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Inequalities on Sasakian Statistical Manifolds in Terms of Casorati Curvatures

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Abstract: A statistical structure is considered as a generalization of a pair of a Riemannian metric and its Levi-Civita connection. With a pair of conjugate connections ∇ and ∇^* in the Sasakian statistical structure, we provide the normalized scalar curvature which is bounded above from Casorati curvatures on C-totally real (Legendrian and slant) submanifolds of a Sasakian statistical manifold of constant φ-sectional curvature. In addition, we give examples to show that the total space is a sphere.

Keywords: Sasakian statistical manifold; conjugate connection; Casorati curvature

1. Introduction

A statistical model in information geometry has a Fisher metric as a Riemannian metric with an affine connection, whose connection is constructed from the average of the probability distribution. In the statistical models, a pair of a Fisher information metric and an affine connection gives the geometric structure, called the Chentsov-Amari connection [1], whose geometric structure is a generalization of a pair of a Riemannian metric and a Levi-Civita connection. By generalizing the geometric structure, a statistical structure has been studied in information geometry. Applying this idea to Sasakian manifolds, one arrived at the definition of a Sasakian statistical structure as a generalization of a Sasakian structure. In other words, it is a triple of an affine connection, a Riemannian metric, and a Sasakian structure on an odd dimensional manifold [2]. The geometry of such a manifold is closely related to affine geometry and Hessian geometry. In such manifolds, there are the fundamental equations such as Gauss formula, Weingarten formula and the equations of Gauss, Codazzi and Ricci in submanifolds of a statistical manifold [3].

On the other hand, it is well-known that the Casorati curvature as a new extrinsic invariant is defined as the normalized square of the length of the second fundamental form, introduced by Casorati ([4,5]). Geometric meanings of Casorati curavature were found in visual perception of shape and appearance ([6–8]). Some optimal inequalities involving Casorati curvatures were proved in [9–15] for several submanifolds in real, complex and quaternionic space forms with various connections. Moreover, Lee et al. established that the normalized scalar curvature is bounded by Casorati curvatures of submanifolds in a statistical manifold of constant curvature [16]. In Kenmotsu statistical manifolds, Decu et al. investigate curvature properties and establish optimizations in terms of a new extrinsic invariant (the normalized δ-Casorati curvature) and an intrinsic invariant (the scalar curvature) [17].

In our paper, we establish optimizations of the normalized scalar curvature (the intrinsic invariant) for a new extrinsic invariant (generalized normalized Casorati curvatures) on Legendrian and slant submanifolds in a Sasakian statistical space form. Moreover, we provide some examples for special Sasakian statistical sphere S^{2m+1} of statistical sectional curvature 1.

2. Preliminaries

Let $(\overline{M}^m, \overline{g})$ be a m-dimensional Riemannian manifold with an affine connection $\overline{\nabla}$. We denote by $\Gamma(T\overline{M})$ the collection of all vector fields on $\overline{M}$.

Definition 1 ([18])**.** *A pair $\left(\overline{\nabla}, \overline{g}\right)$ is called a statistical structure on M if $\overline{\nabla}$ is a torsion free connection on M and the covariant derivative ∇g is symmetric.*

Definition 2. *A statistical manifold $(\overline{M}, \overline{g}, \overline{\nabla})$ is a Riemannian manifold, endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ satisfying*

$$Z\overline{g}\,(X,Y) = \overline{g}\left(\overline{\nabla}_Z X, Y\right) + \overline{g}\left(X, \overline{\nabla}^*_Z Y\right) \tag{1}$$

for any vector fields X, Y and Z. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections.

Remark 1.

(a) $\left(\overline{\nabla}^*\right)^* = \overline{\nabla}$.

(b) *If $\left(\overline{\nabla}, \overline{g}\right)$ is a statistical structure, then so is $\left(\overline{\nabla}^*, g\right)$.*

(c) *Any torsion-free affine connection $\overline{\nabla}$ always has a dual connection satisfying*

$$\overline{\nabla} + \overline{\nabla}^* = 2\overline{\nabla}^0, \tag{2}$$

where $\overline{\nabla}^0$ is the Levi-Civita connection for $\overline{M}$.

Let $\overline{R}$ and $\overline{R}^*$ be the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

Definition 3 ([18,19])**.** *Let $\left(\overline{\nabla}, \overline{g}\right)$ be a statistical structure on $\overline{M}$. We define*

$$S(X,Y)Z = \frac{1}{2}\{\overline{R}(X,Y)Z + \overline{R}^*(X,Y)Z\}$$

for $X, Y, Z \in \Gamma(T\overline{M})$, called the statistical curvature tensor of $\left(\overline{\nabla}, \overline{g}\right)$. In particular, a statistical manifold $\left(\overline{M}, \overline{\nabla}, \overline{g}\right)$ is to be of constant statistical curvature $c \in \mathbb{R}$ if $S(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}$ for $X, Y, Z \in \Gamma(T\overline{M})$.

By the direct calculation, the curvature tensor fields $\overline{R}$ and $\overline{R}^*$ satisfy

$$\overline{g}\left(\overline{R}^*(X,Y)Z, W\right) = -\overline{g}\left(Z, \overline{R}(X,Y)W\right), \quad X, Y, Z, W \in \Gamma(T\overline{M}).$$

Therefore, if $\left(\overline{\nabla}, \overline{g}\right)$ is a statistical structure of constant curvature c, so is $\left(\overline{\nabla}^*, \overline{g}\right)$.

For submanifolds in statistical manifolds, we have pairs of induced connections ∇, ∇^*, second fundamental forms h, h^*, shape operators A, A^*, and normal connections D, D^* satisfying equations analogous to the Gauss and the Weingarten ones for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Moreover, the induced metric g is unique, and (∇, g) and (∇^*, g) are induced dual statistical structures on the submanifold. The fundamental equations for statistical submanifolds are given by Vos ([3]).

Let (M, g) be an n-dimensional submanifold of a statistical manifold $\left(\overline{M}, \overline{g}\right)$ and g the induced metric on M. Then for any vector fields X, Y, the Gauss formulas are given respectively by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X,Y) \\ \overline{\nabla}^*_X Y &= \nabla^*_X Y + h^*(X,Y). \end{aligned}$$

The corresponding Gauss equations with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ are given by the following result.

Theorem 1 ([3]). *Let $\overline{\nabla}$ and $\overline{\nabla}^*$ be dual connections on $(\overline{M}, \overline{g})$ and ∇ and ∇^* the induced dual connections by $\overline{\nabla}$ and $\overline{\nabla}^*$ by a submanifold M of $(\overline{M}, \overline{g})$, respectively. Let $\overline{R}$, R, $\overline{R^*}$ and R^* be the Riemannian curvature tensors of $\overline{\nabla}$, ∇, $\overline{\nabla^*}$ and ∇^*, respectively. Then*

$$\begin{aligned}\overline{g}\left(\overline{R}\left(X,Y\right)Z,W\right) &= \overline{g}\left(R\left(X,Y\right)Z,W\right)\\ &+\overline{g}\left(h\left(X,Z\right),h^*\left(Y,W\right)\right)-\overline{g}\left(h^*\left(X,W\right),h\left(Y,Z\right)\right)\end{aligned} \tag{3}$$

$$\begin{aligned}\overline{g}\left(\overline{R}^*\left(X,Y\right)Z,W\right) &= \overline{g}\left(R^*\left(X,Y\right)Z,W\right)\\ &+\overline{g}\left(h^*\left(X,Z\right),h\left(Y,W\right)\right)-\overline{g}\left(h\left(X,W\right),h^*\left(Y,Z\right)\right)\end{aligned} \tag{4}$$

If $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1}, ..., e_m\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$, then the scalar curvature τ at p is defined as

$$\tau(p) = \sum_{1\le i<j\le n} g\left(S\left(e_i,e_j\right)e_j,e_i\right)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H, H^* the mean curvature vectors, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i), \qquad H^*(p) = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i,e_i) \tag{5}$$

and we also set

$$h_{ij}^{\alpha} = g(h(e_i,e_j),e_\alpha), \quad h_{ij}^{*\alpha} = g(h^*(e_i,e_j),e_\alpha),$$

$i, j \in \{1, ..., n\}$, $\alpha \in \{n+1, ..., m\}$.

Then it is well-known that the squared mean curvatures of the submanifold M in $\overline{M}$ are defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2, \quad \|H^*\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right)^2$$

and the squared norms of h and h^* over dimension n is denoted by $\mathcal{C}$ and $\mathcal{C}^*$ are called the Casorati curvatures of the submanifold M, respectively. Therefore, we have

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2 \text{ and } \mathcal{C}^* = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{*\alpha}\right)^2.$$

The normalized δ-Casorati curvatures $\delta_C(n-1)$ and $\widehat{\delta}_C(n-1)$ of the submanifold M are defined as

$$[\delta_C(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{(n+1)}{2n}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

and

$$\left[\widehat{\delta}_C(n-1)\right]_p = 2\mathcal{C}_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}.$$

Similarly, the dual normalized δ^*-Casorati curvatures $\delta^*_C(n-1)$ and $\widehat{\delta}^*_C(n-1)$ of the submanifold M are defined as

$$[\delta^*_C(n-1)]_p = \frac{1}{2}\mathcal{C}^*_p + \frac{(n+1)}{2n}\inf\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\},$$

and

$$\left[\widehat{\delta}^*_C(n-1)\right]_p = 2\mathcal{C}^*_p - \frac{(2n-1)}{2n}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\}.$$

The generalized normalized δ-Casorati curvatures $\delta_C(t;n-1)$ and $\widehat{\delta}_C(t;n-1)$ of the submanifold M are defined for any positive real number $t \neq n(n-1)$ as

$$[\delta_C(t;n-1)]_p = t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$\left[\widehat{\delta}_C(t;n-1)\right]_p = t\mathcal{C}_p - \frac{(n-1)(n+t)(t-n^2+n)}{nt}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $t > n^2 - n$.

Moreover, the dual generalized normalized δ-Casorati curvatures $\delta^*_C(t;n-1)$ and $\widehat{\delta}^*{}_C(t;n-1)$ of the submanifold M are defined for any positive real number $t \neq n(n-1)$ as

$$[\delta^*_C(t;n-1)]_p = t\mathcal{C}^*_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\inf\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$\left[\widehat{\delta}^*{}_C(t;n-1)\right]_p = t\mathcal{C}^*_p - \frac{(n-1)(n+t)(t-n^2+n)}{nt}\sup\{\mathcal{C}^*(L)|L \text{ a hyperplane of } T_pM\},$$

if $t > n^2 - n$.

The following lemma plays a key role in the proof of our main theorem.

Lemma 1 ([20]). *Let*

$$\Gamma = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = k\}$$

be a hyperplane of $\mathbb{R}^n$*, and* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *a quadratic form, given by*

$$f(x_1, x_2, \cdots, x_n) = a\sum_{i=1}^{n-1}(x_i)^2 + b(x_n)^2 - 2\sum_{1\le i<j\le n} x_i x_j, \qquad a > 0,\ b > 0.$$

Then, the constrained extremum problem $\min_{x\in\Gamma} f(x)$ *has a global solution as follows:*

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{k}{a+1}, \quad x_n = \frac{k}{b+1} = \frac{k(n-1)}{(a+1)b} = (a-n+2)\frac{k}{a+1},$$

provided that

$$b = \frac{n-1}{a-n+2}.$$

Definition 4. *A triple* $(\overline{g}, \varphi, \xi)$ *is called an almost contact metric structure on* $\overline{M}$ *if the following equations hold*

$$\varphi\xi = 0, \quad \overline{g}(\xi, \xi) = 1, \quad \varphi^2 X = -X + \overline{g}(X, \xi)\xi, \quad \overline{g}(\varphi X, Y) + \overline{g}(X, \varphi Y) = 0, \quad X, Y \in \Gamma(T\overline{M})$$

where φ *is a section of* $T\overline{M} \otimes T\overline{M}^*$ *and* ξ *is the structure vector field on* $\overline{M}$.

Definition 5. *A quadraple* $(\overline{\nabla}, \overline{g}, \varphi, \xi)$ *is called a Sasakian statistical structure on* $\overline{M}$ *if* $(\overline{\nabla}, \overline{g})$ *is a statistical structure.*

Theorem 2 ([2])**.** *Let* $(\overline{\nabla}, \overline{g}, \varphi, \xi)$ *be a Sasakian statistical structure on* $\overline{M}$. *Then, so is* $\left(\overline{\nabla}^*, \overline{g}, \varphi, \xi\right)$.

Definition 6. *Let* $(\overline{\nabla}, \overline{g}, \varphi, \xi)$ *be a Sasakian statistical structure on* $\overline{M}$, *and* $c \in \mathbb{R}$. *The Sasakian statistical structure is said to be of constant* φ*-sectional curvature if*

$$\begin{aligned} S(X,Y)Z = \frac{c+3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} + \frac{c-1}{4}\{\overline{g}(\varphi Y, Z)\varphi X \\ - \overline{g}(\varphi X, Z)\varphi Y - 2\overline{g}(\varphi X, Y)\varphi Z - \overline{g}(Y,\xi)\overline{g}(Z,\xi)X \\ + \overline{g}(X,\xi)\overline{g}(Z,\xi)Y + \overline{g}(Y,\xi)\overline{g}(Z,X)\xi - \overline{g}(X,\xi)\overline{g}(Z,Y)\xi\}, \end{aligned} \tag{6}$$

$X, Y, Z \in \Gamma(T\overline{M})$.

A submanifold M^n normal to ξ in a Sasakian statistical manifold $\overline{M}^{2m+1}$ is said to be a *C-totally real* submanifold. In this case, $\varphi\left(T_pM\right) \subset T_p^{\perp}M$, $p \in M$. In particular, if $n = m$, then M is called a *Legendrian* submanifold.

For submanifolds tangent to ξ, there is a θ-slant submanifold of a Sasakian statistical manifold as follows [21]:

A submanifold M^n tangent to ξ in a Sasakian statistical manifold is called a θ*-slant submanifold* if for any vector $X \in T_pM$, linearly independent on ξ_p, the angle between φX and T_pM is a constant $\theta \in [0, \frac{\pi}{2}]$, called the slant angle of M in $\overline{M}$. In particular, if $\theta = 0$ and $\theta = \frac{\pi}{2}$, M is invariant and anti-invariant, respectively.

3. Inequalities with Casorati Curvatures

Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi)$.

Let $p \in M$ and the set $\{e_1, e_2, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{2m}, e_{2m+1} = \xi\}$ be orthonormal bases of T_pM and $T_p^{\perp}M$, respectively. Then, we have the scalar curvature as follows:

$$\begin{aligned} 2\tau(p) &= 2\sum_{1\le i<j\le n} g\left(S\left(e_i, e_j\right)e_j, e_i\right) \\ &= \sum_{1\le i<j\le n}\{\overline{g}\left(R(e_i,e_j)e_j, e_i\right) + \overline{g}\left(R^*(e_i,e_j)e_j, e_i\right)\} \\ &= \sum_{1\le i<j\le n}\{\frac{c+3}{2} + \overline{g}\left(h(e_i,e_i), h^*(e_j,e_j)\right) + \overline{g}\left(h^*(e_i,e_i), h(e_j,e_j)\right) \\ &\quad - 2\overline{g}\left(h^*(e_i,e_j), h(e_i,e_j)\right)\} \\ &= \frac{n(n-1)(c+3)}{4} + n^2\overline{g}(H,H^*) - \sum_{i,j=1}^{n}\overline{g}\left(h^*(e_i,e_j), h(e_i,e_j)\right) \end{aligned} \tag{7}$$

Since $2H^0 = H + H^*$ and the definition of Casorati curvature, $4\|H^0\|^2 = \|H\|^2 + \|H^*\|^2 + 2g(H,H^*)$, we obtain that

$$
\begin{aligned}
2\tau(p) = & \frac{n(n-1)(c+3)}{4} + 2n^2\|H^0\|^2 \\
& - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) - 2n\mathcal{C}^0 + \frac{n}{2}\left(\mathcal{C} + \mathcal{C}^*\right),
\end{aligned} \tag{8}
$$

where $\mathcal{C}^0 = \frac{1}{2}\left(\mathcal{C} + \mathcal{C}^*\right)$.

Define a quadratic polynomial in the components of the second fundamental form h^0 by

$$
\begin{aligned}
\mathcal{P} = & t\mathcal{C}^0 + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}^0(L) + \frac{1}{2}n\left(\mathcal{C} + \mathcal{C}^*\right) \\
& - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) - 2\tau(p) + \frac{n(n-1)(c+3)}{4},
\end{aligned}
$$

where L is a hyperplane of T_pM. Without loss of generality, we can assume that L is spanned by $e_1, \cdots, e_{n-1}$. Then we derive

$$
\begin{aligned}
\frac{1}{2}\mathcal{P} = & \sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}\left[\frac{n^2+n(t-1)-2t}{r}\left(h_{ii}^{0\alpha}\right)^2 + \frac{2(n+t)}{n}\left(h_{in}^{0\alpha}\right)^2\right] \\
& + \sum_{\alpha=n+1}^{m}\left[\frac{2(n+t)(n-1)}{t}\sum_{1=i<j}^{n-1}\left(h_{ij}^{0\alpha}\right)^2 - 2\sum_{1=i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha} + \frac{t}{n}\left(h_{nn}^{0\alpha}\right)^2\right] \\
\geq & \sum_{\alpha=n+1}^{m}\left[\sum_{i=1}^{n-1}\frac{n^2+n(t-1)-2t}{t}\left(h_{ii}^{0\alpha}\right)^2 - 2\sum_{1=i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha} + \frac{t}{n}\left(h_{nn}^{0\alpha}\right)^2\right].
\end{aligned} \tag{9}
$$

For $\alpha = n+1, \cdots, m$, let us consider the quadratic form $f_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$
\begin{aligned}
f_\alpha\left(h_{11}^{0\alpha}, \cdots, h_{nn}^{0\alpha}\right) = & \frac{n^2+n(t-1)-2t}{t}\sum_{i=1}^{n-1}\left(h_{ii}^{0\alpha}\right)^2 \\
& - 2\sum_{1=i<j}^{n} h_{ii}^{0\alpha}h_{jj}^{0\alpha} + \frac{t}{n}\left(h_{nn}^{0\alpha}\right)^2,
\end{aligned} \tag{10}
$$

and the constrained extremum problem

$$
\min f_\alpha
$$

$$
\text{subject to } F^\alpha : h_{11}^{0\alpha} + \cdots + h_{nn}^{0\alpha} = c^\alpha,
$$

where c^α is a real constant. Comparing (10) with the quadratic function in Lemma 1, we see that

$$
a = \frac{n^2+n(t-1)-2t}{t}, \qquad b = \frac{t}{n}.
$$

Therefore, we have the critical point $\left(h_{11}^{0\alpha}, \cdots, h_{nn}^{0\alpha}\right)$, given by

$$
h_{11}^{0\alpha} = h_{22}^{0\alpha} = \cdots = h_{n-1\,n-1}^{0\alpha} = \frac{tc^\alpha}{(n+t)(n-1)}, \qquad h_{nn}^{0\alpha} = \frac{nc^\alpha}{n+t},
$$

is a global minimum point by Lemma 1. Moreover, $f_\alpha\left(h_{11}^{0\alpha}, \cdots, h_{nn}^{0\alpha}\right) = 0$. Therefore, we have

$$
\mathcal{P} \geq 0, \tag{11}
$$

which implies

$$
\begin{aligned}
2\tau(p) \leq & t\mathcal{C}^0 + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}^0(L) + \frac{1}{2}n\left(\mathcal{C} + \mathcal{C}^*\right) \\
& - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{n(n-1)(c+3)}{4}.
\end{aligned}
$$

Therefore, we derive

$$\rho \leq \frac{1}{n(n-1)}\{t\mathcal{C}^0 + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}^0(L)\} + \frac{1}{2(n-1)}(\mathcal{C}+\mathcal{C}^*) - \frac{n}{2(n-1)}\left(\|H\|^2+\|H^*\|^2\right) + \frac{c+3}{4}.$$

Therefore, we have the following theorem:

Theorem 3. *Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi)$. When $0 < t < n^2 - n$, the generalized normalized δ-Casorati curvature $\delta_C^0(t, n-1)$ on M satisfies*

$$\rho \leq \frac{1}{n(n-1)}\delta_C^0(t,n-1) + \frac{1}{2(n-1)}(\mathcal{C}+\mathcal{C}^*) - \frac{n}{2(n-1)}\left(\|H\|^2+\|H^*\|^2\right) + \frac{c+3}{4},$$

where $2\delta_C^0(t,n-1) = \delta_C(t,n-1) + \delta_C^*(t,n-1)$. The equality case holds identically at any point $p \in M$ if and only if $h = -h^*$.

For a unit hypersphere S^{2n+1} in $\mathbb{R}^{2n+2}$, the unit normal vector field N of S^{2n+1} provides the structure vector field $\xi = -JN$ with the standard almost complex structure J on $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. In addition, $\varphi = \pi \circ J$ is the natural projection of the tangent space of $\mathbb{R}^{2n+2}$ onto the tangent space of S^{2n+1}. Then we obtain the standard Sasakian structure (g, φ, ξ) on S^{2n+1}. From [2], we can construct a Sasakian statistical structures on S^{2n+1} of constant statistical sectional curvature 1. Therefore, we have the following optimal inequality:

Example 1. *Let M be an n-dimensional C-totally real submanifold of S^{2m+1}. Then, the generalized normalized δ-Casorati curvature $\delta_C^0(t, n-1)$ on M^n satisfies*

$$\rho \leq \frac{1}{n(n-1)}\delta_C^0(t,n-1) + \frac{1}{2(n-1)}(\mathcal{C}+\mathcal{C}^*) - \frac{n}{2(n-1)}\left(\|H\|^2+\|H^*\|^2\right) + 1.$$

When $t = \frac{n(n-1)}{2}$ in Theorem 3, we have an optimization for a normalized δ-Casoratic curvature as follows:

Corollary 1. *Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi)$. Then, the normalized δ-Casorati curvature $\delta_C^0(n-1)$ on M satisfies*

$$\rho \leq \delta_C^0(n-1) + \frac{1}{2(n-1)}(\mathcal{C}+\mathcal{C}^*) - \frac{n}{2(n-1)}\left(\|H\|^2+\|H^*\|^2\right) + \frac{c+3}{4}.$$

Proof. Taking $t = \frac{n(n-1)}{2}$ in $\delta_C^0(t, n-1)$, we have the following relation:

$$\left[\delta_C^0\left(\frac{n(n-1)}{2}; n-1\right)\right]_p = n(n-1)\left[\delta_C^0(n-1)\right]_p$$

in any point $p \in M$. Therefore, we have an optimal inequality for the normalized δ-Casorati curvature $\delta_C^0(n-1)$. □

Theorem 4. *Let M be an n-dimensional θ-slant submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $\left(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi\right)$. When $0 < t < n^2 - n$, the generalized normalized δ-Casorati curvature $\delta_C^0(t, n-1)$ on M satisfies*

$$\begin{aligned}\rho \leq & \frac{1}{n(n-1)}\delta_C^0(t,n-1) + \frac{1}{2(n-1)}\left(\mathcal{C}+\mathcal{C}^*\right) - \frac{n}{2(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) \\ & + \frac{n(n-1)(c+3)}{4} + \frac{3(n-1)(c-1)\cos^2\theta}{4} - \frac{(n-1)(c-1)}{2}.\end{aligned}$$

Proof. Let $p \in M$ and the set $\{e_1, e_2, \cdots, e_{n-1}, e_n = \xi\}$ and $\{e_{n+1}, e_{n+2}, \cdots, e_{2m}, e_{2m+1}\}$ be orthonormal bases of T_pM and $T_p^{\perp}M$, respectively. Then, we have the scalar curvature as follows:

$$\begin{aligned}2\tau(p) &= 2\sum_{1\leq i<j\leq n} g\left(S\left(e_i,e_j\right)e_j,e_i\right) \\ &= \sum_{1\leq i<j\leq n}\{\overline{g}\left(R(e_i,e_j)e_j,e_i\right) + \overline{g}\left(R^*(e_i,e_j)e_j,e_i\right)\} \\ &= \frac{n(n-1)(c+3)}{4} + \frac{3(n-1)(c-1)\cos^2\theta}{4} - \frac{(n-1)(c-1)}{2} \\ &\quad + n^2\overline{g}(H,H^*) - \sum_{i,j=1}^{n}\overline{g}\left(h^*(e_i,e_j),h(e_i,e_j)\right)\end{aligned} \tag{12}$$

By using a similar argument as in the proof of Theorem 3, we get

$$\begin{aligned}2\tau(p) \leq & t\mathcal{C}^0 + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}^0(L) \\ & + \frac{1}{2}n\left(\mathcal{C}+\mathcal{C}^*\right) - \frac{n^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) \\ & + \frac{n(n-1)(c+3)}{4} + \frac{3(n-1)(c-1)\cos^2\theta}{4} - \frac{(n-1)(c-1)}{2}.\end{aligned}$$

Therefore, we have an ineqaulity as follows:

$$\begin{aligned}\rho \leq & \frac{1}{n(n-1)}\delta_C^0(t,n-1) + \frac{1}{2(n-1)}\left(\mathcal{C}+\mathcal{C}^*\right) - \frac{n}{2(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) \\ & + \frac{n(n-1)(c+3)}{4} + \frac{3(n-1)(c-1)\cos^2\theta}{4} - \frac{(n-1)(c-1)}{2}.\end{aligned}$$

□

If M is an invariant submanifold, then $\theta = 0$. Then we obtain

Corollary 2. *Let M^n be an n-dimensional invariant submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $\left(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi\right)$. When $0 < t < n^2 - n$, we derive*

$$\begin{aligned}\rho \leq & \frac{1}{n(n-1)}\delta_C^0(t,n-1) + \frac{1}{2(n-1)}\left(\mathcal{C}+\mathcal{C}^*\right) \\ & - \frac{n}{2(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{n(n-1)(c+3)}{4} + \frac{(n-1)(c-1)}{4}.\end{aligned}$$

If M is an anti-invariant submanifold, then $\theta = \frac{\pi}{2}$. Then we obtain

Corollary 3. *Let M^n be an n-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional Sasakian statistical manifold $\left(\overline{M}, \overline{\nabla}, \overline{g}, \varphi, \xi\right)$. When $0 < t < n^2 - n$, we derive*

$$\rho \leq \frac{1}{n(n-1)}\delta_C^0(t, n-1) + \frac{1}{2(n-1)}\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{n}{2(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{n(n-1)(c+3)}{4} - \frac{(n-1)(c-1)}{2}.$$

Example 2. *Let M be an n-dimensional θ-slant submanifold of S^{2m+1}. Then, the generalized normalized δ-Casorati curvature $\delta_C^0(t, n-1)$ on M^n satisfies*

$$\rho \leq \frac{1}{n(n-1)}\delta_C^0(t, n-1) + \frac{1}{2(n-1)}\left(\mathcal{C} + \mathcal{C}^*\right) - \frac{n}{2(n-1)}\left(\|H\|^2 + \|H^*\|^2\right) + n(n-1).$$

Remark 2.

(1) *Taking $t = \frac{n(n-1)}{2}$ as Corollary 1, we have optimal inequalities for θ-slant submanifold of a Sasakian statistical manifold.*

(2) *In any optimization throughout our paper, the equality cases hold if and only if a submanifold is totally geodesic from $h = -h^*$.*

(3) *In the case for $t > n^2 - n$, the methods of finding the above inequalities are analogous.*

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