mathematics

Article

Some Properties of the Fuss–Catalan Numbers

Feng Qi [1,2,3,*] **and Pietro Cerone** [4]

[1] Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
[2] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China
[3] School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China
[4] Department of Mathematics and Statistics, La Trobe University, Melbourne (Bundoora) 3086, Victoria, Australia; p.cerone@latrobe.edu.au
* Correspondence: qifeng618@gmail.com

Received: 31 October 2018; Accepted: 14 November 2018; Published: 24 November 2018

Abstract: In the paper, the authors express the Fuss–Catalan numbers as several forms in terms of the Catalan–Qi function, find some analytic properties, including the monotonicity, logarithmic convexity, complete monotonicity, and minimality, of the Fuss–Catalan numbers, and derive a double inequality for bounding the Fuss–Catalan numbers.

Keywords: Fuss–Catalan number; Catalan–Qi function; Catalan number; monotonicity; logarithmic convexity; complete monotonicity; minimality; inequality

MSC: 05A19; 05A99; 11B75; 11B83; 26A48; 33B15

1. Introduction and Main Results

The Catalan numbers C_n for $n \ge 0$ constitute a sequence that is one of the most fascinating sequences in combinatorial number theory with over fifty significant combinatorial interpretations. For details, please refer to monographs [1,2] and closely related references therein.

The Catalan numbers C_n have a generating function

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \cdots.$$

Two explicit formulas for C_n with $n \ge 0$ read that

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0$$

is the classical Euler gamma function. In [1,3,4], it was mentioned that there exists an asymptotic expansion

$$C_x = \frac{4^x \Gamma(x+1/2)}{\sqrt{\pi}\,\Gamma(x+2)} \sim \frac{4^x}{\sqrt{\pi}} \left(\frac{1}{x^{3/2}} - \frac{9}{8}\frac{1}{x^{5/2}} + \frac{145}{128}\frac{1}{x^{7/2}} + \cdots \right), \quad x \to \infty. \tag{2}$$

In the newly published papers [5–10], there are some new results on the Catalan numbers C_n and others.

In [11], an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced as

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge 0. \tag{3}$$

For uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call $C(a,b;x)$ the Catalan–Qi function and, when taking $x=n\ge 0$, call $C(a,b;n)$ the Catalan–Qi numbers. Comparing with the second formula in (1) and the first equality in (2), it is clear that

$$C\left(\frac{1}{2},2;x\right)=C_x,\quad x\ge 0. \tag{4}$$

By the definition (3), we easily see that

$$C(a,b;x)C(b,c;x)=C(a,c;x),\quad a,b,c>0,\quad x\ge 0.$$

In the papers [11–23], the authors discovered many analytic properties, including the monotonicity, a general expression of the asymptotic expansion (2), Schur-convexity, a generalization of the expansion (2), minimality, (logarithmically) complete monotonicity, product inequalities, a generating function, logarithmic convexity, exponential representations, determinantal inequalities, series identities, integral representations, and connections with the Bessel polynomials and the Bell polynomials of the second kind, of the Catalan numbers and function C_n and C_x and the Catalan–Qi function $C(a,b;x)$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined in [24] as numbers of the form

$$A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}. \tag{5}$$

Comparing with the first formula in (1), it is obvious that

$$A_n(2,1)=C_n,\quad n\ge 0. \tag{6}$$

A generalization of the Catalan numbers C_n was defined in [25–27] by

$${}_pd_n=\frac{1}{n}\binom{pn}{n-1}=\frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n\ge 1$. The usual Catalan numbers $C_n={}_2d_n$ are a special case with $p=2$. It is immediate that

$$A_{n-1}(p,p)={}_pd_n$$

for $n\ge 1$. There exists some literature such as [28–38] devoted to the investigation of the Fuss–Catalan numbers $A_n(p,r)$.

Considering the relations (4) and (6), one may ask a question: what is the relation between the Catalan–Qi numbers $C(a,b;n)$ and the Fuss–Catalan numbers $A_n(p,r)$? This question is answered by Theorem 1 below.

Theorem 1. *For $n,r\ge 0$ and $p>1$, we have*

$$A_n(p,r)=r^n\frac{\prod_{k=1}^{p}C\left(\frac{k+r-1}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)}. \tag{7}$$

For $n, p \in \mathbb{N}$ and $r \geq 0$, we have

$$A_n(p,r) = \frac{r}{nB(n(p-1)+1,n)} \frac{(np)^{r-1}}{[n(p-1)+1]^r} C(np, n(p-1)+1; r), \tag{8}$$

where $B(x,y)$ denotes the classical beta function. For $r+1 > n > 0$ and $p \geq 0$, we have

$$A_n(p,r) = \frac{1}{nB(n,r-n+1)} \left(\frac{r}{r-n+1}\right)^{np} C(r, r-n+1; np). \tag{9}$$

When $r+1 > n \geq 1$ and $p \geq 0$, we have

$$A_n(p,r) = \frac{1}{n} \frac{[B(r+1-n,n)]^{p-1}}{[B(r,n)]^p} \prod_{k=0}^{n-1} C\left(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\right). \tag{10}$$

For $n \geq 2$, $r+1 > n$, and $p \in \mathbb{N}$, we have

$$A_n(p,r) = rp^{1/2} B(n-1,2) \frac{[B(r+1-n,n-1)]^{n-1}}{[B(r+1-n+p,n-1)]^n} \prod_{k=0}^{p-1} C\left(\frac{r+k}{p}, \frac{r+k+1-n}{p}; n\right). \tag{11}$$

Recall from ([39], pp. 372–373) and ([40], p. 108, Definition 4) that a sequence $\{\mu_n\}_{0 \leq n \leq \infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences alternate sign, that is,

$$(-1)^k \Delta^k \mu_n \geq 0, \quad n,k \geq 0,$$

where

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}.$$

Further recall from ([40], p. 163, Definition 14a) that a completely monotonic sequence $\{a_n\}_{n \geq 0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Applying the identity (7) and several analytic properties of the Catalan–Qi function $C(a,b;x)$, we find several analytic properties, including monotonicity, logarithmic convexity, complete monotonicity, and minimality, of the Fuss–Catalan sequence $\{A_n(p,r)\}_{n \geq 0}$ and related ones.

Theorem 2. *When $p \geq r > 0$,*

1. *the sequence $\{\mathcal{A}_n(p,r)\}_{n \geq 0}$,*

$$\mathcal{A}_n(p,r) = \begin{cases} 1, & n = 0, \\ \dfrac{1}{\sqrt[n]{A_n(p,r)}}, & n \in \mathbb{N}, \end{cases} \tag{12}$$

   is logarithmically convex, completely monotonic, and minimal;
2. *the sequence of the Fuss–Catalan numbers $\{A_n(p,r)\}_{n \geq 0}$ is increasing and logarithmically convex.*

Finally, by applying a double inequality of the beta function in the papers [41] and ([42], pp. 78–81, Section 3), we derive a double inequality for the Fuss–Catalan numbers $A_n(p,r)$.

Theorem 3. *For $n \geq 2$ and $p, r \in \mathbb{N}$, we have*

$$A_n(p,r) \geq \frac{r[n(p-1)+r+1]}{n}. \tag{13}$$

When $m \triangleq \min\{D(n-1)D(n(p-1)+r+1), b_A(n-1)[n(p-1)+r+1]\} < 1$ *for* $n \ge 2$ *and* $p, r \in \mathbb{N}$, *we have*

$$A_n(p,r) \le \frac{r[n(p-1)+r+1]}{n(1-m)},$$

where

$$D(x) = \frac{x-1}{\sqrt{2x-1}} \quad \text{and} \quad b_A = \max_{x \ge 1}\left[\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)}\right] = 0.08731\ldots \tag{14}$$

for $x \ge 1$.

2. Lemmas

In order to prove Theorems 2 and 3, we need the following notion and lemmas.

Recall from [43–45] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ holds on I for all $k \in \mathbb{N}$.

Lemma 1 ([18], Theorem 6)**.** *The function*

$$\mathcal{C}^{\pm 1}(a,b;x) = \begin{cases} 1, & x = 0 \\ [C(a,b;x)]^{\pm 1/x}, & x > 0 \end{cases}$$

is logarithmically completely monotonic on $[0,\infty)$ *if and only if* $a \gtrless b$.

Lemma 2 ([18], Theorem 7)**.** *Let* $a, b > 0$ *and* $x \ge 0$. *Then*

1. *the unique zero* x_0 *of the equation*
$$\frac{\psi(x+b) - \psi(x+a)}{\ln b - \ln a} = 1$$
satisfies $x_0 \in \left(0, \frac{1}{2}\right)$, *where* ψ *is the logarithmic derivative of the gamma function* Γ;
2. *when* $b > a$, *the function* $C(a,b;x)$ *is decreasing in* $x \in [0, x_0)$, *increasing in* $x \in (x_0, \infty)$, *and logarithmically convex in* $x \in [0,\infty)$;
3. *when* $b < a$, *the function* $C(a,b;x)$ *is increasing in* $x \in [0, x_0)$, *decreasing in* $x \in (x_0, \infty)$, *and logarithmically concave in* $x \in [0,\infty)$.

Lemma 3 ([41] and [42] pp. 78–81, Section 3)**.** *For* $x, y > 1$, *we have*

$$0 \le \frac{1}{xy} - B(x,y) \le \min\left\{\frac{D(x)D(y)}{xy}, b_A\right\}, \tag{15}$$

where $D(x)$ *and* b_A *are defined as in* (14).

3. Proofs of Theorems 1–3

We now start out to prove our theorems.

Proof of Theorem 1. By virtue of the Gauss multiplication formula

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) \tag{16}$$

from ([46], p. 256, 6.1.20), the Fuss–Catalan numbers $A_n(p,r)$ defined by the second expression in (5) can also be rewritten as

$$
\begin{aligned}
A_n(p,r) &= \frac{r\Gamma(p(n+r/p))}{\Gamma(n+1)\Gamma((p-1)(n+(r+1)/(p-1)))} \\
&= \frac{r\frac{p^{np+r-1/2}}{(2\pi)^{(p-1)/2}}\prod_{k=0}^{p-1}\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma(n+1)\frac{(p-1)^{n(p-1)+r+1/2}}{(2\pi)^{(p-2)/2}}\prod_{k=0}^{p-2}\Gamma\left(n+\frac{k+r+1}{p-1}\right)} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r \frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n \frac{\Gamma(1)}{\Gamma\left(\frac{r}{p}\right)}\left(\frac{1}{r/p}\right)^n \frac{\Gamma\left(n+\frac{r}{p}\right)}{\Gamma(n+1)} \\
&\quad\times\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p-1}\right)}{\Gamma\left(\frac{k+r}{p}\right)}\left(\frac{\frac{k+r}{p-1}}{\frac{k+r}{p}}\right)^n \frac{\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma\left(n+\frac{k+r}{p-1}\right)}\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\left(\frac{r/p}{1}\right)^n\prod_{k=1}^{p-1}\left(\frac{\frac{k+r}{p}}{\frac{k+r}{p-1}}\right)^n \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r \frac{r^{n+1}}{[p(p-1)]^{1/2}}\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r \frac{1}{[p(p-1)]^{1/2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{r}{p}+\frac{k}{p}\right)}{\prod_{k=0}^{p-2}\Gamma\left(\frac{r+1}{p-1}+\frac{k}{p-1}\right)}r^{n+1}C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right) \\
&= r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right).
\end{aligned}
$$

Further making use of

$$C(a,b;x) = \frac{1}{C(b,a;x)}, \quad a,b>0, \quad x\geq 0,$$

We can rearrange the above result as

$$
\begin{aligned}
A_n(p,r) &= r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}\left[C\left(\frac{k+r}{p},1;n\right)C\left(1,\frac{k+r}{p-1};n\right)\right] = r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}\frac{C\left(\frac{k+r}{p},1;n\right)}{C\left(\frac{k+r}{p-1},1;n\right)} \\
&= r^nC\left(\frac{r}{p},1;n\right)\frac{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)} = r^n\frac{\prod_{k=0}^{p-1}C\left(\frac{k+r}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)} = r^n\frac{\prod_{k=1}^{p}C\left(\frac{k+r-1}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)}.
\end{aligned}
$$

The identity (7) thus follows.

It is straightforward that

$$
\begin{aligned}
A_n(p,r) &= \frac{r}{\Gamma(n+1)}\frac{\Gamma(r+np)}{\Gamma(r+np-n+1)} \\
&= \frac{r}{\Gamma(n+1)}\frac{\Gamma(np)}{\Gamma(np-n+1)}\left(\frac{np}{np-n+1}\right)^rC(np,np-n+1;r) \\
&= \frac{r\Gamma(n)}{np\Gamma(n+1)}\frac{\Gamma(np+1)}{\Gamma(np-n+1)\Gamma(n)}\left(\frac{np}{np-n+1}\right)^rC(np,np-n+1;r) \\
&= \frac{r}{nB(np-n+1,n)}\frac{(np)^{r-1}}{(np-n+1)^r}C(np,np-n+1;r)
\end{aligned}
$$

and, interchanging the role of r and np,

$$\begin{aligned}
A_n(p,r) &= \frac{r}{\Gamma(n+1)}\frac{\Gamma(np+r)}{\Gamma(np+r-n+1)}\\
&= \frac{r}{\Gamma(n+1)}\frac{\Gamma(r)}{\Gamma(r-n+1)}\left(\frac{r}{r-n+1}\right)^{np}C(r,r-n+1;np)\\
&= \frac{1}{n}\frac{\Gamma(r+1)}{\Gamma(n)\Gamma(r-n+1)}\left(\frac{r}{r-n+1}\right)^{np}C(r,r-n+1;np)\\
&= \frac{1}{nB(n,r-n+1)}\left(\frac{r}{r-n+1}\right)^{np}C(r,r-n+1;np).
\end{aligned}$$

Therefore, the identities (8) and (9) follow.

By the Gauss multiplication formula (16) again, when $r+1>n$, the Fuss–Catalan numbers $A_n(p,r)$ defined by the second expression in (5) can be rearranged as

$$\begin{aligned}
A_n(p,r) &= \frac{r}{\Gamma(n+1)}\frac{\frac{n^{np+r-1/2}}{(2\pi)^{(n-1)/2}}\prod_{k=0}^{n-1}\Gamma\left(p+\frac{r}{n}+\frac{k}{n}\right)}{\frac{n^{n(p-1)+r+1/2}}{(2\pi)^{(n-1)/2}}\prod_{k=0}^{n-1}\Gamma\left(p+\frac{r-n+1}{n}+\frac{k}{n}\right)}\\
&= \frac{n^{n-1}r}{\Gamma(n+1)}\prod_{k=0}^{n-1}\frac{\Gamma\left(\frac{r+k}{n}\right)}{\Gamma\left(\frac{r-n+k+1}{n}\right)}\left(\frac{r+k}{r-n+k+1}\right)^{p}\prod_{k=0}^{n-1}\frac{\Gamma\left(\frac{r-n+k+1}{n}\right)}{\Gamma\left(\frac{r+k}{n}\right)}\left(\frac{r-n+k+1}{r+k}\right)^{p}\frac{\Gamma\left(p+\frac{r+k}{n}\right)}{\Gamma\left(p+\frac{r-n+k+1}{n}\right)}\\
&= \frac{n^{n-1}r}{\Gamma(n+1)}\frac{\frac{(2\pi)^{(n-1)/2}}{n^{r-1/2}}\Gamma(r)}{\frac{(2\pi)^{(n-1)/2}}{n^{r-n+1/2}}\Gamma(r-n+1)}\left[\frac{\Gamma(r+n)\Gamma(r+1-n)}{\Gamma(r)\Gamma(r+1)}\right]^{p}\prod_{k=0}^{n-1}C\left(\frac{r+k}{n},\frac{r-n+k+1}{n};p\right)\\
&= \frac{\Gamma(n)}{\Gamma(n+1)}\left[\frac{\Gamma(r+n)}{\Gamma(r)\Gamma(n)}\right]^{p}\left[\frac{\Gamma(r+1-n)\Gamma(n)}{\Gamma(r+1)}\right]^{p-1}\prod_{k=0}^{n-1}C\left(\frac{r+k}{n},\frac{r-n+k+1}{n};p\right)\\
&= \frac{1}{n}\frac{[B(r+1-n,n)]^{p-1}}{[B(r,n)]^{p}}\prod_{k=0}^{n-1}C\left(\frac{r+k}{n},\frac{r-n+k+1}{n};p\right).
\end{aligned}$$

The identity (10) is thus proved.

Similarly, by virtue of (16) once again, we have

$$\begin{aligned}
A_n(p,r) &= \frac{rp^{n-1}}{\Gamma(n+1)}\prod_{k=0}^{p-1}\frac{\Gamma\left(n+\frac{r+k}{p}\right)}{\Gamma\left(n+\frac{r+k+1-n}{p}\right)}\\
&= \frac{rp^{n-1}}{\Gamma(n+1)}\prod_{k=0}^{p-1}\frac{\Gamma\left(\frac{r}{p}+\frac{k}{p}\right)}{\Gamma\left(\frac{r+1-n}{p}+\frac{k}{p}\right)}\left(\frac{r+k}{r+k+1-n}\right)^{n}\prod_{k=0}^{p-1}C\left(\frac{r+k}{p},\frac{r+k+1-n}{p};n\right)\\
&= \frac{rp^{n-1}}{\Gamma(n+1)}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{r}{p}+\frac{k}{p}\right)}{\prod_{k=0}^{p-1}\Gamma\left(\frac{r+1-n}{p}+\frac{k}{p}\right)}\left(\prod_{k=0}^{p-1}\frac{r+k}{r+k+1-n}\right)^{n}\prod_{k=0}^{p-1}C\left(\frac{r+k}{p},\frac{r+k+1-n}{p};n\right)\\
&= \frac{rp^{n-1}}{\Gamma(n+1)}\frac{p^{r+1-n}\Gamma(r)}{p^{r-1/2}\Gamma(r+1-n)}\left[\frac{\Gamma(r+p)}{\Gamma(r)}\frac{\Gamma(r+1-n)}{\Gamma(r+1-n+p)}\right]^{n}\prod_{k=0}^{p-1}C\left(\frac{r+k}{p},\frac{r+k+1-n}{p};n\right)\\
&= rp^{1/2}B(n-1,2)\frac{[B(r+1-n,n-1)]^{n-1}}{[B(r+1-n+p,n-1)]^{n}}\prod_{k=0}^{p-1}C\left(\frac{r+k}{p},\frac{r+k+1-n}{p};n\right).
\end{aligned}$$

The identity (11) is demonstrated. The proof of Theorem 1 is complete. □

Proof of Theorem 2. We call (see related chapters [39] Chapter XIII, [45] Chapter 1, and [40] Chapter IV) a positive function f defined on an interval I completely monotonic if all of its derivatives satisfy $0 \le (-1)^k f^{(k)}(x) < \infty$ for all $k \ge 0$ on I. The inclusions

$$\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0,\infty)] \quad \text{and} \quad \mathcal{L}[I] \subset \mathcal{C}[I] \tag{17}$$

were found in [44,47,48], where $\mathcal{S}$, $\mathcal{L}[I]$, and $\mathcal{C}[I]$ represent the class of Stieltjes transforms on $(0,\infty)$, the class of logarithmically completely monotonic functions I, and the class of completely monotonic functions on I, respectively. This was mentioned in the monograph [45] and admitted in mathematical community.

By Lemma 1, since $\frac{r}{p} < 1$ and $\frac{k+r}{p} < \frac{k+r}{p-1}$ for all $p > 1$, $p > r > 0$, and $1 \le k \le p-1$, the functions

$$\frac{1}{C(r/p,1;x)} \quad \text{and} \quad \frac{1}{C((k+r)/p,(k+r)/(p-1);x)}$$

are logarithmically completely monotonic on $[0,\infty)$. It is easy to see that the product of finitely many logarithmically completely monotonic functions is still logarithmically completely monotonic. Hence, the function

$$\frac{1}{rC(r/p,1;x)\prod_{k=1}^{p-1} C((k+r)/p,(k+r)/(p-1);x)} \tag{18}$$

is logarithmically completely monotonic on the interval $[0,\infty)$. Consequently, by the definition of logarithmically completely monotonic functions, the sequence (12) is decreasing and logarithmically convex.

Further, by virtue of the first inclusion in (17) and the logarithmically complete monotonicity of the function (18), the function (18) is also completely monotonic on $[0,\infty)$. By Theorem 14b in ([40] p. 164), we know that a sequence $\{a_n\}_0^\infty$ is minimal completely monotonic if and only if there exists a completely monotonic function $f(x)$ on $[0,\infty)$ such that $f(n) = a_n$ for $n \ge 0$. Hence, we arrive at the complete monotonicity and minimality of the sequence $\mathcal{A}_n(p,r)$ defined by (12).

Similarly, by Lemma 2, the identity (7), and $A_0(p,r) = 1$ for all $p > 1$ and $r > 0$, we conclude that the sequence of the Fuss–Catalan numbers $\{A_n(p,r)\}_{n\ge 0}$ is increasing and logarithmically convex. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Applying the inequality (15) to

$$A_n(p,r) = \frac{r}{n(n-1)} \frac{1}{B(n-1,n(p-1)+r+1)}$$

results in

$$\frac{r[n(p-1)+r+1]}{n} \le A_n(p,r)$$

and, when $\min\{D(n-1)D(n(p-1)+r+1), b_A(n-1)[n(p-1)+r+1]\} < 1$,

$$A_n(p,r) \le \frac{r[n(p-1)+r+1]}{n} \frac{1}{1-\min\{D(n-1)D(n(p-1)+r+1), b_A(n-1)[n(p-1)+r+1]\}}$$

which can be rearranged as (13). The proof of Theorem 3 is complete. □

4. Remarks

Finally, we give several remarks on our main results.

Remark 1. *The Catalan–Qi function $C(a,b;z)$, the second expression in (5) of the Fuss–Catalan numbers $A_n(p,r)$, and all the identities from (7) to (11) in Theorem 1 can be extended to real values, even to complex values, that the variables a, b, x and n, p, r can take.*

Remark 2. *The identity (7) is more beautiful and informative than the others in Theorem 1, because the form of the identity (7) is simpler, more regular, and with less restrictions to n, p, and r.*

Remark 3. *Theorem 1 means that the Fuss–Catalan numbers $A_n(p,r)$ can be represented in terms of the Catalan–Qi functions $C(a,b;x)$. This fact shows that introducing the Catalan–Qi function $C(a,b;x)$ in [11] is analytically significant. However, we need the combinatorialists to combinatorially interpret the Catalan–Qi function $C(a,b;x)$ or its special cases.*

Remark 4. *By the definition of the classical beta function, we easily see that*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=\frac{B(b-a,z+a)}{\Gamma(b-a)}. \tag{19}$$

We can use the inequality (15) and the formula (19) to bound the Catalan numbers C_n, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;x)$.

There has been an amount of literature on the ratio of two gamma functions, see, for example, the expository and survey articles [49–55] and plenty of references therein. Applying results in these literature, we can estimate the Catalan numbers C_n, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;x)$ in terms of inequalities and asymptotic formulas of the gamma function $\Gamma(x)$. For example, the double inequality

$$\begin{aligned}\sqrt{\frac{b}{a}}\,[I(a,b)]^{a-b}\exp\left[\sum_{j=1}^{2m}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{a^{2j-1}}-\frac{1}{b^{2j-1}}\right)\right]<\frac{\Gamma(a)}{\Gamma(b)}\\ <\sqrt{\frac{b}{a}}\,[I(a,b)]^{a-b}\exp\left[\sum_{j=1}^{2m-1}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{a^{2j-1}}-\frac{1}{b^{2j-1}}\right)\right]\end{aligned}$$

for $m\in\mathbb{N}$ and $a,b>0$ was derived in ([18] Theorem 11), where B_i for $i\ge 0$ are the classical Bernoulli numbers generated by

$$\frac{x}{e^x-1}=\sum_{i=0}^{\infty}B_i\frac{x^i}{i!}=1-\frac{x}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!},\quad |x|<2\pi$$

and

$$I(\alpha,\beta)=\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}$$

for $\alpha,\beta>0$ with $\alpha\ne\beta$ stands for the identric mean [56]. One more example is the double inequality

$$e^{\psi(L(a,b))}<\left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)}<e^{\psi(I(a,b))},\quad a,b>0,\quad a\ne b$$

in [57,58], where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and

$$L(a,b)=\frac{b-a}{\ln b-\ln a},\quad a,b>0,\quad a\ne b$$

is called the logarithmic mean [56].

In the papers [59,60], some inequalities for the beta function $B(x,y)$ are reviewed and surveyed. The main result in ([60], Theorem) states that, for every $k\ge 1$ and every point $x,y>0$, the quantity $(-1)^{k-1}D_{x,y}^{(k)}(X,Y)$ is positive for $X,Y>0$ and, if k is even, positive definite in X,Y, where $D_{x,y}^{(k)}(X,Y)$ denotes the kth differential of $F(x+X,y+Y)$ in X and Y and

$$F(x,y)=\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)}\frac{(x+y)^{x+y}}{x^xy^y}.$$

As said in ([60], p. 1430), this theorem can produce lower and upper bounds for $F(x,y)$ and every bound for $F(x,y)$ is actually a bound for $B(x,y)$. By inequalities for the beta function $B(x,y)$, we can derive more inequalities for the Fuss–Catalan numbers $A_n(p,r)$.

To the best of our knowledge, we do not find any known inequality for the Fuss–Catalan numbers $A_n(p,r)$ in published literature. It seems that there are more identities than inequalities in combinatorics.

Remark 5. *In order to give a better combinatorial context, we would like to mention three references [2,61,62]. In [61], a collection of combinatorial interpretations for the Fuss–Catalan numbers $A_n(p;1)$ is presented. In [2], a monograph about the Catalan numbers C_n and their generalizations, the Fuss–Catalan numbers $A_n(p,r)$ are studied in Additional Problem A14. In ([62], Corollary 3.4), there is a conclusion about the sequence $\{A_{n-1}(p,p)\}_{n\ge0}$.*

Remark 6. *This paper is a simplified and corrected version of the preprint [63].*

5. Conclusions

In this paper, the authors express the Fuss–Catalan numbers as several forms in terms of the Catalan–Qi function; find some analytic properties, including the monotonicity, logarithmic convexity, complete monotonicity, and minimality, of the Fuss–Catalan numbers; and derive a double inequality for bounding the Fuss–Catalan numbers.

Author Contributions: The authors contributed equally to this work. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors are thankful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Koshy, T. *Catalan Numbers with Applications*; Oxford University Press: Oxford, UK, 2009.
2. Stanley, R.P. *Catalan Numbers*; Cambridge University Press: New York, NY, USA, 2015. [CrossRef]
3. Graham, R.L.; Knuth, D.E.; Patashnik, O. *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed.; Addison-Wesley Publishing Company: Reading, MA, USA, 1994.
4. Vardi, I. *Computational Recreations in Mathematica*; Addison-Wesley: Redwood City, CA, USA, 1991.
5. Qi, F. An improper integral, the beta function, the Wallis ratio, and the Catalan numbers. *Probl. Anal. Issues Anal.* **2018**, *7*, 104–115. [CrossRef]
6. Qi, F. Parametric integrals, the Catalan numbers, and the beta function. *Elem. Math.* **2017**, *72*, 103–110. [CrossRef]
7. Qi, F.; Akkurt, A.; Yildirim, H. Catalan numbers, k-gamma and k-beta functions, and parametric integrals. *J. Comput. Anal. Appl.* **2018**, *25*, 1036–1042.
8. Qi, F.; Guo, B.-N. Integral representations of the Catalan numbers and their applications. *Mathematics* **2017**, *5*, 40. [CrossRef]
9. Sofo, A. Derivatives of Catalan related sums. *J. Inequal. Pure Appl. Math.* **2009**, 10, 69.
10. Sofo, A. Estimates for Sums of Catalan Type Numbers. In *Inequality Theory and Applications*; Cho, Y.J., Kim, J.K., Dragomir, S.S., Eds.; Nova Science Publishers: New York, NY, USA, 2012; Volume 6, Chapter 20, pp. 203–211.
11. Qi, F.; Shi, X.-T.; Mahmoud, M.; Liu, F.-F. The Catalan numbers: A generalization, an exponential representation, and some properties. *J. Comput. Anal. Appl.* **2017**, *23*, 937–944.
12. Liu, F.-F.; Shi, X.-T.; Qi, F. A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function. *Glob. J. Math. Anal.* **2015**, *3*, 140–144. [CrossRef]
13. Mahmoud, M.; Qi, F. Three identities of the Catalan–Qi numbers. *Mathematics* **2016**, *4*, 35. [CrossRef]
14. Qi, F. Some properties of the Catalan numbers. *Ars Comb.* **2021**, *144*, in press.

15. Qi, F.; Guo, B.-N. Logarithmically complete monotonicity of a function related to the Catalan–Qi function. *Acta Univ. Sapientiae Math.* **2016**, *8*, 93–102. [CrossRef]
16. Qi, F.; Guo, B.-N. Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers. *Cogent Math.* **2016**, *3*, 1179379. [CrossRef]
17. Qi, F.; Guo, B.-N. Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers. In *Mathematical Analysis and Applications: Selected Topics*, 1st ed.; Michael, R., Hemen, D., Ravi, P.A., Eds.; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2018; Chapter 5, pp. 101–133. [CrossRef]
18. Qi, F.; Mahmoud, M.; Shi, X.-T.; Liu, F.-F. Some properties of the Catalan–Qi function related to the Catalan numbers. *SpringerPlus* **2016**, *5*, 1126. [CrossRef] [PubMed]
19. Qi, F.; Shi, X.-T.; Liu, F.-F. An integral representation, complete monotonicity, and inequalities of the Catalan numbers. *Filomat* **2018**, *32*, 575–587. [CrossRef]
20. Qi, F.; Shi, X.-T.; Liu, F.-F.; Kruchinin, D.V. Several formulas for special values of the Bell polynomials of the second kind and applications. *J. Appl. Anal. Comput.* **2017**, *7*, 857–871. [CrossRef]
21. Qi, F.; Shi, X.-T.; Mahmoud, M.; Liu, F.-F. Schur-convexity of the Catalan–Qi function related to the Catalan numbers. *Tbilisi Math. J.* **2016**, *9*, 141–150. [CrossRef]
22. Shi, X.-T.; Liu, F.-F.; Qi, F. An integral representation of the Catalan numbers. *Glob. J. Math. Anal.* **2015**, *3*, 130–133. [CrossRef]
23. Yin, L.; Qi, F. Several series identities involving the Catalan numbers. *Trans. A Razmadze Math. Inst.* **2018**, *172*, 466–474. [CrossRef]
24. Fuss, N.I. Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum, per diagonales resolvi queat. *Nova Acta Acad. Sci. Imp. Petropolitanae* **1791**, *9*, 243–251.
25. Hilton, P.; Pedersen, J. Catalan numbers, their generalization, and their uses. *Math. Intell.* **1991**, *13*, 64–75. [CrossRef]
26. Klarner, D.A. Correspondences between plane trees and binary sequences. *J. Comb. Theory* **1970**, *9*, 401–411. [CrossRef]
27. McCarthy, J. Catalan numbers. Letter to the editor: "Catalan numbers, their generalization, and their uses" [Math. Intell. **1991**, 13, 64–75] by P. Hilton and J. Pedersen. *Math. Intell.* **1992**, *14*, 5.
28. Alexeev, N.; Götze, F.; Tikhomirov, A. Asymptotic distribution of singular values of powers of random matrices. *Lith. Math. J.* **2010**, *50*, 121–132. [CrossRef]
29. Aval, J.-C. Multivariate Fuss–Catalan numbers. *Discret. Math.* **2008**, *308*, 4660–4669. [CrossRef]
30. Bisch, D.; Jones, V. Algebras associated to intermediate subfactors. *Invent. Math.* **1997**, *128*, 89–157. [CrossRef]
31. Gordon, I.G.; Griffeth, S. Catalan numbers for complex reflection groups. *Am. J. Math.* **2012**, *134*, 1491–1502. [CrossRef]
32. Lin, C.-H. Some combinatorial interpretations and applications of Fuss–Catalan numbers. *ISRN Discret. Math.* **2011**, *2011*, 534628. [CrossRef]
33. Liu, D.-Z.; Song, C.-W.; Wang, Z.-D. On explicit probability densities associated with Fuss–Catalan numbers. *Proc. Am. Math. Soc.* **2011**, *139*, 3735–3738. [CrossRef]
34. Młotkowski, W. Fuss–Catalan numbers in noncommutative probability. *Doc. Math.* **2010**, *15*, 939–955.
35. Młotkowski, W.; Penson, K.A.; Zyczkowski, K. Densities of the Raney distributions. *Doc. Math.* **2013**, *18*, 1573–1596.
36. Przytycki, J.H.; Sikora, A.S. Polygon dissections and Euler, Fuss, Kirkman, and Cayley numbers. *J. Comb. Theory Ser. A* **2000**, *92*, 68–76. [CrossRef]
37. Stump, C. q,t-Fuß-Catalan numbers for complex reflection groups. In Proceedings of the 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), Vina del Mar, Chile, 23–27 June 2008; pp. 295–306.
38. Stump, C. q,t-Fuß-Catalan numbers for finite reflection groups. *J. Algebraic Comb.* **2010**, *32*, 67–97. [CrossRef]
39. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherland; Boston, TX, USA; London, UK, 1993. [CrossRef]
40. Widder, D.V. *The Laplace Transform*; Princeton Mathematical Series 6; Princeton University Press: Princeton, NJ, USA, 1941.
41. Alzer, H. Sharp inequalities for the beta function. *Indagat. Math.* **2001**, *12*, 15–21. [CrossRef]
42. Cerone, P. Special functions: Approximations and bounds. *Appl. Anal. Discret. Math.* **2007**, *1*, 72–91. [CrossRef]

43. Atanassov, R.D.; Tsoukrovski, U.V. Some properties of a class of logarithmically completely monotonic functions. *Dokl. Bolg. Akad. Nauk.* **1988**, *41*, 21–23.
44. Qi, F.; Chen, C.-P. A complete monotonicity property of the gamma function. *J. Math. Anal. Appl.* **2004**, *296*, 603–607. [CrossRef]
45. Schilling, R.L.; Song, R.; Vondraček, Z. *Bernstein Functions—Theory and Applications*, 2nd ed.; de Gruyter Studies in Mathematics 37; Walter de Gruyter: Berlin, Germany, 2012. [CrossRef]
46. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 10th ed.; National Bureau of Standards, Applied Mathematics Series 55; Dover Publications: New York, NY, USA, 1972.
47. Berg, C. Integral representation of some functions related to the gamma function. *Mediterr. J. Math.* **2004**, *1*, 433–439. [CrossRef]
48. Guo, B.-N.; Qi, F. A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function. *Politeh. Univ. Buchar. Sci. Bull. Ser. A Appl. Math. Phys.* **2010**, *72*, 21–30.
49. Laforgia, A.; Natalini, P. Exponential, gamma and polygamma functions: Simple proofs of classical and new inequalities. *J. Math. Anal. Appl.* **2013**, *407*, 495–504. [CrossRef]
50. Elezović, N. Asymptotic expansions of gamma and related functions, binomial coefficients, inequalities and means. *J. Math. Inequal.* **2015**, *9*, 1001–1054. [CrossRef]
51. Qi, F. Bounds for the ratio of two gamma functions. *J. Inequal. Appl.* **2010**, *2010*, 493058. [CrossRef]
52. Qi, F. Bounds for the ratio of two gamma functions: From Gautschi's and Kershaw's inequalities to complete monotonicity. *Turk. J. Anal. Number Theory* **2014**, *2*, 152–164. [CrossRef]
53. Qi, F.; Li, W.-H. A logarithmically completely monotonic function involving the ratio of gamma functions. *J. Appl. Anal. Comput.* **2015**, *5*, 626–634. [CrossRef]
54. Qi, F.; Luo, Q.-M. Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions. *Banach J. Math. Anal.* **2012**, *6*, 132–158. [CrossRef]
55. Qi, F.; Luo, Q.-M. Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem. *J. Inequal. Appl.* **2013**, *1*, 542. [CrossRef]
56. Bullen, P.S. *Handbook of Means and Their Inequalities*; Mathematics and Its Applications; Springer Science & Business Media: Berlin, Germany, 2003; Volume 560.
57. Qi, F. A new lower bound in the second Kershaw's double inequality. *J. Comput. Appl. Math.* **2008**, *214*, 610–616. [CrossRef]
58. Qi, F.; Guo, S.; Chen, S.-X. A new upper bound in the second Kershaw's double inequality and its generalizations. *J. Comput. Appl. Math.* **2008**, *220*, 111–118. [CrossRef]
59. Cerone, P. Special Functions Approximations and Bounds via Integral Representation. In *Advances in Inequalities for Special Functions*; Pietro, C., Sever, S.D., Eds.; Advances in Mathematical Inequalities; Nova Science Publishers: New York, NY, USA, 2008; pp. 1–35.
60. Grenié, L.; Molteni, G. Inequalities for the beta function. *Math. Inequal. Appl.* **2015**, *18*, 1427–1442. [CrossRef]
61. Heubach, S.; Li, N.Y.; Mansour, T. A Garden of *k*-Catalan Structures. Preprint. 2008. Available online: http://math.haifa.ac.il/toufik/enumerative/kcatalan.pdf (accessed on 16 September 2018).
62. Zhu, B.-X. Higher order log-monotonicity of combinatorial sequences. *arXiv* **2013**, arxiv:1309.6025.
63. Qi, F.; Cerone, P. Some properties of the Fuss–Catalan numbers. *Preprints* **2017**. [CrossRef]

© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).