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Degenerate Daehee Numbers of the Third Kind

Sung-Soo Pyo [1], Taekyun Kim [2] and Seog-Hoon Rim [3,*]

1 Department of Mathematics Education, Silla University, Busan 13557, Korea; ssoopyo@gmail.com
2 Department of Mathematics, Kwangwoon University, Seoul 01897, Korea; tkkim@kw.ac.kr
3 Department of Mathematics Education, Kyungpook National University, Taegu 41566, Korea
* Correspondence: shrim@knu.ac.kr

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Abstract: In this paper, we define new Daehee numbers, the degenerate Daehee numbers of the third kind, using the degenerate log function as generating function. We obtain some identities for the degenerate Daehee numbers of the third kind associated with the Daehee, degenerate Daehee, and degenerate Daehee numbers of the second kind. In addition, we derive a differential equation associated with the degenerate log function. We deduce some identities from the differential equation.

Keywords: degenerate log function; degenerate Daehee numbers of the third kind; nonlinear differential equation

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1. Introduction

After Carlitz [1,2], many mathematicians have studied degenerate functions and numbers (see [3–11]). They mainly used $(1+\lambda t)^{\frac{1}{\lambda}}$ instead of e^t to degenerate polynomials and numbers. In [7], T. Kim and D.S. Kim called $(1+\lambda t)^{\frac{1}{\lambda}}$ the degenerate exponential function and expressed it as e_λ^t. They also presented the degenerate gamma function and degenerate Laplace transformation using e_λ^t. In [12], the authors introduced four degenerate versions of Cauchy numbers. In the degenerate Cauchy numbers of the first and second kind, $(1+\lambda t)^{\frac{1}{\lambda}}$ was used instead of e^t. We call this *degenerate based on the exponential sense*.

In accordance with the exponential sense, $\log(1+\lambda t)^{\frac{1}{\lambda}}$ can be used for t to study degenerate numbers and polynomials. It is natural to think of a degenerate log function as the inverse function of the degenerate exponential function. The degenerate log function, denoted by $\log_\lambda(t)$, is defined by the generating function to be

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1) \text{ (see [9]).} \tag{1}$$

The Cauchy numbers or the second kind Bernoulli numbers, denoted by C_n, are introduced in [5,8,12,13]. Recently, in [8], T. Kim introduced the degenerate Cauchy numbers.

In this case, the author used $\log_\lambda(t)$ instead of $\log t$ for degenerating. We call this *degenerate based on the log sense*.

As is well known, the Bernoulli numbers, denoted by B_n, are defined by the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2}$$

The Bernoulli numbers, which started with a study on the sum of the power series, has many relationships with other special numbers [2,3,13–21]. In [1], the degenerate Bernoulli numbers are presented as follows:

$$\frac{t}{e_\lambda(t) - 1} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!}. \tag{3}$$

The Daehee numbers, denoted by D_n, are defined by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{(see [4,10,11,14,15,20,22])}. \tag{4}$$

The *degenerate Daehee numbers* $D_\lambda(n)$, are introduced as $\frac{\lambda\log(1+\frac{1}{\lambda}\log(1+\lambda t))}{\log(1+\lambda t)} = \sum_{n=0}^{\infty} D_\lambda(n)\frac{t^n}{n!}$ (see [4]). If λ goes to 0, then $D_\lambda(n)$ converges to D_n. These degenerate Daehee numbers $D_\lambda(n)$ are degenerate based on the exponential sense. Recently, D. S. Kim et al. presented degenerate Daehee polynomials and numbers of the second kind as follows [23].

$$\frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_{\lambda,2}(n,x)\frac{t^n}{n!}. \tag{5}$$

When $x=0$, $D_{\lambda,2}(n) = D_{\lambda,2}(n,0)$ are called the degenerate Daehee numbers of the second kind. These degenerate numbers are also based on the exponential sense.

It is natural to think about degenerate Daehee numbers based on the log sense. We define *the degenerate Daehee numbers of the third kind*, denoted by $D_{\lambda,3}(n)$, as follows.

$$\frac{\log_\lambda(1+t)}{t} = \frac{\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)}{t} = \sum_{n=0}^{\infty} D_{\lambda,3}(n)\frac{t^n}{n!}. \tag{6}$$

We note that $\lim_{\lambda\to 0} D_{\lambda,3}(n) = D_n$ for each n.

In this paper, we define the degenerate Daehee numbers based on the log sense. We obtain some identities which are connected with the Daehee, the degenerate Daehee, and the degenerate Daehee numbers of the second kind. Additionally, we deduce a differential equation using the degenerate log function, and we derive some identities related to the degenerate Daehee numbers from this differential equation.

2. Degenerate Daehee Numbers of the Third Kind

From now on, for any real x and non-negative integer n, we denote $(x)_n$ for falling factorial $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ and $(x)_0 = 1$. We use $S_1(m,n)$ and $S_2(m,n)$ to denote the Stirling number of the first kind and the second kind, respectively. From the definition of the degenerate Daehee numbers of the third kind, we get

$$\begin{aligned} \frac{\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)}{t} &= \frac{1}{\lambda t}\sum_{n=1}^{\infty}(\lambda)_n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(\lambda)_{n+1}}{\lambda(n+1)}\frac{t^n}{n!} \end{aligned}. \tag{7}$$

Equation (7) yields the following.

$$D_{\lambda,3}(n) = \frac{(\lambda)_{n+1}}{\lambda(n+1)} = \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n)}{\lambda(n+1)}. \tag{8}$$

From Equation (8), it is easy to show that $\lim_{\lambda\to 0} D_{\lambda,3}(n) = D_n$.

Now let us investigate the relationship between the Daehee numbers and the degenerate Daehee numbers.

$$\begin{aligned}
\frac{\frac{1}{\lambda}\left((1+t)^{\lambda}-1\right)}{t} &= \frac{\log\left(e^{\frac{1}{\lambda}\left((1+t)^{\lambda}-1\right)}-1+1\right)}{t} = \sum_{l=0}^{\infty} D_l \frac{\left(e^{\frac{1}{\lambda}\left((1+t)^{\lambda}-1\right)}-1\right)^l}{l!} \\
&= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m,l)\frac{1}{m!}\frac{1}{\lambda^m}\left((1+t)^{\lambda}-1\right)^m \\
&= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m,l)\frac{1}{m!}\frac{1}{\lambda^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\sum_{n=0}^{\infty}(k\lambda)_n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{D_l S_2(m,l)(k\lambda)_n}{m!\lambda^m}\frac{t^n}{n!}
\end{aligned}. \tag{9}$$

Equation (9) yields a relationship between the Daehee numbers and the degenerate Daehee numbers.

Theorem 1. *For any nonnegative integer n,*

$$D_{\lambda,3}(n) = \sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{D_l S_2(m,l)(k\lambda)_n}{m!\lambda^m}.$$

From the definition of the Daehee numbers, Equation (4), and the degenerate Daehee numbers, Equation (6), we get the following.

$$\begin{aligned}
\frac{\log(1+t)}{t} &= \frac{\frac{1}{\lambda}\left\{\left(1+\left((\lambda\log(1+t)+1)^{\frac{1}{\lambda}}-1\right)\right)^{\lambda}-1\right\}}{t} \\
&= \sum_{l=0}^{\infty} D_{\lambda,3}(l)\frac{\left((\lambda\log(1+t)+1)^{\frac{1}{\lambda}}-1\right)^l}{l!} \\
&= \sum_{l=0}^{\infty}\frac{D_{\lambda,3}(l)}{l!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}(\lambda\log(1+t)+1)^{\frac{m}{\lambda}} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{n}(-1)^{l-m}\binom{l}{m}\left(\frac{m}{\lambda}\right)_k\lambda^k\frac{D_{\lambda,3}(l)S_1(n,k)}{l!}\frac{t^n}{n!}
\end{aligned}. \tag{10}$$

From Equation (10), we have a kind of inversion formula for Theorem 1.

Theorem 2. *For any nonnegative integer n,*

$$D_n = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{n}(-1)^{l-m}\binom{l}{m}\left(\frac{m}{\lambda}\right)_k\lambda^k\frac{D_{\lambda,3}(l)S_1(n,k)}{l!}.$$

In [5], the degenerate Stirling numbers of the first kind, denoted by $S_{1,\lambda}(n,k)$, are introduced by

$$\frac{1}{k!}\left(\log_{\lambda}(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{11}$$

The following show relationships between the Daehee numbers and the Bernoulli numbers.

$$D_n = \sum_{m=0}^{n} B_m S_1(n,m) \tag{12}$$

$$B_m = \sum_{n=0}^{m} D_n S_2(m,n) \tag{13}$$

where $S_1(n,m)$ and $S_2(n,m)$ are the the Stirling numbers of the first kind and the second kind, respectively (see [24]).

In Equation (12), the Daehee numbers can be represented by the Bernoulli and Stirling numbers of the first kind. The next equation is a degenerate version of Equation (12).

$$\begin{aligned} \frac{\log_\lambda(1+t)}{t} &= \frac{\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)}{t} \\ &= \frac{\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)}{\left(1+\lambda\left(\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)\right)\right)^{\frac{1}{\lambda}} - 1} \\ &= \sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{\left(\frac{1}{\lambda}\left((1+t)^\lambda - 1\right)\right)^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \beta_{l,\lambda} S_{1,\lambda}(n,l) \frac{t^n}{n!} \end{aligned}. \tag{14}$$

By comparing the coefficients in Equation (14), we can obtain the following theorem.

Theorem 3. *For any nonnegative integer n,*

$$D_{\lambda,3}(n) = \sum_{l=0}^{n} \beta_{l,\lambda} S_{1,\lambda}(n,l).$$

In [5], the degenerate Stirling numbers of the second kind were defined by the generating function

$$\frac{1}{k!}\left(e_\lambda(t) - 1\right)^k = \frac{\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k}{k!} = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{15}$$

Using Equation (15) and the definition of the degenerate Bernoulli numbers and the degenerate Daehee numbers, we get the following.

$$\begin{aligned} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} &= \frac{\frac{1}{\lambda}\left(\left(1+\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)\right)^\lambda - 1\right)}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &= \sum_{l=0}^{\infty} D_{\lambda,3}(l)\frac{1}{l!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^l \\ &= \sum_{n=l}^{\infty} \sum_{l=0}^{n} D_{\lambda,3}(l) S_{2,\lambda}(n,l)\frac{t^n}{n!} \end{aligned}. \tag{16}$$

Equation (16) gives us an inversion formula of Theorem 3, which is a degenerate version of Equation (13).

Theorem 4. *For any nonnegative integer n,*

$$\beta_{n,\lambda} = \sum_{l=0}^{n} D_{\lambda,3}(l) S_{2,\lambda}(n,l).$$

Now let us observe a relation between the degenerate Daehee numbers of the third kind and the Bernoulli numbers.

$$\begin{aligned}
\frac{\log_\lambda(1+t)}{t} &= \frac{\log_\lambda(1+t)}{e^{\log_\lambda(1+t)}-1}\,\frac{e^{\log_\lambda(1+t)}-1}{t} \\
&= \sum_{l=0}^{\infty} B_l \frac{(\log_\lambda(1+t))^l}{l!} \sum_{m=1}^{\infty} \frac{1}{t}\frac{(\log_\lambda(1+t))^m}{m!} \\
&= \sum_{l=0}^{\infty} B_l \sum_{k=l}^{\infty} S_{1,\lambda}(k,l)\frac{t^k}{k!} \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=1}^{n}\binom{n}{k}\frac{B_l S_{1,\lambda}(n-k,l)S_{1,\lambda}(n+1,m)}{n+1}\right)\frac{t^n}{n!}
\end{aligned} . \tag{17}$$

Obtain the following theorem from the definition of the degenerate Daehee numbers of the third kind (6) and from the previous Equation (17).

Theorem 5. *For any nonnegative integer n,*

$$D_{\lambda,3}(l) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=1}^{n}\binom{n}{k}\frac{B_l S_{1,\lambda}(n-k,l)S_{1,\lambda}(n+1,m)}{n+1}.$$

Substituting $e^{\log_\lambda(1+t)}-1$ by t in the definition of the second kind (5), the left side of Equation (5) becomes

$$\frac{\log(1+(e^{\log_\lambda(1+t)}-1))}{(1+\lambda\log(1+e^{\log_\lambda(1+t)}-1))^{\frac{1}{\lambda}}-1} = \frac{\log_\lambda(1+t)}{t}, \tag{18}$$

and the right side becomes

$$\begin{aligned}
\sum_{k=0}^{\infty} D_{\lambda,2}(k)\frac{(e^{\log_\lambda(1+t)}-1)^k}{k!} &= \sum_{k=0}^{\infty} D_{\lambda,2}(k)\sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\frac{(\log_\lambda(1+t))^l}{l!} \\
&= \sum_{k=0}^{\infty} D_{\lambda,2}(k)\sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\sum_{n=l}^{\infty} S_{1,\lambda}(n,l)\frac{t^n}{n!}. \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l} D_{\lambda,2}(k)S_{2,\lambda}(l,k)S_{1,\lambda}(n,l)\frac{t^n}{n!}
\end{aligned} \tag{19}$$

The following theorem is obtained by comparing the coefficients of Equation (18) with Equation (19).

Theorem 6. *For any nonnegative integer n,*

$$D_{\lambda,3}(n) = \sum_{l=0}^{n}\sum_{k=0}^{l} D_{\lambda,2}(k)S_{2,\lambda}(l,k)S_{1,\lambda}(n,l).$$

To represent $D_{\lambda,3}(n)$ as $D_{\lambda,2}(n)$, substituting $e_\lambda(\log(1+t))-1 = (1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1$ by t in the definition of the degenerate Daehee numbers of the third kind, Equation (6), the left side becomes

$$\frac{\log_\lambda(1+((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1))}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1} = \frac{\frac{1}{\lambda}((1+((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1))^{\lambda}-1)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1} = \frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}, \tag{20}$$

and the right side becomes

$$\begin{aligned}\sum_{k=0}^{\infty} D_{\lambda,3}(k)\frac{(e_\lambda(\log(1+t))-1)^k}{k!} &= \sum_{k=0}^{\infty} D_{\lambda,3}(k)\sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\frac{(\log(1+t))^l}{l!} \\ &= \sum_{k=0}^{\infty} D_{\lambda,3}(k)\sum_{l=k}^{\infty} S_{2,\lambda}(l,k)\sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}. \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l} D_{\lambda,3}(k)S_{2,\lambda}(l,k)S_1(n,l)\frac{t^n}{n!}\end{aligned} \tag{21}$$

From Equations (20) and (21), we obtain an inversion identity of Theorem 6.

Theorem 7. *For any nonnegative integer n,*

$$D_{\lambda,2}(n) = \sum_{l=0}^{n}\sum_{k=0}^{l} D_{\lambda,3}(k)S_{2,\lambda}(l,k)S_1(n,l).$$

3. Differential Equations Arising from the Generating Function of Degenerate Daehee Numbers

In this section, we use techniques similar to those of Martin for our results (see [25–27]). From now on, we use $F = F(t)$ to denote the degenerate log function:

$$F(t) = \log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1) \tag{22}$$

and, for a natural number N, $F^{(N)}$ to denote the N-th derivative of F; that is,

$$F^{(0)} = F(t),\ F^{(N)} = \frac{d}{dt}F^{(N-1)}.$$

Differentiating the two sides of Equation (22) results in the following:

$$\begin{aligned}\frac{d}{dt}F(t) = \frac{d}{dt}\log_\lambda(1+t) &= (1+t)^{\lambda-1} \\ &= \frac{1}{1+t}\left\{\frac{\lambda}{\lambda}((1+t)^\lambda - 1 + 1)\right\} \\ &= \frac{\lambda}{1+t}\frac{1}{\lambda}((1+t)^\lambda - 1) + \frac{1}{1+t} \\ &= \frac{\lambda}{1+t}F(t) + \frac{1}{1+t}\end{aligned}. \tag{23}$$

Further differentiating Equation (23) yields

$$\begin{aligned}\frac{d^2}{dt^2}F(t) &= -\frac{\lambda}{(1+t)^2}F(t) + \frac{\lambda}{1+t}F'(t) - \frac{1}{(1+t)^2} \\ &= -\frac{\lambda}{(1+t)^2}F(t) + \frac{\lambda}{1+t}\left\{\frac{\lambda}{1+t}F(t) + \frac{1}{1+t}\right\} - \frac{1}{(1+t)^2}. \\ &= -\frac{\lambda(\lambda-1)}{(1+t)^2}F(t) + + \frac{\lambda-1}{(1+t)^2}\end{aligned} \tag{24}$$

Based on Equations (23) and (24), we assume that

$$F^{(N)} = \frac{(\lambda)_N}{(1+t)^N}F + \frac{(\lambda)_N}{\lambda(1+t)^N}. \tag{25}$$

Let us differentiate both sides of Equation (25). Then we have

$$\begin{aligned}F^{(N+1)} &= \frac{-N}{(1+t)^{N+1}}(\lambda)_N F + \frac{1}{(1+t)^N}(\lambda)_N F^{(1)} + \frac{-N}{\lambda(1+t)^{N+1}}(\lambda)_N \\ &= \frac{(\lambda)_N}{(1+t)^{N+1}}(\lambda - N)F + \frac{(\lambda)_N}{\lambda(1+t)^{N+1}}(\lambda - N) \\ &= \frac{(\lambda)_{N+1}}{(1+t)^{N+1}}F + \frac{(\lambda)_{N+1}}{\lambda(1+t)^{N+1}}\end{aligned}. \tag{26}$$

From Equations (25) and (26), a mathematical induction gives us the following theorem.

Theorem 8. *For any positive integer N, the differential equation*

$$F^{(N)} = \frac{(\lambda)_N}{(1+t)^N}F + \frac{(\lambda)_N}{\lambda(1+t)^N} \tag{27}$$

has a solution

$$F(t) = \log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1).$$

We note that

$$\begin{aligned}F^{(N)} &= \frac{d^N}{dt^N}\log_\lambda(1+t) = \frac{d^N}{dt^N}\left(\frac{log_\lambda(1+t)}{t}\cdot t\right) \\ &= \frac{d^N}{dt^N}\left(\sum_{n=1}^{\infty} nD_{\lambda,3}(n-1)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}(n+N)D_{\lambda,3}(n+N-1)\frac{t^m}{m!}\end{aligned}. \tag{28}$$

On the other hand,

$$\begin{aligned}\frac{1}{(1+t)^N}F &= \sum_{l=0}^{\infty}\binom{N+l-1}{l}(-1)^l t^l \left(\log_\lambda(1+t)\right) \\ &= \sum_{l=0}^{\infty}\binom{N+l-1}{l}(-1)^l t^l \sum_{m=1}^{\infty}\frac{1}{\lambda}(\lambda)_m\frac{t^m}{m!} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{\lambda}(\lambda)_m\binom{N+n-m-1}{N-1}\binom{n}{m}(n-m)!\frac{t^n}{n!}\end{aligned} \tag{29}$$

and

$$
\begin{aligned}
\frac{(\lambda)_N}{\lambda}\frac{1}{(1+t)^N} &= \frac{(\lambda)_N}{\lambda}\sum_{n=0}^{\infty}(-1)^n\binom{N+n-1}{n}t^n \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(\lambda)_N}{\lambda}n!\binom{N+n-1}{n}\frac{t^n}{n!}
\end{aligned}. \tag{30}
$$

As a result of Equation (25), by comparing the coefficients of Equations (26), (27), and (28), we get the following identity.

Theorem 9. *For any nonnegative integer m and N,*

$$
\begin{aligned}
&(n+N)D_{\lambda,3}(n+N-1) \\
&= \begin{cases} \sum_{m=1}^{n}\frac{1}{\lambda}(\lambda)_m\binom{N+n-m-1}{N-1}\binom{n}{m}(n-m)! + (-1)^n\frac{(\lambda)_N}{\lambda}n! & \text{if } n \geq 1 \\ \frac{(\lambda)_N}{\lambda} & \text{if } n = 0 \end{cases}.
\end{aligned}
$$

In the other view point of the right side of Equation (25), we get

$$
\begin{aligned}
\frac{(\lambda)_N}{(1+t)^N}F + \frac{(\lambda)_N}{\lambda(1+t)^N} &= \frac{(\lambda-1)_{N-1}}{(1+t)^N}(\lambda F+1) \\
&= \frac{(\lambda-1)_{N-1}}{(1+t)^N}\left(\lambda\frac{1}{\lambda}((1+t)^\lambda-1)+1\right) \\
&= (\lambda-1)_{N-1}(1+t)^{\lambda-N} \\
&= \sum_{n=0}^{\infty}(\lambda-1)_{N-1}(\lambda-N)_n\frac{t^n}{n!}
\end{aligned}. \tag{31}
$$

Equations (26) and (31) yield the following.

Theorem 10. *For nonnegative integer n and N,*

$$
(n+N)D_{\lambda,3}(n+N-1) = (\lambda-1)_{N-1}(\lambda-N)_n.
$$

4. Results and Discussion

In this paper, we have studied the degenerate Daehee numbers of the third kind. To define the degenerate function, we used degenerate log function $\log_\lambda(t) = \frac{1}{\lambda}((1+t)^\lambda - 1)$. First we defined the degnerate Daehee numbers of the third kind. We obtained two relations between the Daehee numbers and the degenerate Daehee numbers of the second kind with Stirling numbers in Theorem 1 and Theorem 2. After that, we have two relationships between the degenerate Daehee numbers of the third kind and the degenerate Bernoulli numbers, in Theorem 3 and Theorem 4. We obtained some relations between the degenerate Daehee numbers of the second kind and the third kind in Theorem 6 and Theorem 7.

In Section 3, we derived a differential equation from the degenerate log function, Theorem 6. From the differential equation in Theorem 6, we deduced some identities for the degenerate Daehee numbers of the third kind in Theorem 10.

5. Conclusions

In [1,2], L. Carlitz considered the degenerate exponential function. By using this degenerate exponential function, he studied the degenerate Bernoulli numbers and polynomials, which are given by the generating function. In the degenerate function, if we take λ to 0 and take the appropriate variable t, we can get some form of polynomial or special polynomials number. From this view point, we consider the inverse function of Carlitz's degenerate exponential function, which is called the degenerate logarithmic function. From our degenerate logarithmic function, we derive several identities of special numbers. Through our results, we are able to see that degenerate log function is a useful tool for study of special number theory. Finally we show that degenerate Daehee numbers of third kind are the solution of the non-linear differential equations in Equation (27).

Recently in [28], T. Kim and D.S. Kim introduced some relations of trigonometric numbers through mathematical physics of view points. By taking the appropriate value λ and variable t in this our results, you can derive interesting results for special numbers useful in combinatorial and mathematical physics.

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References

1. Carlitz, L. Extended Stirling and exponential numbers. *Duke Math. J.* **1965**, *32*, 205–224 [CrossRef]
2. Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Math.* **1979**, *15*, 51–88.
3. Dolgy, D.V.; Kim, T.; Jang, L.C.; Kwon, H.I.; Seo, J.J. Some identities of symmetry for the degenerate q-Bernoulli polynomials under symmetry group of degree n. *Glob. J. Pure Appl. Math.* **2016**, *12*, 4385–4394.
4. Jang, G.W.; Kwon, J.; Lee, J.G. Some identities of degenerate Daehee numbers arising from nonlinear differential equation. *Adv. Differ. Equ.* **2017**. [CrossRef]
5. Kim, T. On the degenerate Cauchy numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2015**, *18*, 307–312.
6. Kim, T. λ-analogue of Stirling numbers of the first kind. *Adv. Stud. Contemp. Math.* **2017**, *27*, 423–429.
7. Kim, T; Kim, D.S. Degenerate Laplace Transform and Degenerate Gamma Function. *Russ. J. Math. Phys.* **2017**, *24*, 241–248. [CrossRef]
8. Kim, T.; Kim, D.S.; Jang, G.W. Differential Equations Associated with Degenerate Cauchy numbers. *Iran. J. Sci. Technol. Trans. Sci.* **2018**. [CrossRef]
9. Kim, T.; Kim, D.S.; Kwon, H.I. A note on degenerate Stirling numbers and their applications. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 195–203.
10. Kim, T.; Kim, D. S.. Differential equations associated with Catalan-Daehee numbers and their applications. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* . **2017**, *111*, 1071–1081. [CrossRef]
11. Pyo, S.S.; Kim, T.; Rim, S.H. Identities of the degenerate Daehee numbers with the Bernoulli numbers of the second kind arising from nonlinear differential equation. *J. Nonlinear Sci. Appl.* **2017**, *10*, 6219–6228. [CrossRef]
12. Pyo, S.S.; Kim T.; Rim, S.H. Degenerate Cauchy numbers of the third kind. *J. Inequal. Appl.* **2017**, *10*, 6219–6228. [CrossRef] [PubMed]
13. Kim, T.; Kim, D.S.; Dolgy, D.V.; Seo, J.J. Bernoulli polynomials of the second kind and theire identities arising from umbral calculus. *J. Nonlinear Sci. Appl.* **2016**, *9*, 860–869. [CrossRef]
14. El-Desouky, B.S.; Mustafa, A. New results on higher-order Daehee and Bernoulli numbers and polynomials. *Adv. Differ. Equ.* **2016**. [CrossRef]
15. El-Desouky, B.S.; Mustafa, A.; Abdel-Moneim, F.M. Multiparameter Higher Order Daehee and Bernoulli Numbers and Polynomials. *Appl. Math.* **2017**. [CrossRef]
16. He, Y; Kim, T. A higher-order convolution for Bernoulli polynomials of the second kind. *Appl. Math. Comput.* **2018**, *324*, 51-58. [CrossRef]
17. Kim, D. S.; Kim, T. Identities arising from higher-order Daehee polynomial bases. *Open Math.* **2015**, *13*, 196–208. [CrossRef]
18. Kim, T.; Kim, D.S. Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind. *Sci. China Math.* **2018**. [CrossRef]

19. Kim, D.S.; Kim, T. Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation. *Bull. Korean Math. Soc.* **2015**, *52*, 2001–2010. [CrossRef]
20. Ozden, H.; Cangul, I.N.; Simsek, Y. Remarks on q-Bernoulli numbers associated with Daehee numbers. *Adv. Stud. Contemp. Math.* **2009**, *18*, 41–48.
21. Simsek, Y. Identities and relations related to combinatorial numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 127–135.
22. Jang, G.W.; Kim, T. Revisit of identities of Daehee numbers arising from nonlinear differential equations. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 163–177.
23. Kim, D.S.; Kim, T.; Kwon, H.I.; Jang, G.W. Degenerate Daehee polynomials of the second kind.*Proc. Jangjeon Math. Soc.* **2018**, *21*, 83–97.
24. Kim, D.S.; Kim, T. Daehee Numbers and Polynomials. *Appl. Math. Sci.* **2013**, *7*, 5969–5976. [CrossRef]
25. Marin, M. Contributions on uniqueness in thermoelastodynamics on bodies with voids. *Cienc. Mat.* **1998**, *16*, 101–109.
26. Marin, M. A temporally evolutionary equation in elasticity of micropolar bodies with voids. *UPB Sci. Bull. Ser. A Appl. Math. Physic.* **1998**, *60*, 3–12.
27. Marin, M. Weak Solutions in Elasticity of Dipolar Porous Materials. *Math. Probl. Eng.* **2008**. [CrossRef]
28. Kim, D.S.; Kim, T. Some p-Adic Integrals on $\mathbb{Z}_p$ Associated with Trigonometric Functions. *Russ. J. Math. Phys.* **2018**, *25*, 300–308. [CrossRef]