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Construction Algorithm for Zero Divisor Graphs of Finite Commutative Rings and Their Vertex-Based Eccentric Topological Indices

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Abstract: Chemical graph theory is a branch of mathematical chemistry which deals with the non-trivial applications of graph theory to solve molecular problems. Graphs containing finite commutative rings also have wide applications in robotics, information and communication theory, elliptic curve cryptography, physics, and statistics. In this paper we discuss eccentric topological indices of zero divisor graphs of commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$, where p_1, p_2, and q are primes. To enhance the importance of these indices a construction algorithm is also devised for zero divisor graphs of commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$.

Keywords: topological index; zero divisor graphs; commutative ring

MSC: 05C12; 05CO5; 05C90

1. Introduction

A single number that can be utilized to depict properties of the graph of a molecule is known as a topological descriptor for that graph. There are different topological descriptors that have found a number of applications in theoretical science. Topological descriptors are numerical parameters of a graph that characterize its topology and are usually graph-invariant. Topological descriptors are utilized within the improvement of quantitative structure-activity connections (QSARs) and quantitative structure-property connections (QSPR) in which the organic movement or other properties of atoms are connected with their chemical structure. Topological descriptors are utilized in QSPR/QSAR. These days, there exists a variety of topological descriptors that have some applications in chemistry, physics, robotics, statistics, and computer networks. The topological descriptors deal with the distance among the vertices in a graph are "distance-based topological indices". Other topological descriptors that deal with the degree of vertices in graph are "degree-based topological indices". The Wiener index is the first distance based topological index and it has eminent applications in chemistry. Wiener index is based on topological distance of vertices within the individual graph, the Hosoya index is calculated by checking of non-incident edges in a graph, the energy and the Estrada index are based on the range of the graph, the Randic connectivity index is calculated utilizing the degrees of vertices. For further detail about other indices [1–9] can be explored.

2. Definitions and Notations

Let G be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A numerical quantity related to a graph that is invariant under graph automorphisms is topological index or topological descriptor. For a graph G, the degree of a vertex v is the number of edges incident with v and denoted by $d(v)$. The maximum degree of a graph G, denoted by $\Delta(G)$, and the minimum degree of a graph, denoted by $\delta(G)$, are the maximum and minimum degree of its vertices. The sum of degree of all vertices u which are adjacent to vertex v is denoted by $S(v)$. The distance between the vertices u and v of G is denoted by $d(u,v)$ and it is defined as the number of edges in a minimal path connecting them.

Connectivity descriptors are important among topological descriptors and used in various fields like chemistry, physics, and statistics. Let $x_1, x_2 \in V(G)$, the distance $d(x_1, x_2)$ between x_1 and x_2, be defined as the length of any shortest path in G connecting x_1 and x_2. In mathematical, eccentricity is defined as:

$$e(u) = max\{d(u,v) : \forall\ v \in V(G)\} \tag{1}$$

In 1997 eccentric connectivity index was introduced by Sharma [10]. By using eccentric connectivity index, the mathematical modeling of biological activities of diverse nature is done. The general formula of eccentric connectivity index is defined as:

$$\xi(G) = \sum_{v \in V} d_v\, e(v) \tag{2}$$

where $e(v)$ is the eccentricity of vertex v in G. Some applications and mathematical properties of eccentric connectivity index can be found in [11–14]. The total eccentricity index is the sum of eccentricity of the all the vertex v in G. Total eccentricity index is introduced by Farooq and Malik [15], which is defined as:

$$\zeta(G) = \sum_{v \in V} e(v) \tag{3}$$

The first Zagreb index of a graph G is studied in [16] based on degree and the new version of the first Zagreb index based on eccentricity was introduced by Ghorbani and Hosseinzadeh [17], as follows:

The eccentric connectivity polynomial is the polynomial version of the eccentric-connectivity index which was proposed by Alaeiyan, Mojarad, and Asadpour [18] and some graph operations can be found in [19]. The eccentric connectivity polynomial of a graph G is defined as:

$$ECP(G,x) = \sum_{v \in V} d(v) x^{e(v)} \tag{4}$$

Gupta, Singh and Madan [20] defined the augmented eccentric connectivity index of a graph G as follows:

$$\xi^{ac}(G) = \sum_{v \in V} \frac{M(v)}{e(v)} \tag{5}$$

where $M(v)$ denotes the product of degrees of all vertices u which are adjacent to vertex v. Some interesting results on augmented eccentric connectivity index are discussed in [21,22]. Another very relevant and special eccentricity based topological index is connective eccentric index. The connective eccentric index was defined by Gupta, Singh, and Madan [20] defined as follows:

$$\xi^{C}(G) = \sum_{v \in V} \frac{d(v)}{e(v)} \tag{6}$$

Ediz [23,24] introduced the Ediz eccentric connectivity index and reverse eccentric connectivity index of graph G, which is used in various branches of sciences, molecular science, and chemistry etc. The Ediz eccentric connectivity index and reverse eccentric connectivity index are defined as:

$$E\zeta(G) = \sum_{v \in V(G)} \frac{S(v)}{e(v)} \tag{7}$$

$$^{Re}\zeta(G) = \sum_{v \in V(G)} \frac{e(v)}{S(v)} \tag{8}$$

where $S(v)$ is the sum of degrees of all vertices, u, adjacent to vertex v, $e(v)$ is the eccentricity of v.

Let R be a commutative ring with identity and $Z(R)$ is the set of all zero divisors of R. $G(R)$ is said to be a zero divisor graph if $x, y \in V(G(R)) = Z(R)$ and $(x, y) \in E(G(R))$ if and only if $x.y = 0$. Beck [25] introduced the notion of zero divisor graph. Anderson and Livingston [26] proved that $G(R)$ is always connected if R is commutative. Anderson and Badawi [27] introduced the total graph of R as: There is an edge between any two distinct vertices $u, v \in R$ if and only if $u + v \in Z(R)$. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [28–32]. Similar problems were investigated in [33,34].

Let p_1, p_2, and q be prime numbers, with $p_2 > p_1$ and $\Gamma(\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q)$ being zero divisor graph of the commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$. In this paper, we investigate the eccentric topological descriptors namely, eccentric connectivity index, total eccentric index, first Zagreb eccentricity index, connective eccentric index, Ediz eccentric index, eccentric connectivity polynomial, and augmented eccentric connectivity index of zero divisor graphs $\Gamma(\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q)$. Now onward, we use G as a zero divisor graph of the commutative rings $\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$.

3. Methods

In this paper, we adopted interdisciplinary methods by combining algorithmic approach for graph construction and outcome of algorithm are aligned with eccentric topological indices. For prime numbers p_1, p_2, q with $p_2 > p_1$, we consider the commutative ring $R = \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ with usual addition and multiplication. The zero divisor graph $G = \Gamma(\mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q)$ associated with ring R is defined as: For $a \in \mathbb{Z}_{p_1p_2}$, $b \in \mathbb{Z}_q$, $(a, b) \notin V(G)$ if and only if $a \neq kp_1, a \neq sp_2$ for $k = 1, 2, \cdots, p_2 - 1, s = 1, 2, \cdots, p_1 - 1$ and $y \neq 0$. Let $J = \{(a, b) \notin V(G) : a \neq kp_1, a \neq sp_2, k = 1, 2, \cdots, p_2 - 1, s = 1, 2, \cdots, p_1 - 1 \,\&\, y \neq 0\}$, then $|J| = (p_1p_2 - p_1 - p_2 + 1)(q - 1)$. The elements of the set J are the non zero divisors of R. Also $(0, 0) \in \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ is a non zero divisor. Therefore, $|J| + 1 = (p_1p_2 - p_1 - p_2 + 1)(q - 1) + 1$ are the total number of non zero divisors of R and the total number of elements of R are p_1p_2q. Hence, $p_1p_2q - (p_1p_2 - p_1 - p_2 + 1)(q - 1) + 1 = (p_1 + p_2 - 1)(q - 1) + p_1p_2 - 1$ are the total number of zero divisors. This implies that $|V(G)| = (p_1 + p_2 - 1)(q - 1) + p_1p_2 - 1$. We can construct the zero divisor graph of commutative ring $R = \mathbb{Z}_{p_1p_2} \times \mathbb{Z}_q$ by the following algorithm:

Input: p_1, p_2 and q are three prime numbers.
Output: ordered pairs for zero divisor.

Algorithm 1 ZeroDivisorGraph(p_1, p_2, q)

```
if (p1 < p2)
  for x1 ← 0 to p1 × p2
    for y1 ← 0 to q
      if (x1 ≠ 0 OR y1 ≠ 0)
        createGraph( x1, y1, p1, p2, q)
```

Algorithm 2 createGraph (x_1, y_1, p_1, p_2, q)

```
for x2 ← x1 to p1 × p2
  for y2 ← y1 to q
    if (x1 ≠ x2 AND y1 ≠ y2)
      if (x1 ≠ 0 OR y1 ≠ 0)
        k1 = 0
      else
        k1 = x1 × x2
      if (y1 ≠ 0 OR y2 ≠ 0)
        k2 = 0
      else
        k2 = y1 × y2
      if (k1 mod p1 = 0 AND k1 mod p2 = 0 AND k2 = 0)
        return x1, y1, x1, y2
```

Outcomes of above algorithm, the degree of each vertex $(a, b) \in V(G)$ can be depicted mathematically in the following cases:

Case 1: If $a = 0$ and any $b \in \mathbb{Z}_q \setminus \{0\}$, then each such type of vertex $(0, b)$ is adjacent to the vertices $(a', 0)$ for every $a' \in \mathbb{Z}_{p_1p_2} \setminus \{0\}$. Hence the degree of each vertex $(0, b)$ is $p_1p_2 - 1$.

Case 2: If $a = kp_1, k = 1, 2, \cdots, p_2 - 1$ and $b = 0$, then each such type of vertex $(a, 0)$ is adjacent to the vertices $(0, b), (a', 0)$ & (a', b') for every $b' = \{1, 2, \cdots, q - 1\}$, and $a' = sp_2, s = 1, 2, \cdots, p_1 - 1$. Hence the degree of each vertex $(a, 0)$ is $q - 1 + p_1 - 1 + (p_1 - 1)(q - 1) = p_1q - 1$. Similarly, if $a = sp_2, s = 1, 2, \cdots, p_1 - 1$ and $b = 0$, then the degree of each such type of vertices $(a, 0)$ is $p_2q - 1$.

Case 3: If $a \in \mathbb{Z}_{p_1p_2} \setminus \{0, kp_1, sp_2$ with $k = 1, 2, \cdots, p_2 - 1, s = 1, 2, \cdots, p_1 - 1\}$ and $b = 0$, then each such type of vertex $(a, 0)$ is adjacent with only $(0, b')$ for every $b' \in \mathbb{Z}_q \setminus \{0\}$. Hence the degree of each vertex $(a, 0)$ is $q - 1$.

Case 4: If $a = kp_1, k = 1, 2, \cdots, p_2 - 1$ and $b \in \mathbb{Z}_q \setminus \{0\}$, then each such type of vertex (a, b) is adjacent with only $(a', 0)$ for every $a' = sp_2, s = 1, 2, \cdots, p_1 - 1$. Therefore, the degree of each vertex (a, b) is $p_1 - 1$. Similarly, if $a = sp_2, s = 1, 2, \cdots, p_1 - 1$ and $b \in \mathbb{Z}_q \setminus \{0\}$, then degree of each such type of vertices (a, b) is $p_2 - 1$.

From the above discussion and our convenance, let us partitioned the vertex set of G based one their degrees as follows:

$$
\begin{aligned}
V_1 =& \{(0, x) : x \in \mathbb{Z}_q, x \neq 0\} \\
V_2 =& \{(x, 0) : x = kp_1, k = 1, 2, \ldots, p_2 - 1\} \\
V_3 =& \{(x, 0) : x = sp_2, s = 1, 2, \ldots, p_1 - 1\} \\
V_4 =& \{(x, 0) : x \in \mathbb{Z}_{p_1p_2} \setminus \{0\}, x \neq kp_1, x \neq sp_1, k = 1, 2, \ldots, p_2 - 1, s = 1, 2, \ldots, p_1 - 1\} \\
V_5 =& \{(x, y) : x = kp_1, k = 1, 2, \ldots, p_2 - 1, y \in \mathbb{Z}_q \setminus \{0\}\} \\
V_6 =& \{(x, y) : x = sp_2, s = 1, 2, \ldots, p_1 - 1, y \in \mathbb{Z}_q \setminus \{0\}\}
\end{aligned}
$$

This shows that $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$. It is easy to see that $|V_1| = q-1$, $|V_2| = p_2-1$, $|V_3| = p_1-1$, $|V_4| = (p_1-1)(p_2-1)$, $|V_5| = (p_2-1)(q-1)$, and $|V_1| = (p_1-1)(q-1)$.

4. Main Results

Let $d_U(u)$ denote the degree of a vertex u in U and $d(U,V)$ denotes the distance between the vertices of two sets U and V. In the following theorem, we determined the eccentricity of the vertices of G.

Theorem 1. *Let G be the zero divisor graph of the commutative ring R, then the eccentricity of the vertices of G is 2 or 3.*

Proof. From case 1, the vertices of the set V_1 are at distance 1 with the vertices of the sets V_2, V_3, & V_4 i.e., $d(V_1,V_2) = d(V_1,V_3) = d(V_1,V_4) = 1$. From Case 4, the vertices of the sets V_2 and V_3 are adjacent with the vertices of the sets V_6 and V_5, respectively. This implies that $d(V_1,V_5) = d(V_1,V_6) = 2$. The distance between any two different vertices of the set V_1 is also 2. Therefore the eccentricity of the vertices of set V_1 is 2, i.e., $e(V_1) = 2$. Similarly, it is easy to see that $e(V_2) = e(V_3) = 2$.

As $d(V_1,V_2) = d(V_1,V_3) = d(V_1,V_4) = 1$ and $d(V_1,V_5) = d(V_1,V_6) = 2$. This implies that $d(V_4,V_5) = d(V_4,V_1) + d(V_1,V_5) = 3$. This shows that $e(V_4) = 3$. Similarly, it is easy to calculate that $e(V_5) = e(V_6) = 3$. This completes the proof. □

Summarizing the above cases, partition of vertices and their cardinality and Theorem 1 in Table 1.

Table 1. The representation of vertices, their degree, eccentricity, and frequency of the vertices in G.

Representatives of Vertices	Degree	Eccentricity	Frequency
V_1	p_1p_2-1	2	$q-1$
V_2	p_1q-1	2	p_2-1
V_3	p_2q-1	2	p_1-1
V_4	$q-1$	3	$(p_1-1)(p_2-1)$
V_5	p_1-1	3	$(p_2-1)(q-1)$
V_6	p_2-1	3	$(p_1-1)(q-1)$

In the following theorem, we determined the eccentric connectivity index of the graph G.

Theorem 2. *Let $p_1 < p_2$, q be prime numbers, then eccentric connectivity index of graph G is $\xi(G) = p_1p_2(15q-11) - (p_1+p_2)(11q-7) + 7q - 3$.*

Proof. By using the degree of each vertex partition and corresponding their eccentricity from Table 1 in the Equation (2), we obtain:

$$\begin{aligned}\xi(G) = &\sum_{v\in V} d_v\, e(v)\\ =&2(p_1p_2-1)(q-1) + 2(p_1q-1)(p_2-1) + 2(p_2q-1)(p_1-1)\\ &+3(p_1-1)(p_2-1)(q-1) + 3(p_1-1)(p_2-1)(q-1) + 3(p_1-1)(p_2-1)(q-1)\end{aligned}$$

After simplification, we get:
$\xi(G) = p_1p_2(15q-11) - (p_1+p_2)(11q-7) + 7q - 3$.
This completes the proof. □

The eccentricity of the vertices and their frequency is given in the Table 1 of the graph G, by putting these values and after simplification we obtain the following two corollaries.

Corollary 1. *Let $p_1 < p_2$, q be prime numbers, then the total-eccentricity index of G is given by* $\zeta(G) = 3(p_1p_2 + p_1q + p_2q + 1) - 4(p_1 + p_2 + q)$.

Corollary 2. *Let $p_1 < p_2$, q be prime numbers, then the first Zagreb eccentricity index of G is given by* $M_1{}^*(G) = 9(p_1p_2 + p_1q + p_2q) - 14(p_1 + p_2 + q) + 15$.

Theorem 3. *Let $p_1 < p_2$, q be prime numbers, then the connective eccentric index of graph G is* $\xi^C(G) = (p_1 - 1)(p_2 - 1)(q - 1) + 2 + \frac{p_1p_2(3q-1)-(p_1+p_2+1)(q+1)}{2}$.

Proof. By using the values of degrees and their eccentricity in the Equation (6), we obtain the following:

$$\begin{aligned}
\xi^C(G) &= \sum_{v \in V} \frac{d(v)}{e(v)} \\
&= \frac{(p_1p_2 - 1)(q - 1)}{2} + \frac{(p_1q - 1)(p_2 - 1)}{2} + \frac{(p_2q - 1)(p_1 - 1)}{2} \\
&+ \frac{(p_1 - 1)(p_2 - 1)(q - 1)}{3} + \frac{(p_1 - 1)(p_2 - 1)(q - 1)}{3} \\
&+ \frac{(p_1 - 1)(p_2 - 1)(q - 1)}{3} \\
&= (p_1 - 1)(p_2 - 1)(q - 1) + 2 + \frac{p_1p_2(3q - 1) - (p_1 + p_2 + 1)(q + 1)}{2}.
\end{aligned}$$

After simplification, we get
$\xi^C(G) = (p_1 - 1)(p_2 - 1)(q - 1) + 2 + \frac{p_1p_2(3q-1)-(p_1+p_2+1)(q+1)}{2}$.
This completes the proof. □

Theorem 4. *Let $p_1 < p_2$, q be prime numbers, then the Ediz eccentric connectivity index of graph G is* $E\zeta(G) = \frac{9(p_1-1)(p_2-1)(q-1)+8[(p_1-1)(p_2q-1)+(p_2-1)(p_1q-1)+(p_1p_2-1)(q-1)]}{6}$.

Proof. $S(v)$ is the sum of degrees of all vertices u which are adjacent to vertex v. Calculate the values of $S(v)$ with the help of Table 1. The eccentricity of each vertex is also given in Table 1. Putting these vales in Equation (7), we obtain the following:

$$\begin{aligned}
E\zeta(G) &= \sum_{v \in V(G)} \frac{S(v)}{e(v)} \\
&= \frac{(p_1 - 1)(p_2 - 1)(q - 1) + (p_1 - 1)(p_2q - 1) + (p_2 - 1)(p_1q - 1)}{2} \\
&+ \frac{(p_1 - 1)(p_2 - 1)(q - 1) + (p_1 - 1)(p_2q - 1) + (q - 1)(p_1p_2 - 1)}{2} \\
&+ \frac{(p_1 - 1)(p_2 - 1)(q - 1) + (p_2 - 1)(p_1q - 1) + (q - 1)(p_1p_2 - 1)}{2} \\
&+ \frac{(q - 1)(p_1p_2 - 1)}{3} + \frac{(p_1 - 1)(p_2q - 1)}{3} + \frac{(p_2 - 1)(p_1q - 1)}{3}
\end{aligned}$$

After simplification, we get
$E\zeta(G) = \frac{9(p_1-1)(p_2-1)(q-1)+8[(p_1-1)(p_2q-1)+(p_2-1)(p_1q-1)+(p_1p_2-1)(q-1)]}{6}$.
This completes the proof. □

Theorem 5. *Let $p_1 < p_2$, q be prime numbers, then the eccentric connectivity polynomial of graph G is* $ECP(G, x) = (3p_1p_2q - p_1p_2 - p_1q - p_1 - p_2q - p_2 - q + 3)x^2 + 3(p_1 - 1)(p_2 - 1)(q - 1)x^3$.

Proof. By using the degree of each vertex partition and their corresponding eccentricities from Table 1 Equation (4), we obtain:

$$
\begin{aligned}
ECP(G,x) = \sum_{v\in V} d(v)x^{e(v)} \\
&= (p_1p_2-1)(q-1)x^2 + (p_1q-1)(p_2-1)x^2 \\
&+ (p_2q-1)(p_1-1)x^2 + (p_1-1)(p_2-1)(q-1)x^3 \\
&+ (p_1-1)(p_2-1)(q-1)x^3 + (p_1-1)(p_2-1)(q-1)x^3 \\
&= (3p_1p_2q - p_1p_2 - p_1q - p_1 - p_2q - p_2 - q + 3)x^2 \\
&+ 3(p_1-1)(p_2-1)(q-1)x^3.
\end{aligned}
$$

After simplification, we get
$ECP(G,x) = (3p_1p_2q - p_1p_2 - p_1q - p_1 - p_2q - p_2 - q + 3)x^2 + 3(p_1-1)(p_2-1)(q-1)x^3$.
This completes the proof. □

Theorem 6. *Let $p_1 < p_2$, q be prime numbers, then augmented eccentric connectivity index of graph G is* $\xi^{ac}(G) = \frac{(p_1-1)(p_2-1)(p_1p_2-1)^{q-1}+(p_1-1)(q-1)(p_1q-1)^{p_2-1}+(p_2-1)(q-1)(p_2q-1)^{p_1-1}}{3} + \frac{(p_1-1)^{p_2q-q-p_2+2}(p_1q-1)^{p_2-1}(p_1p_2-1)^{q-1}+(p_2-1)^{p_1q-p_1-q+2}(p_1p_2-1)^{q-1}(p_2q-1)^{p_1-1}}{2} + \frac{(q-1)^{p_1p_2-p_1-p_2+2}(p_1q-1)^{p_2-1}(p_2q-1)^{p_1-1}}{2}$.

Proof. $M(v)$ is the product of degrees of all vertices u which are adjacent to vertex v. Calculate the values of $M(v)$ with the help of Table 1. The eccentricity of each vertex is also given in the Table 1. Putting these vales in Equation (5), we obtain the following:

$$
\begin{aligned}
\xi^{ac}(G) &= \sum_{v\in V} \frac{M(v)}{e(v)} \\
&= \frac{(p_1-1)(p_2-1)(p_1p_2-1)^{q-1}}{3} + \frac{(p_1-1)(q-1)(p_1q-1)^{p_2-1}}{3} \\
&+ + \frac{(p_2-1)(q-1)(p_2q-1)^{p_1-1}}{3} \\
&+ \frac{(p_1-1)(p_1-1)^{(p_1-1)(q-1)}(p_1q-1)^{p_2-1}(p_1p_2-1)^{q-1}}{2} \\
&+ \frac{(p_2-1)(p_2-1)^{(p_2-1)(q-1)}(p_1p_2-1)^{q-1}(p_2q-1)^{p_1-1}}{2} \\
&+ \frac{(q-1)(q-1)^{(p_1-1)(p_2-1)}(p_1q-1)^{p_2-1}(p_2q-1)^{p_1-1}}{2}
\end{aligned}
$$

After simplification, we get
$\xi^{ac}(G) = \frac{(p_1-1)(p_2-1)(p_1p_2-1)^{q-1}+(p_1-1)(q-1)(p_1q-1)^{p_2-1}+(p_2-1)(q-1)(p_2q-1)^{p_1-1}}{3} + \frac{(p_1-1)^{p_2q-q-p_2+2}(p_1q-1)^{p_2-1}(p_1p_2-1)^{q-1}+(p_2-1)^{p_1q-p_1-q+2}(p_1p_2-1)^{q-1}(p_2q-1)^{p_1-1}}{2} + \frac{(q-1)^{p_1p_2-p_1-p_2+2}(p_1q-1)^{p_2-1}(p_2q-1)^{p_1-1}}{2}$
This completes the proof. □

If p_1, p_2 and q are prime numbers with $p_1 = p_2 = p$, then Ahmad et al. [35] determined the vertex-based eccentric topological indices of zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ as follows:

Theorem 7 ([35])**.** *Let p, q be prime numbers. If $G(R)$ is the zero divisor graph of the commutative ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, then*

- $\xi(G(R)) = 10p^2q - 8p^2 - 11pq + 5p + q + 3.$

- $\zeta(G(R)) = 3p^2 + 3pq - 4p - q - 1$
- $M_1{}^*(G(R)) = 9p^2 + 9pq - 14p - 5q + 1.$
- $\xi^C(G) = \frac{(p-1)(10pq-7p+q-7)}{6}$
- $\xi^{ac}(G(R)) = (\frac{p-1}{3} + \frac{(q-1)^{p^2-p}}{2})(pq-2)^{p-1}(q-1) + (\frac{p}{3} + \frac{p^{(p-1)(q-1)}(pq-2)^{p-2}}{2})(p^2-1)^{q-1}(p-1).$

5. Conclusions

In this paper, we discussed the vertex-based eccentric topological indices, namely eccentric connectivity index, total-eccentricity index, first Zabreb eccentricity index, connective eccentric index, Ediz eccentric connectivity index, eccentric connectivity polynomial, and augmented eccentric index for zero divisor graphs of commutative rings $\mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_q$ where p_1, p_2 and q are primes. These indices are helpful in understanding the characteristics of different physical structures like carbon nanostructures, hexagonal belts and chains, Fullerence and Nanocone, structure-boiling point, and the relationships of various alkanes. They can be used in estimating and trouble shooting computer network problems regarding distance, speed, and time. They can also be helpful in developing efficient physical structure in robotics.

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