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Edge Version of Metric Dimension and Doubly Resolving Sets of the Necklace Graph

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Abstract: Consider an undirected and connected graph $G = (V_G, E_G)$, where V_G and E_G represent the set of vertices and the set of edges respectively. The concept of edge version of metric dimension and doubly resolving sets is based on the distances of edges in a graph. In this paper, we find the edge version of metric dimension and doubly resolving sets for the necklace graph.

Keywords: necklace graph; resolving sets; edge version of metric dimension; edge version of doubly resolving sets

1. Introduction and Preliminaries

Let G be a connected, simple and undirected graph consisting of nonempty finite sets V_G of vertices and E_G of edges. The order of a graph G is $|V_G|$ and $|E_G|$ is the size of G . The number of vertices joining to v , where $v \in V_G$ is called a degree of that vertex and written as d_v . $\Delta(G)$ is the maximum degree in a graph G .

For any two vertices $x, y \in V_G$, the distance $d(x, y)$ is the length of a shortest path between x and y . Let $R = \{r_1, r_2, \dots, r_l\} \subset V_G$ be an ordered set and let $x \in V_G$, then $r(x, R)$ representation of x with respect to R is the l -tuple $(d(x, r_1), d(x, r_2), \dots, d(x, r_l))$. R is said to be a resolving set if different vertices of G have different representations with respect to R . The minimum number of vertices in a resolving set is called a basis for G and the cardinality of the basis is known as the metric dimension of G , represented by $\dim(G)$. For $R = \{r_1, r_2, \dots, r_l\} \subset V_G$, the i th component of $r(x, R)$ is 0 if and only if $x = r_i$. Hence, to prove that R is a resolving set it is enough to show that $r(x, R) \neq r(y, R)$ for each pair $x \neq y \in V_G \setminus R$.

The following lemma is very helpful for determining resolving set for $\dim(G)$:

Lemma 1. Let R be a resolving set for a simple connected graph G and $x, y \in V_G$. If $d(x, r) = d(y, r)$ for all vertices $r \in V_G \setminus \{x, y\}$, then $\{x, y\} \cap R \neq \emptyset$.

The idea of resolving sets and metric dimension was presented by Slater in [1] and also by Hararay and Melter in [2]. Metric dimension is applied in different branches of navigation [3], robotics [3], chemistry [4], and network discovery and verification. It is well-known in [5] that computing the metric dimension of a graph is an NP-hard problem. Metric dimension has been deeply elaborated in surveys [6,7]. The line graph $L(G)$ of a graph G is defined as, the graph whose vertices are the edges of G , with two adjacent vertices if the corresponding edges share the common vertex in G . Also, metric properties of line graphs were studied to a great extent in [8–18]. The line graph of a graph G is helpful to find edge distances using the same technique of finding vertex distances of the graph G .

The concept of edge metric dimension was set up by Kelenc, Tratnik and Yero in [19] in 2016. They computed the edge metric dimension of different families of graphs and showed edge metric dimension i.e., $edim(G)$ can be less, equal to or more than $dim(G)$. They also showed computing $edim(G)$ is NP-hard in general. Since in literature edge metric dimension exists and that is entirely different from edge metric dimension defined in [20], so we renamed edge metric dimension proposed in [20] as an edge version of metric dimension. The edge version of metric dimension is defined as:

Definition 1.

1. The edge distance $d_E(f, g)$ between two edges $f, g \in E_G$ is the length of a shortest path between vertices f and g in the line graph $L(G)$.
2. If $d_E(e, f) \neq d_E(e, g)$, then the edge $e \in E_G$ is said to resolve two edges f and g of E_G .
3. Suppose that $R_E = \{f_1, f_2, \dots, f_k\} \subset E_G$ is an ordered set and e is an edge of E_G , then $r_E(e, R_E)$ the edge version of representation of e with respect to R_E is the k -tuple $(d_E(e, f_1), d_E(e, f_2), \dots, d_E(e, f_k))$.
4. If different edges of G have different edge version of representations with respect to R_E , then the set R_E is said to be a an edge version of resolving set of G .
5. The edge version of the metric basis of G is basically an edge version of the resolving set having minimum cardinality. The cardinality of the edge version of metric basis is represented by $dim_E(G)$, and is called the edge version of metric dimension of G .

In literature the edge version of metric dimension is known for few classes of graphs. Bounds of an edge version of metric dimension are also known and these bounds are given in the next theorem:

Theorem 1 ([21]). *If G is a connected graph with $|V_G| \geq 5$, then*

$$\lceil \log_2 \Delta(G) \rceil \leq dim_E(G) \leq |V_G| - 2.$$

Table 1, represents all those graphs for which the edge version of metric dimension is known. In the table P_n, C_n and K_n represent the path graph, the cycle graph and the complete graph on n vertices respectively. $W_{1,n} = K_1 + C_n$ is a wheel graph on $n + 1$ vertices, $K_{s,t}$ is a complete bipartite graph on $s + t$ vertices and for $n \geq 2, B_n = (k_1, k_2, \dots, k_n)$ is a bouquet of circles C^1, C^2, \dots, C^n with a cut-vertex where k_i is the number of vertices of C^i ($1 \leq i \leq n$). Also S_n represents the n -sunlet graph and D_n is the prism graph on $2n$ vertices.

Table 1. Edge version of the metric dimension of graphs.

G	$dim_E G$
P_n	1 [4]
C_n	2
K_n	$n - 1$ [6]
$K_{s,t}$	$\begin{cases} \lfloor \frac{2(s+t-1)}{3} \rfloor & \text{if } s \leq t \leq 2s; \\ t - 1 & \text{if } t \geq 2s. \end{cases}$ [22]
$W_{1,n}$	$\begin{cases} 3 & \text{if } n = 3, 4; \\ 4 & \text{if } n = 5; \\ n - \lceil \frac{n}{3} \rceil & \text{if } n \geq 6. \end{cases}$ [23]
B_n	$2n - 1$ [23]
S_n	$\begin{cases} 2 & \text{if } n \text{ is even;} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$ [20]
D_n	3 [20]

Caceres et al. define the notion of a doubly resolving set in [22]. The doubly resolving sets present a valuable source for finding upper bounds of the metric dimension of graphs. Let the vertices a and b of the graph G with order $|V_G| \geq 2$ doubly resolve vertices c and d of the graph G if $d(c, a) - d(c, b) \neq d(d, a) - d(d, b)$. A subset D of vertices doubly resolves G if every two vertices in G are doubly resolved by some two vertices of D . Moreover, in G there do not exist any two different vertices having the same difference between their corresponding metric coordinates with respect to D . A doubly resolving set with minimum cardinality is called the minimal doubly resolving set. The minimum cardinality of a doubly resolving set for G is represented by $\psi(G)$. In case of some convex prism, hamming and polytopes graphs, the minimal doubly resolving sets have been obtained in [24–26] respectively.

Clearly, every doubly resolving set is a resolving set, which implies $\dim(G) \leq \psi(G)$ for all graphs G . Also, if a and b doubly resolve c and d , then $d(c, a) - d(d, a) \neq 0$ or $d(f, b) - d(g, b) \neq 0$, and thus a or b resolve c and d , this shows that a doubly resolving set is also a resolving set.

Ahmed et al. in [27] proposed the idea of minimal edge version of doubly resolving sets of graph G , based on the distances of the edges of graph G which is defined as follows:

Definition 2.

1. The edges f and g of the graph G with size $|E_G| \geq 2$ are supposed to edge doubly resolve edges f_1 and f_2 of the graph G if $d_E(f_1, f) - d_E(f_1, g) \neq d_E(f_2, f) - d_E(f_2, g)$ in G .
2. Let $D_E = \{e_1, e_2, \dots, e_k\}$ be an ordered set of the edges of G . If any two edges $e \neq f \in E_G$ are doubly resolved by any two edges of set D_E in G , then the set $D_E \subset E_G$ is said to be an edge version of doubly resolving set of G . The minimum cardinality of an edge version of doubly resolving set of G is represented by $\psi_E(G)$.

Note that every edge version of a doubly resolving set is an edge version of a resolving set, which implies $\dim_E(G) \leq \psi_E(G)$ for all graphs G .

In this paper we compute the edge version of metric dimension and doubly resolving set for the necklace graph. At the end, we conclude that edge version of metric dimension and doubly resolving set are independent of choice of n .

2. The Edge Version of Metric Dimension for N_{e_n}

The necklace graph (see Figure 1) denoted by N_{e_n} [28] is a cubic Halin graph [29] obtained by joining a cycle with all vertices of degree 1 of a caterpillar (also called a comb) having n vertices of degree 3 and $n + 2$ vertices of degree 1, denoted by x_0, x_1, \dots, x_{n+1} and y_1, y_2, \dots, y_n , respectively. We have $V_{N_{e_n}} = \{x_0, \dots, x_{n+1}, y_1, \dots, y_n\}$ and $E_{N_{e_n}} = F \cup H \cup K$, where $F = \{f_1, f_2, \dots, f_{n+1}\}$, $H = \{h_1, h_2, \dots, h_{n+1}\}$ and $K = \{g_1, g_2, \dots, g_{n+1}\}$. The necklace graph is 3-regular graph with constant metric dimension, which is computed in [30] given below:

$$\dim(N_{e_n}) = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

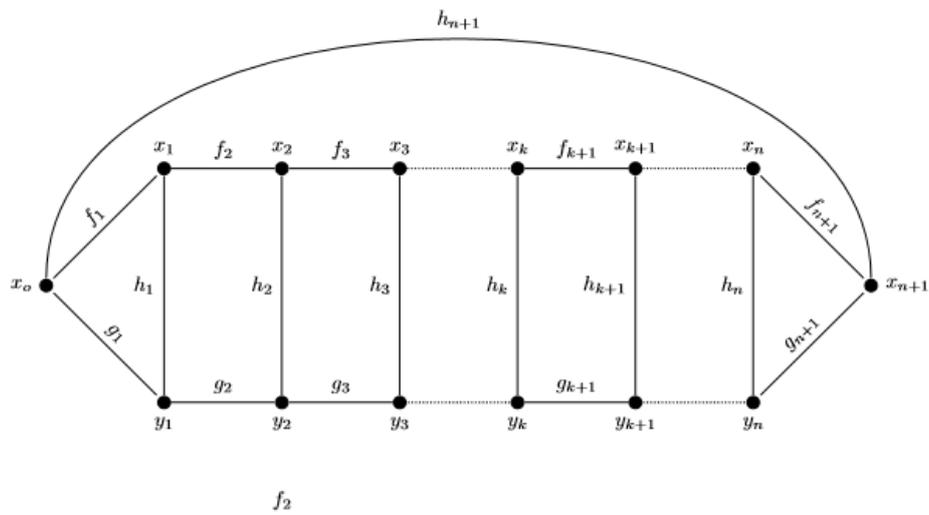


Figure 1. The necklace graph N_{e_n} .

For the edge version of metric dimension of the necklace graph, we have to construct a line graph $L(N_{e_n})$ of N_{e_n} with $n \geq 2$. (see Figure 2).

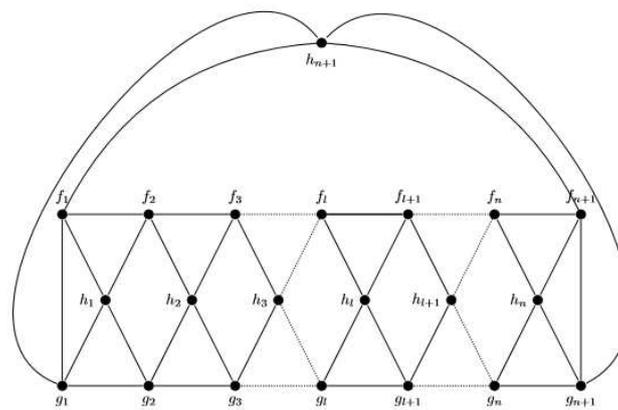


Figure 2. The line graph of a necklace graph: $L(N_{e_n})$.

Theorem 2. The edge version of metric dimension of N_{e_n} is 3 for $n \geq 2$.

Proof. Let $l = \lfloor \frac{n}{2} \rfloor$. For $n = 2$ and 3 , consider the set $R_E = \{f_1, f_n, f_{n+1}\} \subset E_{N_{e_n}}$, then the edge version of representation of each edge of N_{e_n} with respect to R_E is given below:

The edge version of representation of the edges $f_p \in F$ with respect to R_E is:

$$r_E(f_p, R_E) = \begin{cases} (0, n - 1, 2) & \text{if } p = 1; \\ (p - 1, |n - p|, n - p + 1) & \text{if } 2 \leq p \leq n; \\ (2, l, 0) & \text{if } p = n + 1. \end{cases}$$

The edge version of representation of the inner edges $h_p \in H$ with respect to R_E is:

$$r_E(h_p, R_E) = \begin{cases} (p, n - p, n - p + 1), & \text{if } 1 \leq p \leq n - 1; \\ (n, 1, 1), & \text{if } p = n; \\ (1, l + 1, 1), & \text{if } p = n + 1. \end{cases}$$

The edge version of representation of the edges $g_p \in K$ with respect to R_E is:

$$r_E(g_p, R_E) = \begin{cases} (1, n, 2), & \text{if } p = 1; \\ (p, 2, n - p + 2), & \text{if } 2 \leq p \leq n; \\ (2, l + 1, 1), & \text{if } p = n + 1. \end{cases}$$

For $n \geq 4$, take a set $R_E = \{f_1, f_l, f_n\} \subset E_{N_{e_n}}$, we will show that R_E is an edge version of resolving set for N_{e_n} .

The edge version of representation of the edges $f_p \in F$ with respect to R_E is:

$$r_E(f_p, R_E) = \begin{cases} (p - 1, l - p, p + 2) & \text{if } 1 \leq p \leq l - 1; \\ (p - 1, |l - p|, n - p) & \text{if } l \leq p \leq l + 2; \\ (n - p + 3, |l - p|, |n - p|) & \text{if } l + 3 \leq p \leq n; \\ (2, l + 1, 1), & \text{if } p = n + 1. \end{cases}$$

The edge version of representation of the inner edges $h_p \in H$ with respect to R_E is:

$$r_E(h_p, R_E) = \begin{cases} (p, l - p, p + 3), & \text{if } 1 \leq p \leq l - 2; \\ (p, |l - p| + 1, n - p), & \text{if } l - 1 \leq p \leq l + 1; \\ (n - p + 3, |l - p| + 1, n - p), & \text{if } l + 2 \leq p \leq n - 1; \\ (3, |l - n| + 1, 1), & \text{if } p = n; \\ (1, l, 2), & \text{if } p = n + 1. \end{cases}$$

The edge version of representation of the edges $g_p \in K$ with respect to R_E is:

$$r_E(g_p, R_E) = \begin{cases} (p, l - p + 1, p + 2), & \text{if } 1 \leq p \leq l - 1; \\ (p, 2, n - p + 1), & \text{if } l \leq p \leq l + 1; \\ (n - p + 3, p - l + 1, n - p + 1), & \text{if } l + 2 \leq p \leq n - 1; \\ (3, n - l + 1, 2), & \text{if } p = n; \\ (2, l + 1, 2), & \text{if } p = n + 1. \end{cases}$$

From the above representations it is clear that no two edges of N_{e_n} have the same edge version of representations, which implies R_E is the edge version of resolving set and hence $\dim_E(N_{e_n}) \leq 3$. Next, we have to show that $\dim_E(N_{e_n}) \geq 3$. Suppose on the contrary that $\dim_E(N_{e_n}) = 2$, then we have the following possibilities:

1. Let two edges h_1 and h_p from the edge set H with $2 \leq p \leq l + 1$. For $R_E = \{h_1, h_p\} \subset E_{N_{e_n}}$, we have $r_E(f_1, R_E) = r_E(g_1, R_E) = (1, p)$, a contradiction.

2. Now suppose that both edges are from the edge set F . Suppose without loss of generality that $R_E = \{f_1, f_p\} \subset E_{N_{e_n}}$, where $2 \leq p \leq l + 1$. Then, $r_E(g_1, R_E) = r_E(h_{n+1}, R_E) = (1, p)$, a contradiction.
3. Now suppose that both edges are from the edge set K . Suppose without loss of generality that $R_E = \{g_1, g_p\} \subset E_{N_{e_n}}$, where $2 \leq p \leq l + 1$. Since $r_E(f_1, R_E) = r_E(h_{n+1}, R_E) = (1, p)$, so we have a contradiction.
4. If one edge belongs to the set F and the second edge is from H , without loss of generality we can take $R_E = \{f_1, h_p\} \subset E_{N_{e_n}}$ with $1 \leq p \leq l$. For $2 \leq n \leq 5$, we have $r_E(g_{n+1}, R_E) = r_E(f_{n+1}, R_E) = (2, 3)$ for $p = 1$ and $r_E(g_{n+1}, R_E) = r_E(f_{n+1}, R_E) = (2, n - 1)$ for $p = 2$. For $n \geq 6$, we have $r_E(f_n, R_E) = r_E(g_n, R_E) = (3, p + 3)$, a contradiction.
5. If one edge belongs to the set G and the second edge is from H . i.e. $R_E = \{g_1, h_p\} \subset E_{N_{e_n}}$. This case is similar to (4).
6. If one edge belongs to the set F and the second edge is from K , then we have the following five subcases:
 - (a) Let $R_E = \{f_1, g_1\} \subset E_{N_{e_n}}$. For $2 \leq n \leq 5$, we have $r_E(h_1, R_E) = r_E(h_{n+1}, R_E) = (1, 1)$. For $n \geq 6$ and $l + 4 \leq p \leq n + 1$, we have $r_E(g_p, R_E) = r_E(f_p, R_E) = (n - p + 3, n - p + 3)$, a contradiction.
 - (b) If $R_E = \{f_1, g_p\} \subset E_{N_{e_n}}$ and $2 \leq p \leq l + 1$, then we have $r_E(g_1, R_E) = r_E(h_1, R_E) = (1, p - 1)$, a contradiction.
 - (c) Let $R_E = \{g_1, f_p\} \subset E_{N_{e_n}}$ with $2 \leq p \leq l + 1$. This case is similar to 6 (b).
 - (d) Let $R_E = \{f_2, g_p\} \subset E_{N_{e_n}}$ with $1 \leq p \leq l$. For $2 \leq n \leq 5$ and $p = 1$, we have $r_E(f_1, R_E) = r_E(h_1, R_E) = (1, 1)$ and when $p = 2$, then $r_E(h_1, R_E) = r_E(h_2, R_E) = (1, 1)$. For $n \geq 6$, we have $r_E(f_n, R_E) = r_E(h_n, R_E) = (4, p + 2)$, a contradiction.
 - (e) Let $R_E = \{g_2, f_p\} \subset E_{N_{e_n}}$. This case is similar to 6 (e).

All the above possibilities lead to a contradiction. Hence, there is no edge version of resolving set of cardinality 2 for edges $E_{N_{e_n}}$, which implies that $dim_E(N_{e_n}) = 3$. \square

3. The Minimal Edge Version of Doubly Resolving Sets for N_{e_n}

The minimum doubly resolving set for the necklace graph N_{e_n} has been discussed in [31]. In this section, we determine minimal edge version of doubly resolving sets for the necklace graph. Define $S_j(h_{n+1}) = \{g \in E_{N_{e_n}} : d_E(h_{n+1}, g) = j\}$ be the set of edges in N_{e_n} at edge distance j from edge h_{n+1} . The Table 2 can be easily formulated for $S_j(h_{n+1})$ and it will be used to get the edge distances between two arbitrary edges in $E_{N_{e_n}}$.

Table 2. $S_j(h_{n+1})$ for N_{e_n} .

n	j	$S_j(h_{n+1})$
	1	$\{g_1, f_1, g_{n+1}, f_{n+1}\}$
	$2 \leq j \leq t$	$\{g_j, f_j, g_{n+2-j}, f_{n+2-j}, h_{j-1}, h_{n+2-j}\}$
$2t(t \geq 2)$	$t + 1$	$\{h_t, f_{t+1}, g_{t+1}\}$
$2t + 1(t \geq 2)$	$t + 1$	$\{h_t, f_{t+1}, g_{t+1}\}$

By the symmetry of the necklace graph N_{e_n} , it is clear that $d_E(f_j, f_s) = d_E(g_j, g_s) = d_E(h_{n+1}, f_{s-j}) = d_E(h_{n+1}, g_{s-j})$ for $s > j$
 For $n = 2t$

$$d_E(g_j, f_s) = \begin{cases} d_E(h_{n+1}, g_{|s-j|}) + 1 & \text{if } |j - s| \leq t, 1 \leq j, s \leq n; \\ d_E(h_{n+1}, g_{|s-j|}) & \text{if } |j - s| > t, 1 \leq j, s \leq n; \\ d_E(h_{n+1}, g_s) & \text{if } j = s = 1, \text{ or } n + 1; \\ d_E(h_j, g_{j+2}) & \text{if } 1 < j = s < n; \\ d_E(h_{n+1}, f_n) & \text{if } j = s = n. \end{cases}$$

For $n = 2t + 1$

$$d_E(g_j, f_s) = \begin{cases} d_E(h_{n+1}, g_{|s-j|}) & \text{if } s > j; \\ d_E(h_{n+1}, g_s) & \text{if } j = s = 1, \text{ or } n + 1; \\ d_E(h_j, g_{j+2}) & \text{if } 1 < j = s < n; \\ d_E(h_{n+1}, f_n) & \text{if } j = s = n. \end{cases}$$

Lemma 2. $\psi_E(N_{e_n}) = 3$, whenever $n = 2t, t \geq 2$.

Proof. The Table 3 represents the vectors of edge version of representations of N_{e_n} with respect to $D_E = \{f_1, f_{t+1}, f_{n+1}\}$

Table 3. Vectors of edge version of representations of $N_{e_n}, n = 2t$.

j	$S_j(h_{n+1})$	$D_E = \{f_1, f_{t+1}, f_{n+1}\}$
0	h_{n+1}	$(1, t + 1, 1)$
1	g_1	$(1, t + 2, 2)$
	f_1	$\{0, t, 2\}$
	g_{n+1}	$\{2, t + 1, 1\}$
$2 \leq j \leq k$	f_{n+1}	$\{2, t, 0\}$
	g_j	$\{j, t + 2 - j, j + 1\}$
	f_j	$\{j, t + 1 - j, j + 1\}$
	g_{n+2-j}	$\{n - j + 3, j - t, n - j + 2\}$
	f_{n+2-j}	$\{n - j + 3, t + 1 - j , n - j + 1\}$
	h_{j-1}	$\{j, t + 1 - j, j + 2\}$
	h_{n+2-j}	$\{n - j + 3, t - j, n - j + 1\}$
	h_t	$\{t, 1, t\}$
	h_{t+1}	$\{t + 1, 1, t + 1\}$
	f_{k+1}	$\{t, 0, t\}$
	g_{t+1}	$\{t + 1, 2, t + 1\}$

It can be verified that for each $j \in \{1, 2, 3, \dots, t + 1\}$, no two edges $f, g \in S_j(h_{n+1})$ exist such that $r_E(f, D_E) - r_E(g, D_E) = 0$ holds. Also for any $j, s \in \{1, 2, 3, \dots, t + 1\}$, there do not exist any two edges $f \in S_i(w_{n+1})$ and $g \in S_j(h_{n+1})$ such that $r_E(f, D_E) - r_E(g, D_E) = i - j$. So, $D_E = \{f_1, f_{t+1}, f_{n+1}\}$ becomes the minimal edge version of doubly resolving set for $n = 2t, t \geq 2$ and therefore the Lemma 2 holds. \square

Lemma 3. $\psi_E(N_{e_n}) = 3$ whenever, $n = 2t + 1, t \geq 2$.

Proof. As we know that $dim_E(N_{e_n}) \leq \psi_E(N_{e_n})$ holds. Now the Table 4 represents the vectors of edge version of representations of N_{e_n} with respect to $D_E = \{f_1, f_n, h_{t+1}\}$.

Table 4 shows that for $j \in \{1, 2, 3, \dots, t + 1\}$ there do not exist two edges $f, g \in S_j(h_{n+1})$ such that the following condition $r_E(f, D_E) - r_E(g, D_E) = 0$ holds. Also, for any $i, j \in \{1, 2, 3, \dots, t + 1\}$, there do not exist any two edges $f \in S_i(h_{n+1})$ and $g \in S_j(h_{n+1})$ such that $r_E(f, D_E) - r_E(g, D_E) = i - j$. So, $D_E = \{f_1, f_n, h_{t+1}\}$ becomes the minimal edge version of doubly resolving set for $n = 2t + 1, t \geq 2$ and therefore the Lemma 3 holds. \square

Table 4. Vectors of edge version of representations of N_{e_n} , for $n = 2t + 1$.

j	$S_j(h_{n+1})$	$D_E = \{f_1, f_n, h_{t+1}\}$	
0	h_{n+1}	$(1, 2, t + 2)$	
1	g_1	$(1, 3, t + 1)$	
	f_1	$\{0, 3, t + 1\}$	
	g_{n+1}	$\{2, 2, t + 1\}$	
2	f_{n+1}	$\{2, 1, t + 1\}$	
	g_n	$\{j + 1, 2, t + 2 - j\}$	
	h_n	$\{j + 1, 1, t + 3 - j\}$	
	$2 \leq j \leq t$	g_j	$\{j, j + 2, t + 2 - j\}$
		f_j	$\{j, j + 2, j\}$
$t + 1$	g_{n+1-j}	$\{j + 1, j - 1, t - j + 2\}$	
	f_{n+2-j}	$\{j + 1, j - 2, t - j + 2\}$	
	h_{j-1}	$\{j - 1, j + 2, j\}$	
	h_{n+1-j}	$\{j + 1, j - 2, t - j + 3\}$	
	f_{t+1}	$\{t, t, 1\}$	
	g_{t+1}	$\{t + 1, t + 1, 1\}$	
	f_{t+2}	$\{t + 1, t - 2, 1\}$	
	g_{t+2}	$\{t + 2, t + 1, 1\}$	
	h_{t+2}	$\{t + 2, t - 1, 2\}$	

Note: A counting technique determines that $\psi_E(N_{e_n}) = 3$ for $n = 2$ and 3 . The sets $\{f_1, f_2, f_3\}$ and $\{f_1, f_3, h_2\}$ are the minimal edge version of doubly resolving sets for N_{e_2} and N_{e_3} respectively. When Lemma 2 and Lemma 3 is combined, then the following main theorem is formulated.

Theorem 3. Let N_{e_n} be the necklace graph. Then $\psi_E(N_{e_n}) = 3$ for $n \geq 2$.

4. Conclusions

In this paper, we have extended the notion of metric dimension to the edge version of metric dimension for the necklace graph N_{e_n} which is the least cardinality over all edge versions of resolving sets. We also calculated the minimal edge version of doubly resolving sets for N_{e_n} . It is interesting to consider the necklace graph because its edge version of metric dimension and the minimal edge version of doubly resolving set are independent of parity of n . In previous work on necklace graphs, (see [30,31]) resolving sets were based on vertices and distances were calculated between vertices only. While, in this paper edges have been considered for getting resolving sets and distances have been calculated between edges. Finally, we get $dim_E(N_{e_n}) = \psi_E(N_{e_n}) = 3$ for every n .

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Open Problem: Characterize all classes of graphs in which metric dimension/doubly resolving sets coincides with edge version of metric dimension/doubly resolving sets.

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