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Article

Pinching Theorems for a Vanishing C-Bochner Curvature Tensor

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Abstract: The main purpose of this article is to construct inequalities between a main intrinsic invariant (the normalized scalar curvature) and an extrinsic invariant (the Casorati curvature) for some submanifolds in a Sasakian manifold with a zero C-Bochner tensor.

Keywords: C-Bochner tensor; generalized normalized δ-Casorati curvature; Sasakian manifold; slant; invariant; anti-invariant

1. Introduction

Bochner [1] introduced the Bochner tensor in Kähler manifolds by analogy to the Weyl conformal curvature tensor. The Bochner tensor is equal to the 4-th order Chern–Moser curvature tensor in CR-manifolds by Webster [2]. In contact manifolds, the Bochner tensor was reinterpreted by Matsumoto and Chuman [3] as a C-Bochner curvature tensor in Sasakian manifolds. They showed that a Sasakian space form is a space with a vanishing C-Bochner curvature tensor. A Sasakian manifold with a non-constant φ-sectional curvature and a vanishing C-Bochner curvature tensor was constructed by Kim [4]. Tano showed that the C-Bochner curvature tensor is invariant in terms of D-homothetic deformations [5].

On the other hand, F. Casorati introduced a new extrinsic invariant of submanifolds in a Riemannian manifold, called the Casorati curvature. This curvature is defined as the normalized square of the length of the second fundamental form ([6,7]). Moreover, there are very interesting optimizations involving Casorati curvatures, proved in [8–19] for various basic submanifolds in different spaces (real, complex, and quaternionic space forms) with several connections.

In our paper, we investigate new optimal inequalities involving Casorati curvatures for some submanifolds of a Sasakian manifold with a zero C-Bochner curvature tensor and characterize those submanifolds for which the equalities hold.

2. Preliminaries

In this section, we recall some results on almost contact manifolds and give a brief review of basic facts of C-Bochner curvature tensor.

A manifold $\overline{M} = (\overline{M}, \varphi, \xi, \eta, \overline{g})$ is called *an almost contact metric manifold* if there exist structure tensors $(\varphi, \xi, \eta, \overline{g})$, where φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form, and $\overline{g}$ is the Riemannian metric on $\overline{M}$ satisfying [20]

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1$$

$$\varphi^2 = -I + \eta \otimes \xi, \quad \text{and} \quad \overline{g}(\varphi X, \varphi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y)$$

where $I : T\overline{M} \longrightarrow T\overline{M}$ is the identity endomorphism, and X, Y are vector fields on $\overline{M}$. In particular, if $\overline{M}$ is Sasakian [21], then we have

$$(\nabla_X \varphi)Y = -\overline{g}(X,Y)\xi + \eta(Y)X \quad \text{and} \quad \overline{\nabla}_X \xi = \varphi X$$

where $\overline{\nabla}$ is the Levi–Civita connection on $\overline{M}$.

Let M^n be an n-dimensional submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$. If ∇ is the induced covariant differentiation on M of the Levi–Civita connection $\overline{\nabla}$ on $\overline{M}$, then we have the Gauss and Weingarten formulas:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \forall X, Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \forall X \in \Gamma(TM), \forall N \in \Gamma(T^{\perp} M)$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on $T^{\perp}M$, and A_N is the shape operator of M with respect to a normal section N. If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇, respectively, then we have the Gauss equation:

$$\begin{aligned} \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \overline{g}(h(X,W), h(Y,Z)) \\ - \overline{g}(h(X,Z), h(Y,W)) \end{aligned} \tag{1}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

Let M^n be an n-dimensional Riemannian submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$. A plane section $\pi \subset T_pM,\ p \in M$ of a Sasakian manifold $\overline{M}$ is called a φ-section if $\pi = span\{X, \varphi X\}$ for $X \in \Gamma(TM)$ orthogonal to ξ at each point $p \in M$. The sectional curvature $K(\pi)$ with respect to a φ-section π is called a φ-sectional curvature. If $\{e_1, ..., e_n, \xi\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, ..., e_m\}$ is an orthonormal basis of $T_p^{\perp}M$, then the scalar curvature τ and the normalized scalar curvature ρ at p are defined, respectively, as

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) \qquad \rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

and we also set

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha),\ i, j \in \{1, ..., n\},\ \alpha \in \{n+1, ..., m\}.$$

It is well-known that an intrinsic invariant of the submanifold M in $\overline{M}$ is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

and the squared norm of h over the dimension n is denoted by $\mathcal{C}$, called the Casorati curvature of the submanifold M. That is,

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha} \right)^2.$$

The submanifold M is said to be *invariantly quasi-umbilical* if there exist $m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, ..., \xi_m$ such that the shape operator with respect to each direction ξ_α has an eigenvalue of multiplicity $n - 1$ and the distinguished eigendirection is the same for each ξ_α.

Suppose now that L is a s-dimensional subspace of T_pM, and $s \geq 2$. Let $\{e_1, ..., e_s\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the s-plane section L is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{s} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{s} \left(h_{ij}^{\alpha}\right)^2.$$

The normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ of the submanifold M^n are given by

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\left[\widehat{\delta}_c(n-1)\right]_p = 2\mathcal{C}_p - \frac{2n-1}{2n}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}.$$

The generalized normalized δ-Casorati curvatures $\delta_C(t; n-1)$ and $\widehat{\delta}_C(t; n-1)$ of the submanifold M^n are defined for any positive real number $t \neq n(n-1)$ as

$$[\delta_C(t; n-1)]_p = t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$\left[\widehat{\delta}_C(t; n-1)\right]_p = t\mathcal{C}_p - \frac{(n-1)(n+t)(t-n^2+n)}{nt}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

if $t > n^2 - n$.

The C-Bochner curvature tensor [22] on a Sasakian manifold is defined by

$$\begin{aligned}
B(X,Y)Z = {} & \overline{R}(X,Y)Z + \frac{1}{2n+4}\{\overline{g}(X,Z)QY - Ric(Y,Z)X \\
& - \overline{g}(Y,Z)QX + Ric(X,Z)Y + \overline{g}(\varphi X,Z)Q\varphi Y \\
& - Ric(\varphi Y,Z)\varphi X - \overline{g}\varphi Y,Z)Q\varphi X + Ric(\varphi X,Z)\varphi Y \\
& + 2Ric(\varphi X,Y)\varphi Z + 2\overline{g}(\varphi X,Y)Q\varphi Z + \eta(Y)\eta(Z)QX \\
& - \eta(Y)Ric(X,Z)\xi + \eta(X)Ric(Y,Z)\xi - \eta(X)\eta(Z)QY\} \\
& - \frac{D+2n}{2n+4}\{\overline{g}(\varphi X,Z)\varphi Y - \overline{g}(\varphi Y,Z)\varphi X + 2\overline{g}(\varphi X,Y)\varphi Z\} \\
& + \frac{D+2n}{2n+4}\{\eta(Y)\overline{g}(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\
& - \eta(X)\overline{g}(Y,Z)\xi\} - \frac{D-4}{2n+4}\{\overline{g}(X,Z)Y - \overline{g}(Y,Z)X\}
\end{aligned} \tag{2}$$

for all $X, Y, Z, W \in \Gamma(T\overline{M})$, where $D = \frac{\tau+2n}{2n+2}$, and $\overline{R}$, Ric, and Q are the Riemannian curvature tensor, the Ricci tensor, and the Ricci operator, respectively. If the C-Bochner curvature tensor vanishes, from Equation (5), we have

$$\begin{aligned}\overline{R}(X,Y,Z,W) = &-\frac{1}{2n+4}\{\overline{g}(X,Z)Ric(Y,W) - Ric(Y,Z)\overline{g}(X,W)\\&- \overline{g}(Y,Z)Ric(X,W) + Ric(X,Z)\overline{g}(Y,W)\\&+ \overline{g}(\varphi X,Z)Ric(\varphi Y,W) - Ric(\varphi Y,Z)\overline{g}(\varphi X,W)\\&- \overline{g}(\varphi Y,Z)Ric(\varphi X,W) + Ric(\varphi X,Z)\overline{g}(\varphi Y,W)\\&+ 2Ric(\varphi X,Y)\overline{g}(\varphi Z,W) + 2\overline{g}(\varphi X,Y)Ric(\varphi Z,W)\\&+ \eta(Y)\eta(Z)Ric(X,W) - \eta(Y)\eta(W)Ric(X,Z)\\&+ \eta(X)\eta(W)Ric(Y,Z) - \eta(X)\eta(Z)Ric(Y,W)\}\\&+ \frac{D+2n}{2n+4}\{\overline{g}(\varphi X,Z)\overline{g}(\varphi Y,W) - \overline{g}(\varphi Y,Z)\overline{g}(\varphi X,W)\\&+ 2\overline{g}(\varphi X,Y)\overline{g}(\varphi Z,W)\} - \frac{D+2n}{2n+4}\{\eta(Y)\eta(W)\overline{g}(X,Z)\\&- \eta(Y)\eta(Z)\overline{g}(X,W) + \eta(X)\eta(Z)\overline{g}(Y,W)\\&- \eta(X)\eta(W)\overline{g}(Y,Z)\} + \frac{D-4}{2n+4}\{\overline{g}(X,Z)\overline{g}(Y,W)\\&- \overline{g}(Y,Z)\overline{g}(X,W)\}\end{aligned} \quad . \tag{3}$$

Now, we recall some definitions from literature on submanifolds.

Definition 1. *Let $(\overline{M}, \varphi, \xi, \eta)$ be an almost contact metric manifolds and M be a submanifold isometrically immersed in $\overline{M}$ tangent to the structure vector field ξ. Then M is said to be invariant (anti-invariant) if $\varphi(T_pM) \subseteq T_pM$ $\left(\varphi(T_pM) \subset T_p^{\perp}M\right)$ for every $p \in M$, where T_pMs denote the tangent space of M at the point p. Moreover, M is called a slant submanifold if for all non-zero vector $U \in T_pM$ at a point p, and the angle of $\theta(U)$ between φU and T_pM is constant (i.e., it does not depend on the choice of $p \in M$ and $U \in \Gamma\left(T_pM\right) - < \xi(p) >$).*

Let M^n be an n-dimensional submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$. For $X \in \Gamma(TM)$, we can write $\varphi X = PX + QX$, where PX and QX are the tangential and the normal components of φX, respectively. The submanifold is said to be *an anti-invariant (invariant) submanifold* if $P = 0 (Q = 0$, respectively$)$. The squared norm of P at $p \in M$ is defined as

$$||P||^2 = \sum_{i,j=1}^{n} \overline{g}^2(\varphi e_i, e_j)$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_pM. The structure vector field ξ can be decomposed as

$$\xi = \xi^{\top} + \xi^{\perp}$$

where $\xi^{\top}$ and $\xi^{\perp}$ are the tangential and the normal components of ξ, respectively.

The following constrained extremum problem plays a key role in the proof of our theorems.

Lemma 1. *[23] Let*

$$\Gamma = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = k\}$$

be a hyperplane of $\mathbb{R}^n$, *and* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *a quadratic form given by*

$$f(x_1, x_2, \cdots, x_n) = a \sum_{i=1}^{n-1} (x_i)^2 + b\,(x_n)^2 - 2 \sum_{1 \le i < j \le n} x_i x_j, \qquad a > 0,\ b > 0.$$

Then, f has the global extreme at the following point:

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{k}{a+1}, \quad x_n = \frac{k}{b+1} = \frac{k(n-1)}{(a+1)b} = (a-n+2)\frac{k}{a+1}$$

provided that

$$b = \frac{n-1}{a-n+2}$$

by the constrained extremum problem.

3. Inequalities Involving a Vanishing C-Bochner Curvature Tensor

Let M be a submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. Let $p \in M$ and the set $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_m\}$ be orthonormal bases of T_pM and $T_p^{\perp}M$, respectively. From Equation (3), we have

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{R}(e_i, e_j, e_j, e_i) = {} & \frac{7n^2 + n - 8 + 2(n-1)||\xi^{\perp}||^2}{4(n+1)(n+2)}\tau \\ & - \frac{3}{n+2} \sum_{i,j=1}^{n} \overline{g}(\varphi e_i, e_j) Ric(e_i, \varphi e_j) \\ & + \frac{n(n-1)(2n+3)}{(n+1)(n+2)}||\xi^{\perp}||^2 - \frac{3n+4}{2(n+1)(n+2)}\end{aligned} \quad . \tag{4}$$

Combining Equation (1) and Equation (4), we obtain

$$\begin{aligned}2\tau = {} & n^2||H||^2 - n\mathcal{C} + \frac{7n^2 + n - 8 + 2(n-1)||\xi^{\perp}||^2}{4(n+1)(n+2)}\tau \\ & - \frac{3}{n+2} \sum_{i,j=1}^{n} \overline{g}(\varphi e_i, e_j) Ric(e_i, \varphi e_j) \\ & + \frac{n(n-1)(2n+3)}{(n+1)(n+2)}||\xi^{\perp}||^2 - \frac{3n+4}{2(n+1)(n+2)}\end{aligned} \quad . \tag{5}$$

We now consider a quadratic polynomial in the components of the second fundamental form:

$$\begin{aligned}\mathcal{P} = {} & t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) - \frac{n^2 + 23n + 24 - 2(n-1)||\xi^{\perp}||^2}{4(n+1)(n+2)}\tau \\ & - \frac{3}{n+2} \sum_{i,j=1}^{n} \overline{g}(\varphi e_i, e_j) Ric(e_i, \varphi e_j) + \frac{n(n-1)(2n+3)}{(n+1)(n+2)}||\xi^{\perp}||^2 - \frac{3n+4}{2(n+1)(n+2)}\end{aligned}$$

where L is a hyperplane of T_pM. Without loss of generality, we may assume that $L = span\{e_1, ..., e_{n-1}\}$. Then we derive

$$\begin{aligned}
\mathcal{P} = & \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \left[\frac{n^2 + n(t-1) - 2t}{r} (h_{ii}^{\alpha})^2 + \frac{2(n+t)}{n} (h_{in}^{\alpha})^2 \right] \\
& + \sum_{\alpha=n+1}^{m} \left[\frac{2(n+t)(n-1)}{t} \sum_{1=i<j}^{n-1} \left(h_{ij}^{\alpha}\right)^2 - 2 \sum_{1=i<j}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{t}{n} (h_{nn}^{\alpha})^2 \right]. \\
\geq & \sum_{\alpha=n+1}^{m} \left[\sum_{i=1}^{n-1} \frac{n^2 + n(t-1) - 2t}{t} (h_{ii}^{\alpha})^2 - 2 \sum_{1=i<j}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{t}{n} (h_{nn}^{\alpha})^2 \right]
\end{aligned} \tag{6}$$

For $\alpha = n+1, \cdots, m$, we consider the quadratic form $f_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f_\alpha \left(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}\right) = \frac{n^2 + n(t-1) - 2t}{t} \sum_{i=1}^{n-1} (h_{ii}^{\alpha})^2 - 2 \sum_{i<j=1}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{t}{n} (h_{nn}^{\alpha})^2. \tag{7}$$

We then have the constrained extremum problem

$$\min f_\alpha$$

$$\text{subject to } F^{\alpha} : h_{11}^{\alpha} + \cdots + h_{nn}^{\alpha} = c^{\alpha}$$

where c^{α} is a real constant. Comparing Equation (7) with the quadratic function in Lemma 1, we get

$$a = \frac{n^2 + n(t-1) - 2t}{t}, \qquad b = \frac{t}{n}.$$

Therefore, we have the critical point $\left(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}\right)$, given by

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1\,n-1}^{\alpha} = \frac{tc^{\alpha}}{(n+t)(n-1)}, \qquad h_{nn}^{\alpha} = \frac{nc^{\alpha}}{n+t},$$

which is a global minimum point by Lemma 1. Moreover, $f_\alpha \left(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}\right) = 0$. Therefore, we have

$$\mathcal{P} \geq 0,$$

which implies

$$\begin{aligned}
\frac{n^2 + 23n + 24 - 2(n-1)||\xi^{\perp}||^2}{4(n+1)(n+2)} \tau \leq & \; t\mathcal{C} + \frac{(n-1)(n+t)(n^2 - n - t)}{nt} \mathcal{C}(L) \\
& - \frac{3}{n+2} \sum_{i,j=1}^{n} \overline{g}(\varphi e_i, e_j) Ric(e_i, \varphi e_j) \\
& + \frac{n(n-1)(2n+3)}{(n+1)(n+2)} ||\xi^{\perp}||^2 - \frac{3n+4}{2(n+1)(n+2)}
\end{aligned}.$$

Therefore, we derive

$$
\begin{aligned}
\rho \leq\ & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}\left(t\mathcal{C}+\frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L)\right) \\
& -\frac{24(n+1)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}\sum_{i,j=1}^{n}\overline{g}(\varphi e_i,e_j)Ric(e_i,\varphi e_j) \\
& +\frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2}-\frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}
.
$$

Summing up, we obtain the following theorem:

Theorem 1. *Let M be a submanifold of a Sasakian manifold $(\overline{M},\overline{g},\varphi,\xi,\eta)$ with a vanishing C-Bochner curvature tensor. When $0 < t < n^2-n$, the generalized normalized δ-Casorati curvature $\delta_C(t,n-1)$ on M^n satisfies*

$$
\begin{aligned}
\rho \leq\ & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}\delta_C(t,n-1) \\
& -\frac{24(n+1)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}\sum_{i,j=1}^{n}\overline{g}(\varphi e_i,e_j)Ric(e_i,\varphi e_j) \\
& +\frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2}-\frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}
.
$$

Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with the trivial normal connection in a Sasakian manifold $(\overline{M},\overline{g},\varphi,\xi,\eta)$, such that the shape operators $A_r \equiv A_{\xi_r}$ and $r \in \{n+1,\cdots,m\}$ take the following forms:

$$
A_{n+1}=\begin{pmatrix}
a & 0 & 0 & \dots & 0 & 0 \\
0 & a & 0 & \dots & 0 & 0 \\
0 & 0 & a & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & a & 0 \\
0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}a
\end{pmatrix},\ A_{n+2}=\cdots=A_m=0 \tag{8}
$$

with respect to a suitable orthonormal tangent frame $\{\xi_1,\cdots,\xi_n\}$ and a normal orthonormal frame $\{\xi_{n+1},\cdots,\xi_m\}$.

When a submanifold M is Einstein of a Sasakian manifold $(\overline{M},\overline{g},\varphi,\xi,\eta)$, the Ricci curvature tensor $\rho(X,Y)=\lambda g(X,Y)$ for $X,Y \in \Gamma(TM)$, where λ is some constant. Therefore, we have the following corollary:

Corollary 1. *Let M be an Einstein submanifold of a Sasakian manifold $(\overline{M},\overline{g},\varphi,\xi,\eta)$ with a vanishing C-Bochner curvature tensor. Then, for a Ricci curvature λ, we obtain*

$$
\begin{aligned}
\rho \leq\ & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}\delta_C(t,n-1) \\
& +\frac{24(n+1)||P||^2\lambda}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)} \\
& +\frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2}-\frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}
.
$$

Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with the trivial normal connection in a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$, such that with respect to a suitable orthonormal tangent frame $\{\xi_1, \cdots, \xi_n\}$ and a normal orthonormal frame $\{\xi_{n+1}, \cdots, \xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$ and $r \in \{n+1, \cdots, m\}$ take the form of Equation (8).

For a slant submanifolds ($\overline{g}(\varphi e_i, e_j) = \cos\theta$ with the slant angle θ) of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor, we have following corollaries.

Corollary 2. *Let M be a slant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}\rho \leq & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}\delta_C(t, n-1) \\ & + \frac{24(n+1)\cos\theta}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}\sum_{i,j=1}^{n} Ric(e_i, \varphi e_j) + \frac{4(2n+3)||\xi^\perp||^2}{n^2+23n+24-2(n-1)||\xi^\perp||^2} \\ & - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}\end{aligned}$$

where θ is a slant function. Moreover, the equality case holds if and only if, with respect to a suitable frames $\{e_1, ..., e_n\}$ on M and $\{e_{n+1}, ..., e_m\}$ on $T_p^\perp M$, $p \in M$, the components of h satisfy

$$\begin{aligned} h^\alpha_{11} = h^\alpha_{22} = \cdots = h^\alpha_{n-1\ n-1} = \tfrac{t}{n(n-1)} h^\alpha_{nn}, \qquad & \alpha \in \{n+1, \cdots, m\}, \\ h^\alpha_{ij} = 0, \quad i, j \in \{1, 2, \cdots, n\} (i \neq j), \quad & \alpha \in \{n+1, \cdots, m\}. \end{aligned}$$

When the slant angle is zero in Corollary 2, we have the following corollary:

Corollary 3. *Let M be an invariant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}\rho \leq & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}\delta_C(t, n-1) \\ & + \frac{4(6n^2-3n-10)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)} + \frac{4(2n+3)||\xi^\perp||^2}{n^2+23n+24-2(n-1)||\xi^\perp||^2}.\end{aligned}$$

Moreover, the equality case holds if and only if, with respect to a suitable frames $\{e_1, ..., e_n\}$ on M and $\{e_{n+1}, ..., e_m\}$ on $T_p^\perp M$, $p \in M$, the components of h satisfy

$$\begin{aligned} h^\alpha_{11} = h^\alpha_{22} = \cdots = h^\alpha_{n-1\ n-1} = \tfrac{t}{n(n-1)} h^\alpha_{nn}, \qquad & \alpha \in \{n+1, \cdots, m\}, \\ h^\alpha_{ij} = 0, \quad i, j \in \{1, 2, \cdots, n\} (i \neq j), \quad & \alpha \in \{n+1, \cdots, m\}. \end{aligned}$$

When the slant angle is $\frac{\pi}{2}$ in Corollary 1, we have the following corollary:

Corollary 4. *Let M be an anti-invariant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}\rho \leq & \frac{8(n+1)(n+2)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}\delta_C(t, n-1) \\ & + \frac{4(2n+3)||\xi^\perp||^2}{n^2+23n+24-2(n-1)||\xi^\perp||^2} - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^\perp||^2\right)}.\end{aligned}$$

Moreover, the equality case holds if and only if, with respect to a suitable frames $\{e_1, ..., e_n\}$ on M and $\{e_{n+1}, ..., e_m\}$ on $T_p^{\perp}M$, $p \in M$, the components of h satisfy

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1\ n-1}^{\alpha} = \tfrac{t}{n(n-1)} h_{nn}^{\alpha}, \qquad \alpha \in \{n+1, \cdots, m\},$$
$$h_{ij}^{\alpha} = 0, \quad i, j \in \{1, 2, \cdots, n\} (i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.$$

Remark 1. *In the case for $t > n^2 - n$, the methods of finding the above inequailities is analogous. Thus, we leave these problems for readers.*

Taking $t = \frac{n(n-1)}{2}$ in $\delta_C(t, n-1)$, we have the following relation:

$$\left[\delta_C\left(\frac{n(n-1)}{2}; n-1\right)\right]_p = n(n-1)\left[\delta_C(n-1)\right]_p$$

in any point $p \in M$. Therefore, we have following optimal inequalities for the normalized δ-Casorati curvature $\delta_C(n-1)$.

Corollary 5. *Let M be a submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. The normalized δ-Casorati curvature $\delta_C(n-1)$ on M^n satisfies*

$$\begin{aligned}
\rho \leq & \frac{8(n+1)(n+2)}{(n^2+23n+24-2(n-1)||\xi^{\perp}||^2)}\delta_C(n-1) \\
& - \frac{24(n+1)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)} \sum_{i,j=1}^{n} \overline{g}(\varphi e_i, e_j) Ric(e_i, \varphi e_j) \\
& + \frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2} - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}.$$

Corollary 6. *Let M be an Einstein submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. Then, for a Ricci curvature λ, we obtain*

$$\begin{aligned}
\rho \leq & \frac{8(n+1)(n+2)}{(n^2+23n+24-2(n-1)||\xi^{\perp}||^2)}\delta_C(n-1) \\
& + \frac{24(n+1)||P||^2\lambda}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)} + \frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2}. \\
& - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}$$

Corollary 7. *Let M be a slant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}
\rho \leq & \frac{8(n+1)(n+2)}{(n^2+23n+24-2(n-1)||\xi^{\perp}||^2)}\delta_C(n-1) \\
& + \frac{24(n+1)\cos\theta}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)} \sum_{i,j=1}^{n} Ric(e_i, \varphi e_j) \\
& + \frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2} - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}
\end{aligned}$$

where θ is a slant function.

Corollary 8. *Let M be an invariant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}\rho \leq & \frac{8(n+1)(n+2)}{(n^2+23n+24-2(n-1)||\xi^{\perp}||^2)}\delta_C(n-1) \\ & + \frac{4(6n^2-3n-10)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)} + \frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2}.\end{aligned}$$

Corollary 9. *Let M be an anti-invariant submanifold of a Sasakian manifold $(\overline{M}, \overline{g}, \varphi, \xi, \eta)$ with a vanishing C-Bochner curvature tensor. We then obtain*

$$\begin{aligned}\rho \leq & \frac{8(n+1)(n+2)}{(n^2+23n+24-2(n-1)||\xi^{\perp}||^2)}\delta_C(n-1) \\ & + \frac{4(2n+3)||\xi^{\perp}||^2}{n^2+23n+24-2(n-1)||\xi^{\perp}||^2} - \frac{4(3n+4)}{n(n-1)\left(n^2+23n+24-2(n-1)||\xi^{\perp}||^2\right)}.\end{aligned}$$

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