthermore, the operator $\frac{\partial}{\partial s}$ is given in terms of $u$ by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$, and the pseudo arc length parameter is given by $ds = vdu$.

On the other hand, a flow $\frac{\partial \gamma}{\partial t}$ of $\gamma$ can be given by:

$$\frac{\partial \gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g} \tag{13}$$

in terms of the Darboux frames $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$ of the lightlike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$, where $f_1, f_2, f_3$ are scalar speeds of the lightlike curve $\gamma$, respectively. We put $s(u, t) = \int_0^u v du$, it is called the pseudo arc length variation of $\gamma$. From this, we have the following condition:

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0 \tag{14}$$

for all $u \in [0, l]$.

**Definition 2.** *A curve evolution $\gamma(u,t)$ and its flow $\frac{\partial\gamma}{\partial t}$ of a lightlike curve $\gamma$ in $\mathbb{R}^3_1$ are said to be inextensible if:*

$$\frac{\partial}{\partial t}\left\langle \frac{\partial^2\gamma}{\partial u^2}, \frac{\partial^2\gamma}{\partial u^2}\right\rangle^{\frac{1}{4}} = 0.$$

**Theorem 5.** *Let $M$ be a lightlike surface in Minkowski three-space $\mathbb{R}^3_1$ and $\{\mathbf{t},\mathbf{n},\mathbf{g}\}$ be the Darboux frames along a lightlike curve $\gamma$ on $M$. If $\frac{\partial\gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ is a flow of $\gamma$ on a lightlike surface $M$, then we have the following equation:*

$$\frac{\partial v}{\partial t} = \frac{1}{2v^3}\left[ \left(\frac{\partial v}{\partial u} + v^2\kappa_n\right)\left(\frac{\partial \Phi_2}{\partial u} - v\kappa_n\Phi_2 - v\kappa_g\Phi_3\right) + v^2\kappa_g\left(\frac{\partial \Phi_3}{\partial u} + v\kappa_g\Phi_1 + v\tau_g\Phi_2\right)\right], \tag{15}$$

*where:*

$$\begin{aligned}
\Phi_1 &= \frac{\partial f_1}{\partial u} + vf_1\kappa_n - vf_3\tau_g,\\
\Phi_2 &= \frac{\partial f_2}{\partial u} - vf_2\kappa_n - vf_3\kappa_g,\\
\Phi_3 &= \frac{\partial f_1}{\partial u} + vf_1\kappa_g + vf_2\tau_g.
\end{aligned}$$

**Proof.** From the definition of a lightlike curve $\gamma$, we have $v^4 = \left\langle \frac{\partial^2\gamma}{\partial u^2}, \frac{\partial^2\gamma}{\partial u^2}\right\rangle$. By differentiating $v^4$, we have:

$$4v^3\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial^2\gamma}{\partial u^2}, \frac{\partial^2\gamma}{\partial u^2}\right\rangle = 2\left\langle \frac{\partial^2\gamma}{\partial u^2}, \frac{\partial^2}{\partial u^2}\left(\frac{\partial\gamma}{\partial t}\right)\right\rangle. \tag{16}$$

On the other hand,

$$\frac{\partial^2\gamma}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial\gamma}{\partial u}\right) = \frac{\partial}{\partial u}(v\mathbf{t}) = \left(\frac{\partial v}{\partial u} + v^2\kappa_n\right)\mathbf{t} + v^2\kappa_g\mathbf{g}$$

and:

$$\begin{aligned}
&\frac{\partial^2}{\partial u^2}\left(\frac{\partial\gamma}{\partial t}\right) = \frac{\partial^2}{\partial u^2}(f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g})\\
&= \left[\frac{\partial \Phi_1}{\partial u} + v\kappa_n\Phi_1 - v\tau_g\Phi_3\right]\mathbf{t} + \left[\frac{\partial \Phi_2}{\partial u} - v\kappa_n\Phi_2 - v\kappa_g\Phi_3\right]\mathbf{n} + \left[\frac{\partial \Phi_3}{\partial u} + v\kappa_g\Phi_1 + v\tau_g\Phi_2\right]\mathbf{g}.
\end{aligned}$$

Thus, (16) implies (15). This completes the proof. □

**Theorem 6.** *Let $\frac{\partial\gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ be a flow of a lightlike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. Then, the flow is inextensible if and only if:*

$$\left(\frac{\partial v}{\partial s} + v\kappa_n\right)\frac{\partial \Phi_2}{\partial s} + v\kappa_g\frac{\partial \Phi_3}{\partial s} = \left(\frac{\partial v}{\partial s} + v\kappa_n\right)(\kappa_n\Phi_2 + \kappa_g\Phi_3) - v\kappa_g(\kappa_g\Phi_1 + \tau_g\Phi_2). \tag{17}$$

**Proof.** Suppose that the flow of a lightlike curve $\gamma$ on $M$ is inextensible. By using (15) and $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$, (14) gives (17). Conversely, by following a similar way as above, the proof is completed. □

Next, we give the time evolution equations of the Darboux frame of a lightlike curve on a lightlike surface.

**Theorem 7.** *Let $\frac{\partial\gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ be a flow of a lightlike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. If the flow is inextensible, then a time evolution of the Darboux frame $\{\mathbf{t},\mathbf{n},\mathbf{g}\}$ along a curve $\gamma$ on a lightlike surface $M$ is given by:*

$$\frac{d}{dt}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{g}\end{pmatrix} = \begin{pmatrix}\frac{\Phi_1}{v} & 0 & \frac{\Phi_3}{v}\\ 0 & -\frac{\Phi_1}{v} & \Theta\\ -\Theta & -\frac{\Phi_3}{v} & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{g}\end{pmatrix}, \tag{18}$$

*where $\Theta = \langle \frac{\partial \mathbf{n}}{\partial t}, \mathbf{g}\rangle$.*

**Proof.** The proof can be obtained by using a similar method of proof of Theorem 3. □

**Theorem 8.** *Let $\frac{\partial\gamma}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$ be a flow of a lightlike curve $\gamma$ on a lightlike surface $M$ in $\mathbb{R}^3_1$. Then, the time evolution equations of the functions $\kappa_g$, $\kappa_n$ and $\tau_g$ for the inextensible spacelike curve $\gamma$ are given by:*

$$\begin{aligned}\frac{\partial\kappa_g}{\partial t} &= \frac{\partial}{\partial s}(\frac{1}{v}\Phi_3) + \frac{1}{v}\left(\kappa_g\Phi_1 - \kappa_n\Phi_3\right),\\ \frac{\partial\kappa_n}{\partial t} &= \frac{\partial}{\partial s}(\frac{1}{v}\Phi_1) + \kappa_g\Theta - \frac{1}{v}\tau_g\Phi_3,\\ \frac{\partial\tau_g}{\partial t} &= \frac{\partial\Theta}{\partial s} + \kappa_n\Theta - \frac{1}{v}\tau_g\Phi_1.\end{aligned} \tag{19}$$

**Proof.** The proof can be obtained by using a similar method of proof of Theorem 4. □

## 5. Lightlike Ruled Surfaces

In this section, we investigate inextensible flows of ruled surfaces, in particular lightlike ruled surfaces in Minkowski three-space $\mathbb{R}^3_1$.

Let $I$ be an open interval on the real line $\mathbb{R}$. Let $\alpha$ be a curve in $\mathbb{R}^3_1$ defined on $I$ and $\beta$ a transversal vector field along $\alpha$. For an open interval $J$ of $\mathbb{R}$, we have the parametrization for $M$:

$$X(u,v) = \alpha(u) + v\beta(u), \quad u \in I, \quad v \in J.$$

Here, $\alpha$ is called a base curve and $\beta$ a director vector field. In particular, the director vector field $\beta$ can be naturally chosen so that it is orthogonal to $\alpha$, that is $\langle\alpha',\beta\rangle = 0$. It is well known that the ruled surface is developable if $\det(\alpha'\beta\beta')$ is identically zero. A developable surface is a surface whose Gaussian curvature of the surface is everywhere zero.

On the other hand, the tangent vectors are given by:

$$X_u = \frac{\partial X}{\partial u} = \alpha'(u) + v\beta'(u), \quad X_v = \frac{\partial X}{\partial v} = \beta(u),$$

which imply that the coefficients of the first fundamental form of the surface are given by:

$$\begin{aligned}E &= \langle X_u, X_u\rangle = \langle\alpha',\alpha'\rangle + 2v\langle\alpha',\beta'\rangle + v^2\langle\beta',\beta'\rangle,\\ F &= \langle X_u, X_v\rangle = 0,\\ G &= \langle X_v, X_v\rangle = \langle\beta,\beta\rangle.\end{aligned}$$

Suppose that the ruled surface is lightlike. Then, we get $E = 0$ or $G = 0$.

First of all, we consider $E = 0$; it implies that:

$$\langle\alpha',\alpha'\rangle = 0, \quad \langle\alpha',\beta'\rangle = 0, \quad \langle\beta',\beta'\rangle = 0. \tag{20}$$

Thus, a base curve $\alpha$ is lightlike, and a director vector $\beta$ is constant or $\beta'$ is lightlike.

Case 1: If $\beta$ is constant, from $\langle \alpha', \beta \rangle = 0$, $\beta$ is a lightlike vector or a spacelike vector. If $\beta$ is lightlike, there exists a smooth function $k$ such that $\beta = k\alpha'$. This is a contradiction because $G = 0$. If $\beta$ is spacelike as a constant vector, then the lightlike cylindrical ruled surface is parametrized by:

$$X(u,v) = \alpha(u) + v\beta,$$

where $\alpha$ is a lightlike curve and $\beta$ is a constant spacelike vector.

Case 2: Let $\beta'$ be a lightlike vector. Since $\langle \alpha', \beta' \rangle = 0$, there exists a smooth function $k$ such that $\beta' = k\alpha'$. Thus, a lightlike non-cylindrical ruled surface is parametrized by:

$$X(u,v) = \alpha(u) + v\beta(u), \tag{21}$$

where $\alpha$ and $\beta$ satisfy the condition (20).

Next, we consider $G = \langle \beta, \beta \rangle = 0$, since $\beta \neq \mathbf{0}$, a director vector $\beta$ must be lightlike. Furthermore, since $\langle \alpha', \beta \rangle = 0$, $\alpha$ is a spacelike curve or a lightlike curve.

Case 1: If $\alpha$ is a spacelike curve, then a lightlike non-cylindrical ruled surface is parametrized by:

$$X(u,v) = \alpha(u) + v\beta(u), \tag{22}$$

where $\alpha$ is a spacelike curve and $\beta$ is a lightlike vector.

Case 2: Let $\alpha$ be a lightlike curve. Then, there exists a smooth function $k$ such that $\beta' = k\alpha'$, and a lightlike ruled surface as a tangent developable surface is parametrized by:

$$X(u,v) = \alpha(u) + vk\alpha'(u), \tag{23}$$

where $\alpha$ and $\alpha''$ are a lightlike curve and a spacelike vector, respectively.

In [5], the authors gave the following:

**Definition 3.** *A surface evolution $X(u,v,t)$ and its flow $\frac{\partial X}{\partial t}$ are said to be inextensible if the coefficients of the first fundamental form of the surface satisfy:*

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0.$$

This definition states that the surface $X(u,v,t)$ is, for all time $t$, the isometric image of the original surface $X(u,v,t_0)$ defined at some initial time $t_0$.

Now, we study inextensible flows of a lightlike tangent developable surface in Minkowski three-space.

Consider a lightlike tangent developable surface parametrized by:

$$X(u,v) = \alpha(u) + v\alpha'(u), \tag{24}$$

where $\alpha$ is a lightlike curve. Suppose that the parameter $u$ is a pseudo-arc length of $\alpha$. In this case, we get $E = v^2||\alpha''||^2$ and $F = G = 0$.

Thus, we have:

**Theorem 9.** *Let $X(u,v)$ be a lightlike tangent developable surface given by (24). The surface evolution $X(u,v,t) = \alpha(u,t) + v\alpha'(u,t)$ is inextensible if and only if:*

$$\frac{\partial}{\partial t}||\alpha''||^2 = 0.$$

As a consequence, we have the following results:

**Theorem 10.** *Let $X(u,v,t) = \alpha(u,t) + v\alpha'(u,t)$ be a surface evolution of a lightlike tangent developable surface given by (24) in $\mathbb{R}^3_1$. Then, we have the following statements:*

(1) *$\alpha(u,t)$ is an inextensible evolution of a lightlike curve $\alpha(u)$ in $\mathbb{R}^3_1$.*

(2) *An inextensible evolution of a lightlike tangent developable surface can be completely characterized by the inextensible evolutions of a lightlike curve $\alpha(u)$ in $\mathbb{R}^3_1$.*

**Proof.** In fact, $0 = \frac{\partial}{\partial t}||\alpha''||^2 = 2||\alpha''||\frac{\partial}{\partial t}||\alpha''||$ and $\alpha'' \neq \mathbf{0}$, and we get $\frac{\partial}{\partial t}||\alpha''|| = 0$; it implies $\frac{\partial}{\partial t}||\alpha''||^{\frac{1}{2}} = 0$. This means that $\alpha(u,t)$ satisfies the condition for Definition 2. □

**Theorem 11.** *Let $X(u,v,t) = \alpha(u,t) + v\alpha'(u,t)$ be a surface evolution of a lightlike tangent developable surface given by (24) in $\mathbb{R}^3_1$, and $\frac{\partial \alpha}{\partial t} = f_1\mathbf{t} + f_2\mathbf{n} + f_3\mathbf{g}$, where $\mathbf{t}, \mathbf{n}, \mathbf{g}$ are the Darboux frames along a lightlike curve $\alpha$ on a lightlike surface. If the surface evolution $X(u,v,t)$ is inextensible, then $f_1, f_2, f_3$ satisfy Equation (19).*

## 6. Conclusions

We study an inextensible flow of a spacelike or a lightlike curve on a lightlike surface in Minkowski three-space and investigate a time evolution of the Darboux frame $\{\mathbf{t}, \mathbf{n}, \mathbf{g}\}$ (see Theorems 3 and 7) and the functions $\kappa_n, \kappa_g$ and $\tau_g$ (see Theorems 4 and 8). Furthermore, in Theorems 2 and 6, we give a necessary and sufficient condition of inextensible flows of a spacelike curve and a lightlike curve on a lightlike surface in terms of a partial differential equation involving the curvatures of the curve on a lightlike surface. Finally, we completely classify lightlike ruled surfaces in Minkowski three-space and characterize an inextensible evolution of a lightlike curve on a lightlike tangent developable surface (see Theorems 9 and 10).

**Author Contributions:** D.W.Y. gave the idea of inextensible flows of a spacelike curve and a lightlike curve on a lightlike surface. Z.K.Y. checked and polished the draft.

**Funding:** The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07046979).

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Chirikjian, G.S.; Burdick, J.W. Kinematics of hyper-redundant manipulation. In Proceedings of the ASME Mechanisms Conference, Chicago, IL, USA, 16–19 September 1990; pp. 391–396.
2. Desbrun, M.; Cani-Gascuel, M.-P. Active implicit surface for animation. In *Graphics Interface*; The Canadian Information Processing Society: Mississauga, ON, Canada, 1998; pp. 143–150.
3. Kass, M.; Witkin, A.; Terzopoulos, D. Snakes: Active contour models. In Proceedings of the 1st International Conference on Computer Vision, London, UK, 8–11 June 1987; pp. 259–268.
4. Schief, W.K.; Rogers, C. Binormal motion of curves of constant curvature and torsion. generation of soliton surfaces. *Proc. R. Soc. Lond. A* **1999**, *455*, 3163–3188. [CrossRef]
5. Kwon, D.Y.; Park, F.C. Inextensible flows of curves and developable surfaces. *Appl. Math. Lett.* **2005**, *18*, 1156–1162. [CrossRef]
6. Mohamed, S.G. Binormal motions of inextensible curves in de-sitter space $\mathbb{S}^{2,1}$. *J. Egypt. Math. Soc.* **2017**, *25*, 313–318. [CrossRef]
7. Andrews, B. Classification of limiting shapes for isotropic curve flows. *J. Am. Math. Soc.* **2003**, *16*, 443–459. [CrossRef]
8. Gurbuz, N. Inextensible flows of spacelike, timelike and null curves. *Int. J. Contemp. Math. Sci.* **2009**, *4*, 1599–1604.
9. Hussien, R.A.; Mohamed, S.G. Generated surfaces via inextensible flows of curves in $\mathbb{R}^3$. *J. Appl. Math.* **2016**, *2016*, 6178961. [CrossRef]
10. Yeneroglu, M. On new characterization of inextensible flows of space-like curves in de Sitter space. *Open Math.* **2016**, *14*, 946–954. [CrossRef]

11. Zhu, X.-P. Asymptotic behavior of anisotropic curve flows. *J. Differ. Geom.* **1998**, *48*, 225–274. [CrossRef]
12. Gurbuz, N. Three clasess of non-lightlike curve evolution according to Darboux frame and geometric phase. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15*, 1850023. [CrossRef]