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# On a Length Problem for Univalent Functions

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**Abstract:** Let $g$ be an analytic function with the normalization in the open unit disk. Let $L(r)$ be the length of $g(\{z : |z| = r\})$. In this paper we present a correspondence between $g$ and $L(r)$ for the case when $g$ is not necessary univalent. Furthermore, some other results related to the length of analytic functions are also discussed.

**Keywords:** analytic functions; starlike functions; univalent functions; length problems

**MSC:** 30C45; 30C80

## 1. Introduction

Let $\mathcal{A}$ be the family of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subfamily of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$.

Let $C(r)$ denote the image curve of the $|z| = r < 1$ under the function $g \in \mathcal{A}$ which bound the area $A(r)$. Furthermore, let $L(r)$ be the length of $C(r)$ and $M(r) = \max_{|z|=r<1} |g(z)|$.

If $g \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0,\ z \in \mathbb{D},$$

then $g$ is said to be starlike with respect to the origin in $\mathbb{D}$ and we write $g \in \mathcal{S}^*$. It is known (for details, see [1,2]) that $\mathcal{S}^* \subset \mathcal{S}$.

The aim of the present paper is to prove, using a modified methodology, that in the following implication

$$g \in \mathcal{S}^* \quad \Rightarrow \quad L(r) = \mathcal{O}\left(M(r) \log \frac{1}{1-r}\right) \quad \text{as} \quad r \to 1, \tag{2}$$

where $\mathcal{O}$ denotes the Landau's symbol, the assumption that $g$ is starlike univalent can be changed by a weaker one. Result (2) was proved by Keogh [3]. Moreover, some other length problems for analytic functions are investigated. Several interesting developments related to length problems for univalent functions were considered in [4–15].

## 2. Main Results

**Theorem 1.** *Let $g$ be of the form* (1) *and suppose that*

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \left|\frac{1+z}{1-z}\right|, \quad z \in \mathbb{D}. \tag{3}$$

*Then*

$$L(r) = \mathcal{O}\left(M(r)\log\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1,$$

*where*

$$M(r) = \max_{|z|=r<1} |g(z)|$$

*and $\mathcal{O}$ means Landau's symbol.*

**Proof.** Let $z = re^{i\nu}$. We have $g \neq 0$ in $\mathbb{D} \setminus \{0\}$. In fact, if $g = 0$ in $\mathbb{D}$, it contradicts hypothesis (3). Applying [3] (Theorem 1) and the hypothesis of Theorem 1, we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zg'(z)| \mathrm{d}\nu = \int_0^{2\pi} \left|\frac{zg'(z)}{g(z)}\right| |g(z)| \mathrm{d}\nu \\ &\leq M(r) \int_0^{2\pi} \left|\frac{zg'(z)}{g(z)}\right| \mathrm{d}\nu \leq M(r) \int_0^{2\pi} \left|\frac{1+re^{i\nu}}{1-re^{i\nu}}\right| \mathrm{d}\nu \\ &\leq M(r)\left(2\pi + 4\log\frac{1+r}{1-r}\right) \quad \text{as} \quad r \to 1. \end{aligned}$$

□

**Remark 1.** *If $g$ satisfies the condition of Theorem 1, then $g$ is not necessary univalent in $\mathbb{D}$. It is well known that if $g \in \mathcal{S}$, then it follows that*

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1+|z|}{1-|z|}, \ z \in \mathbb{D}$$

*(for details, see [1] (Vol. 1, p. 69)).*

*If $g \in \mathcal{A}$ satisfies*

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g^{1-\gamma}(z)h^{\gamma}(z)}\right\} > 0, \ z \in \mathbb{D}$$

*for some $h \in \mathcal{S}^*$ and some $\gamma \in (0, \infty)$, then $g$ is said to be a Bazilevič function of type $\gamma$ [13]. The class of Bazilevič functions of type $\gamma$ is denoted by $g \in \mathcal{B}(\gamma)$ . We note that Theorem 1 improves the implication* (2) *by Keogh [3] and it is also related to Theorem 3 given by Thomas [13].*

We will need the following Tsuji's result.

**Lemma 1** ([16] (p. 226))**.** *(Theorem 3) If $0 \leq r < R$ and $z = e^{i\nu}$, then*

$$\frac{R-r}{R+r} \leq \mathfrak{Re}\left\{\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right\} = \frac{R^2-r^2}{R^2-2Rr\cos(\phi-\nu)+r^2} \leq \frac{R+r}{R-r}. \tag{4}$$

*Moreover,*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\phi-\nu)+r^2} \mathrm{d}\nu = 1. \tag{5}$$

**Theorem 2.** *Let $g$ be of the form (1) and suppose that*

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \left|\frac{1+z}{1-z}\right|, \quad z \in \mathbb{D} \tag{6}$$

*and*

$$M(r,\beta) = \max_{|z|=r<1} |g(z)| \leq \left|\frac{1+z}{1-z}\right|^{\beta}, \tag{7}$$

*where $1 < \beta$. Then*

$$L(r) = \mathcal{O}\left(\frac{1}{(1-r)^{\beta}}\right) \quad \text{as} \quad r \to 1,$$

*where $\mathcal{O}$ means Landau's symbol.*

**Proof.** From the hypotheses (6) and (7), it follows that

$$\begin{aligned}
L(r) &= \int_0^{2\pi} |zg'(z)| \mathrm{d}\nu = \int_0^{2\pi} \left|\frac{zg'(z)}{g(z)}\right| |g(z)| \mathrm{d}\nu \\
&\leq \int_0^{2\pi} \left|\frac{1+z}{1-z}\right| \left|\frac{1+z}{1-z}\right|^{\beta} \mathrm{d}\nu \leq 2^{1+\beta} \int_0^{2\pi} \frac{1}{|1-z|^{1+\beta}} \mathrm{d}\nu \\
&= \frac{2^{1+\beta}}{(1-r)^{\beta-1}} \int_0^{2\pi} \frac{1}{1-2r\cos\nu + r^2} \mathrm{d}\nu.
\end{aligned}$$

From (5), we have

$$\int_0^{2\pi} \frac{1}{1-2r\cos\nu+r^2} \mathrm{d}\nu = \frac{2\pi}{1-r^2}.$$

Hence, we obtain

$$\begin{aligned}
L(r) &\leq \frac{2^{1+\beta}}{(1-r)^{\beta-1}} \frac{2\pi}{1-r^2} \\
&= \mathcal{O}\left(\frac{1}{(1-r)^{\beta}}\right) \quad \text{as} \quad r \to 1.
\end{aligned}$$

Therefore, we complete the proof of Theorem 2. □

Let us recall the following Fejér-Riesz's result.

**Lemma 2** ([16])**.** *Let $h$ be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then*

$$\int_{-1}^{1} |h(z)|^p |\mathrm{d}z| \leq \frac{1}{2} \int_{|z|=1} |h(z)|^p |\mathrm{d}z|,$$

*where $p > 0$.*

**Theorem 3.** *Let $g$ be of the form (1) and suppose that*

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D}. \tag{8}$$

*Then*

$$\mathcal{O}\left(m(r)\log\frac{1}{1-r}\right) \leq L(r) \leq \mathcal{O}\left(\frac{M(r)}{1-r}\right) \quad \text{as} \quad r \to 1,$$

*where*

$$m(r) = \min_{|z|=r<1} |g(z)|, \quad M(r) = \max_{|z|=r<1} |g(z)| \tag{9}$$

*and $\mathcal{O}$ means Landau's symbol.*

**Proof.** From the assumption, we have

$$L(r) = \int_0^{2\pi} |zg'(z)| \mathrm{d}\nu = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \mathrm{d}\nu$$
$$\geq m(r) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| \mathrm{d}\nu$$

because $g(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. In fact, if $g(z) = 0$ in $\mathbb{D}$, it contradicts hypothesis (8). Applying Fejér-Riesz's Lemma 2, we have

$$L(r) \geq m(r) \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| \mathrm{d}\nu \geq 2m(r) \int_{-r}^{r} \frac{1-\rho}{1+\rho} \mathrm{d}\rho$$
$$\geq 2m(r) \log \frac{1+r}{1-r} - 2r$$
$$= \mathcal{O}\left( m(r) \log \frac{1}{(1-r)} \right) \quad \text{as} \quad r \to 1.$$

While, we obtain

$$L(r) = \int_0^{2\pi} |zg'(z)| \mathrm{d}\nu = \int_0^{2\pi} \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \mathrm{d}\nu$$
$$= M(r) \int_0^{2\pi} \frac{1+|z|}{1-|z|} \mathrm{d}\nu = 2\pi M(r) \frac{1+r}{1-r}$$
$$= \mathcal{O}\left( \frac{M(r)}{1-r} \right) \quad \text{as} \quad r \to 1.$$

Therefore, we complete the proof of Theorem 3. □

From Theorem 3, we have the following result.

**Corollary 1.** *Let $g$ be of the form (1) and suppose that $g$ is univalent in $\mathbb{D}$. Then we have*

$$\mathcal{O}\left( m(r) \log \frac{1}{1-r} \right) \leq L(r) \leq \mathcal{O}\left( \frac{M(r)}{1-r} \right) \quad \text{as} \quad r \to 1,$$

*where $m(r)$ and $M(r)$ are given by (9), respectively.*

**Proof.** From the hypothesis, we have

$$\frac{1-|z|}{1+z|} \leq \left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathbb{D},$$

which completes the proof. □

**Lemma 3** ([17] (p. 280) and [18] (p. 491))**.**

$$\int_0^{2\pi} \frac{\mathrm{d}\nu}{|1-re^{i\nu}|^{\beta}} = \begin{cases} \mathcal{O}\left( (1-r)^{1-\beta} \right) & \text{for the case } 1 < \beta, \\ \mathcal{O}\left( \log \frac{1}{1-r} \right) & \text{for the case } \beta = 1, \\ \mathcal{O}(1) & \text{for the case } 0 \leq \beta < 1, \end{cases}$$

*where* $0 < r < 1, 0 \leq \nu \leq 2\pi, 0 \leq \beta$ *and* $\mathcal{O}$ *means Landau's symbol.*

**Theorem 4.** *Let* $g$ *be of the form* (1) *and suppose that*

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1}{1-|z|}, \quad z \in \mathbb{D} \tag{10}$$

*and*

$$|g(z)| \leq \frac{1}{|1-z|^{\beta}}, \quad z \in \mathbb{D}. \tag{11}$$

*Then*

$$L(r) \leq \begin{cases} \mathcal{O}\left((1-r)^{-3/2}\right) & \text{for } 1 < \beta \leq 3/2, \\ \mathcal{O}\left((1-r)^{-3/2} \log \frac{1}{1-r}\right) & \text{for the case } \beta = 3/2, \\ \mathcal{O}\left((1-r)^{-\beta}\right) & \text{for the case } 3/2 < \beta, \end{cases}$$

*where* $0 < |z| = r < 1$ *and* $\mathcal{O}$ *means Landau's symbol.*

**Proof.** From the hypothesis (10), it follows that $g(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. Then we have

$$\begin{aligned} L(r) &= \int_0^{2\pi} \left|re^{i\nu} g'(re^{i\nu})\right| \mathrm{d}\nu = \int_0^{2\pi} \left|\frac{zg'(z)}{g(z)}\right| |g(z)| \mathrm{d}\nu \\ &< \int_0^{2\pi} \left(\frac{1}{1-|z|}\right)\left(\frac{1}{|1-z|^{\beta}}\right) \mathrm{d}\nu \\ &= \int_0^{2\pi} \left(\frac{1}{|1-z|}\right)\left(\frac{1}{|1-z|^{\beta-1}}\right)\left(\frac{1}{1-|z|}\right) \mathrm{d}\nu \\ &\leq \left(\int_0^{2\pi} \frac{1}{|1-z|^2} \mathrm{d}\nu\right)^{1/2} \left(\int_0^{2\pi} \left(\frac{1}{|1-z|^{2\beta-2}}\right) \frac{1}{(1-|z|)^2} \mathrm{d}\nu\right)^{1/2}. \end{aligned}$$

Applying Hayman's Lemma 3, we have

$$\begin{aligned} L(r) &\leq \left(\frac{1}{1-r^2}\right)^{1/2} \left(\frac{1}{1-r}\right) \mathcal{O}(1) \\ &= \mathcal{O}\left(\frac{1}{(1-r)^{3/2}}\right) \quad \text{as} \quad r \to 1 \end{aligned}$$

for the case $1 < \beta < 3/2$,

$$\begin{aligned} L(r) &\leq \left(\frac{1}{1-r^2}\right)^{1/2} \left(\frac{1}{1-r}\right) \mathcal{O}\left(\log \frac{1}{1-r}\right) \\ &= \mathcal{O}\left(\frac{1}{(1-r)^{3/2}} \log \frac{1}{1-r}\right) \quad \text{as} \quad r \to 1 \end{aligned}$$

for the case $\beta = 3/2$ and

$$L(r) = \left(\frac{1}{1-r^2}\right)^{1/2} \left(\frac{1}{1-r}\right) \left(\frac{1}{1-r}\right)^{(2\beta-3)/2} \quad \text{as} \quad r \to 1$$

for the case $3/2 < \beta$. □

**Lemma 4** ([16] (p. 227))**.** *If* $g(z) = u(z) + iv(z)$ *is analytic in* $|z| \leq R$*, then*

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\phi}) \frac{Re^{i\phi} + z}{Re^{i\phi} - z} \mathrm{d}\phi + iv(0). \tag{12}$$

*Moreover, if* $|z| < R$ *and* $v(0) = 0$*, then*

$$|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} |u(Re^{i\phi})| \left|\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right| \mathrm{d}\phi.$$

**Theorem 5.** *Let* $g$ *be of the form* (1)*. Then*

$$M(r) = \mathcal{O}\left(A(r)\log\frac{1}{1-r}\right) \quad as \quad r \to 1, \tag{13}$$

*where* $0 < |z| = r < 1$ *and* $\mathcal{O}$ *means Landau's symbol.*

**Proof.** It follows that

$$M(r) = \max_{|z|=r<1}\left|\int_0^z g'(s)\mathrm{d}s\right| = \max_{|z|=r<1}\left|\int_0^r g'(\rho e^{i\nu})\mathrm{d}\rho\right|.$$

Applying (12), we have

$$\begin{aligned} M(r) &= \max_{|z|=r<1}\left|\frac{1}{2\pi}\int_0^r\int_0^{2\pi}\Re\mathfrak{e}\, g'(te^{i\nu})\frac{te^{i\phi}+\rho e^{i\nu}}{te^{i\phi}-\rho e^{i\nu}}\mathrm{d}\phi\mathrm{d}\rho\right| \\ &\le \max_{|z|=r<1}\frac{1}{2\pi}\int_0^r\int_0^{2\pi}\left|g'(te^{i\nu})\right|\left|\frac{te^{i\phi}+\rho e^{i\nu}}{te^{i\phi}-\rho e^{i\nu}}\right|\mathrm{d}\phi\mathrm{d}\rho, \end{aligned}$$

where $0 \le \rho \le r < t < 1$. Then, applying Schwarz's lemma, we have

$$\begin{aligned} M(r) &\le \max_{|z|=r<1}\left(\frac{1}{2\pi}\int_0^r\int_0^{2\pi}\left|g'(te^{i\nu})\right|^2\mathrm{d}\phi\mathrm{d}\rho\right)^{1/2}\left(\int_0^r\int_0^{2\pi}\left|\frac{te^{i\phi}+\rho e^{i\nu}}{te^{i\phi}-\rho e^{i\nu}}\right|^2\mathrm{d}\phi\mathrm{d}\rho\right)^{1/2} \\ &\le \max_{|z|=r<1}(I_1)^{1/2}(I_2)^{1/2}, \quad \text{say.} \end{aligned}$$

Putting $0 < r_1 < r$ and $t = \sqrt{(1+\rho^2)/2}$, we have

$$\rho\mathrm{d}\rho = 2\sqrt{\frac{1+\rho^2}{2}}\mathrm{d}t < 2\mathrm{d}t.$$

Then we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_0^{r_1}\int_0^{2\pi}\left|g'(te^{i\phi})\right|^2\mathrm{d}\phi\mathrm{d}\rho + \frac{1}{2\pi r_1^2}\int_{\sqrt{(1+r_1^2)/2}}^{\sqrt{(1+r^2)/2}}\int_0^{2\pi} t\left|g'(te^{i\phi})\right|^2\mathrm{d}\phi\mathrm{d}t \\ &\le C + \frac{1}{2\pi r_1^2}A\left(\sqrt{\frac{1+r^2}{2}}\right) \\ &= C + \frac{1}{2\pi r_1^2}A\left(\sqrt{\frac{1+r^2}{2r^2}}r\right) \\ &= \mathcal{O}(A(r)) \quad \text{as} \quad r \to 1, \end{aligned}$$

where $C$ is a bounded positive constant. On the other hand, putting $t \to 1^-$, we have

$$\begin{aligned} I_2 &= \int_0^r \int_0^{2\pi} \left| \frac{te^{i\phi} + \rho e^{iv}}{te^{i\phi} - \rho e^{iv}} \right|^2 \mathrm{d}\phi \mathrm{d}\rho \\ &\leq \int_0^r \int_0^{2\pi} \frac{4}{|te^{i\phi} - \rho e^{iv}|^2} \mathrm{d}\phi \mathrm{d}\rho \\ &= \int_0^r \int_0^{2\pi} \frac{4}{t^2 - 2\rho t \cos(\phi - v) + \rho^2} \mathrm{d}\phi \mathrm{d}\rho. \end{aligned}$$

Using (5), we have

$$\begin{aligned} I_2 &\leq 8\pi \int_0^r \frac{1}{t^2 - \rho^2} \mathrm{d}\rho \\ &= \frac{4\pi}{t} \int_0^r \left( \frac{1}{t+\rho} + \frac{1}{t-\rho} \right) \mathrm{d}\rho \\ &= \frac{4\pi}{t} \log \frac{t+r}{t-r} \to \mathcal{O}\left( \log \frac{1}{1-r} \right) \quad \text{as } r \to 1. \end{aligned}$$

Therefore we complete the proof of (13). □

**Remark 2.** *In Theorem 5, we do not suppose that $g$ is univalent in $|z| < 1$ and therefore, it improves the result by Pommerenke [2].*

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