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On the Generalization of a Class of Harmonic Univalent Functions Defined by Differential Operator

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Abstract: In this article, a new class of harmonic univalent functions, defined by the differential operator, is introduced. Some geometric properties, like, coefficient estimates, extreme points, convex combination and convolution (Hadamard product) are obtained.

Keywords: harmonic univalent function; coefficient inequality; extreme points; convex combination; Hadamard product

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic. In any simply connected domain $\mathcal{B} \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathcal{B}$. We call h and g are analytic part and co-analytic part of f respectively. Clunie and Sheil-Small [1] observed that a necessary and sufficient condition for the harmonic functions $f = h + \overline{g}$ to be locally univalent and sense-preserving in $\mathcal{B}$ is that $|h'(z)| > |g'(z)|$, $(z \in \mathcal{B})$.

Denote by S_H the family of harmonic functions $f = h + \overline{g}$, which are univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ where h and g are analytic in $\mathcal{B}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; g(z) = \sum_{n=1}^{\infty} b_n z^n, \; |b_1| < 1. \tag{1}$$

Note that S_H reduces to the class of normalized analytic univalent functions if the co-analytic part of its members equals to zero.

Also, denote by $S_{\overline{H}}$ the subclass of S_H consisting of all functions $f_k(z) = h(z) + \overline{g_k(z)}$, where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \; and \; g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n, \; |b_1| < 1. \tag{2}$$

In 1984 Clunie and Sheil-Small [1] investigated the class S_H, as well as its geometric subclass and obtained some coefficient bounds. Many authors have studied the family of harmonic univalent function (see References [2–7]).

In 2016 Makinde [8] introduced the differential operator F^k such that

$$F^k f(z) = z + \sum_{n=2}^{\infty} C_{nk} z^n, \tag{3}$$

where

$$C_{nk} = \frac{n!}{|n-k|!}, \; F^k f(z) = z^k \left[z^{-(k-1)} + \sum_{n=2}^{\infty} C_{nk} z^n \right], \; k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and

$$F^0 f(z) = f(z), \; F^1 f(z) = z + \sum_{n=2}^{\infty} C_{n1} z^n.$$

Thus, it implies that $F^k f(z)$ is identically the same as $f(z)$ when $k = 0$. Also, it reduced the first differential coefficient of the Salagean differential operator when $k = 1$.

For $f = h + \overline{g}$ given by Equation (1), Sharma and Ravindar [9] considered the differential operator which defined by Equation (3) of f as

$$F^k f(z) = F^k h(z) + (-1)^k \overline{F^k g(z)}, \; k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{C}, \tag{4}$$

where

$$F^k h(z) = z + \sum_{n=2}^{\infty} C_{nk} a_n z^n, \; F^k g(z) = \sum_{n=1}^{\infty} C_{nk} b_n z^n \text{ and } C_{nk} = \frac{n!}{|n-k|!}.$$

In this paper, motivated by study in [9], a new class $A_H(k, \alpha, \gamma)(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \, 0 \le \gamma \le 1, \, 0 \le \alpha < 1,)$ of harmonic univalent functions in $U = \{z \in \mathbb{C} : |z| < 1\}$ is introduced and studied. Furthermore, coefficient conditions, distortion bounds, extreme points, convex combination and radii of convexity for this class are obtained.

2. Main Results

2.1. The Class $A_H(k, \alpha, \gamma)$

Definition 1. *Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic function, where $h(z)$ and $g(z)$ are given by Equation (1). Then$f(z) \in A_H(k, \alpha, \gamma)$ it satisfies*

$$Re\left\{ \frac{F^{k+1} f(z)}{(1-\gamma)z + \gamma F^k f(z)} \right\} > \alpha, \tag{5}$$

for$k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \; 0 \le \gamma \le 1, \; 0 \le \alpha < 1, \; z \in U$, and $F^k f(z)$ defined by Equation (4)

Let $A_{\overline{H}}(k, \alpha, \gamma)$ be the subclass of $A_H(k, \alpha, \gamma)$, where $A_{\overline{H}}(k, \alpha, \gamma) = S_{\overline{H}} \cap A_H(k, \alpha, \gamma)$.

Remark 1. *The class $A_{\overline{H}}(k, \alpha, \gamma)$ reduces to the class $B_{\overline{H}}(k, \alpha)$ [9], when $\gamma = 1$.*

Here, we give a sufficient condition for a function f to be in the class $A_H(k, \alpha, \gamma)$.

Theorem 1. *Let $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ were given by (1). If*

$$\sum_{n=2}^{\infty} \varnothing(n, k, \alpha, \gamma)|a_n| + \sum_{n=1}^{\infty} \psi(n, k, \alpha, \gamma)|b_n| \le 1, \tag{6}$$

where

$$\varnothing(n,k,\alpha,\gamma) = \frac{(|n-k| - \alpha\gamma)C_{nk}}{(1-\alpha)}$$

$$\psi(n,k,\alpha,\gamma) = \frac{(|n-k| + \alpha\gamma)C_{nk}}{(1-\alpha)}$$

$$(\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1,\ n \in \mathbb{N}),$$

then $f(z)$ is harmonic univalent and sense-preserving in U and $f(z) \in A_H(k,\alpha,\gamma)$.

Proof. Firstly, to show that $f(z)$ is harmonic univalent in U, suppose that $z_1, z_2 \in U$ for $\lfloor z_1 \rfloor \le \lfloor z_2 \rfloor < 1$, we have by inequality so that $z_1 \neq z_2$, then

$$\begin{aligned} &\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \\ &\qquad \ge 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{n=1}^{\infty} b_n\left(z_1^n - z_2^n\right)}{(z_1-z_2) - \sum_{n=2}^{\infty} a_n\left(z_1^n - z_2^n\right)}\right| \\ &\qquad \ge 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} \ge 1 - \frac{\sum_{n=1}^{\infty} \frac{(|n-k|+\alpha\gamma)C_{nk}}{(1-\alpha)} |b_n|}{1-\sum_{n=2}^{\infty} \frac{(|n-k|-\alpha\gamma)C_{nk}}{(1-\alpha)} |a_n|} \ge 0. \end{aligned}$$

Thus f is a univalent function in U.

Note that f is sense-preserving in U. This is because

$$\begin{aligned} |h'(z)| &\ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n| \ge 1 - \sum_{n=2}^{\infty} \frac{(|n-k|-\alpha\gamma)C_{nk}}{(1-\alpha)} |a_n| \\ &\ge \sum_{n=1}^{\infty} \frac{(|n-k|+\alpha\gamma)C_{nk}}{(1-\alpha)} |b_n| \ge \sum_{n=1}^{\infty} n|b_n| \ge \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \ge |g'(z)|. \end{aligned}$$

According to the condition of Equation (5), we only need to show that if Equation (6) holds, then

$$Re\left\{\frac{F^{k+1}f(z)}{(1-\gamma)z + \gamma F^k f(z)}\right\} = Re\left(w = \frac{A(z)}{B(z)}\right) > \alpha$$

where $z = re^{i\theta}$, $0 \le \theta \le 2\pi$, $0 \le r < 1$ and $0 \le \alpha < 1$.

Note that $A(z) = F^{k+1}f(z)$ and $B(z) = (1-\gamma)z + \gamma F^k f(z)$.

Using the fact that $Re(w) > \alpha$ if and only if $|w - (1+\alpha)| \le |w + (1-\alpha)|$, it suffices to show that

$$|A(z) - (1+\alpha)B(z)| - |A(z) + (1-\alpha)B(z)| \le 0 \tag{7}$$

Substituting for $A(z)$ and $B(z)$ in $|A(z) - (1+\alpha)B(z)|$, we obtain

$$\begin{aligned} |A(z) - (1+\alpha)B(z)| &= \left|F^{k+1}f(z) - (1+\alpha)\left[(1-\gamma)z + \gamma F^k f(z)\right]\right| \\ &= \Bigg|\left[z + \sum_{n=2}^{\infty} C_{n(k+1)} a_n z^n + (-1)^{(k+1)} \sum_{n=1}^{\infty} C_{n(k+1)} \overline{b_n z^n}\right] \\ &\quad -(1+\alpha)\left[(1-\gamma)z + \gamma z + \gamma \sum_{n=2}^{\infty} C_{nk} a_n z^n + \gamma(-1)^k \sum_{n=1}^{\infty} C_{nk} \overline{b_n z^n}\right]\Bigg| \\ &\le \alpha|z| + \sum_{n=2}^{\infty} |(\gamma(1+\alpha)) - |n-k||C_{nk}|a_n||z|^n \\ &\quad + \sum_{n=1}^{\infty} |(\gamma(1+\alpha)) + |n-k||C_{nk}|a_n||\overline{z}|^n. \end{aligned} \tag{8}$$

Now, substituting for $A(z)$ and $B(z)$ in $|A(z)+(1-\alpha)B(z)|$, we obtain

$$\begin{aligned}|A(z)+(1-\alpha)B(z)| &= \left|F^{k+1}f(z)+(1-\alpha)\left[(1-\gamma)z+\gamma F^k f(z)\right]\right| \\ &= \Bigg|\left[z+\sum_{n=2}^{\infty}C_{n(k+1)}a_nz^n+(-1)^{(k+1)}\sum_{n=1}^{\infty}C_{n(k+1)}\overline{b_nz^n}\right] \\ &\quad +(1-\alpha)\left[(1-\gamma)z+\gamma z+\gamma\sum_{n=2}^{\infty}C_{nk}a_nz^n+\gamma(-1)^k\sum_{n=1}^{\infty}C_{nk}\overline{b_nz^n}\right]\Bigg| \\ &\geq (2-\alpha)|z|-\sum_{n=2}^{\infty}|(\gamma(\alpha-1))-|n-k||C_{nk}|a_n||z|^n \\ &\quad -\sum_{n=1}^{\infty}||n-k|-(\gamma(1-\alpha))|C_{nk}|a_n||\overline{z}|^n.\end{aligned} \tag{9}$$

Substituting for Equations (8) and (9) in the inequality we obtain

$$\begin{aligned}&|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)| \\ &\leq \alpha|z|+\sum_{n=2}^{\infty}|(\gamma(1+\alpha))-|n-k||C_{nk}|a_n||z|^n \\ &\quad +\sum_{n=1}^{\infty}|(\gamma(1+\alpha))+|n-k||C_{nk}|b_n||\overline{z}|^n \\ &\quad +(\alpha-2)|Z|+\sum_{n=2}^{\infty}|(\gamma(\alpha-1))-|n-k||C_{nk}|a_n||z|^n \\ &\quad +\sum_{n=1}^{\infty}||n-k|-(\gamma(1-\alpha))|C_{nk}|b_n||\overline{z}|^n. \\ &= 2\sum_{n=2}^{\infty}(|n-k|-\alpha\gamma)C_{nk}|a_n|+2\sum_{n=1}^{\infty}(|n-k|+\alpha\gamma)C_{nk}|b_n|-2(1-\alpha) \\ &\quad \leq 0.\ \text{(by hypothesis)}.\end{aligned}$$

Therefore, we have

$$\sum_{n=2}^{\infty}(|n-k|-\alpha\gamma)C_{nk}|a_n|+\sum_{n=1}^{\infty}(|n-k|+\alpha\gamma)C_{nk}|b_n|\leq(1-\alpha).$$

□

The harmonic univalent function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1}{\varnothing(n,k,\alpha,\gamma)}\mathcal{X}_nz^n+\sum_{n=1}^{\infty}\frac{1}{\psi(n,k,\alpha,\gamma)}\overline{\mathcal{Y}_nz^n}, \tag{10}$$

where $k\in\mathbb{N}_0$ and $\sum_{k=2}^{\infty}|\mathcal{X}_n|+\sum_{k=1}^{\infty}|\mathcal{Y}_n|=1$, shows that the coefficient bound given by Equation (6) is sharp. Since

$$\begin{aligned}&\sum_{n=2}^{\infty}\varnothing(n,k,\alpha,\gamma)|a_n|+\sum_{n=1}^{\infty}\psi(n,k,\alpha,\gamma)|b_n| \\ &=\sum_{n=2}^{\infty}\varnothing(n,k,\alpha,\gamma)\tfrac{1}{\varnothing(n,k,\alpha,\gamma)}|\mathcal{X}_n|+\sum_{n=1}^{\infty}\psi(n,k,\alpha,\gamma)\tfrac{1}{\psi(n,k,\alpha,\gamma)}|\mathcal{Y}_n| \\ &=\sum_{n=2}^{\infty}|\mathcal{X}_n|+\sum_{n=1}^{\infty}|\mathcal{Y}_n|=1.\end{aligned}$$

Now, we show that the condition of Equation (6) is also necessary for functions $f_k=h+\overline{g_k}$, where h and g_n are given by Equation (6).

Theorem 2. *Let $f_k=h+\overline{g_k}$ be given by Equation (6). Then $f_k(z)\in A_{\overline{H}}(k,\alpha,\gamma)$ if and only if the coefficient in condition of Equation (6) holds.*

Proof. We only need to prove the "only if" part of the theorem because of $A_{\overline{H}}(k,\alpha,\gamma) \subset A_H(k,\alpha,\gamma)$. Then by Equation (5), we have

$$Re\left\{\frac{F^{k+1}f(z)}{(1-\gamma)z+\gamma F^k f(z)}\right\} > \alpha$$

or, equivalently

$$Re\left[\frac{\begin{array}{c} z-\sum_{n=2}^{\infty} C_{n(k+1)}|a_n|z^n+(-1)^{2k+1}\sum_{n=1}^{\infty} C_{n(k+1)}|b_n|\overline{z}^n \\ -\alpha\left\{(1-\gamma)z+\gamma z+\gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^n+\gamma(-1)^{2k}\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^n\right\}\end{array}}{(1-\gamma)z+\gamma z-\gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^n+\gamma(-1)^{2k}\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^n}\right] \geq 0 \tag{11}$$

We observe that the above-required condition of Equation (11) must behold for all values of z *in* U. If we choose z to be real and $z \to 1^-$, we get

$$\frac{\begin{array}{c}(1-\alpha)-\sum_{n=2}^{\infty}(|n-k|-\alpha\gamma)C_{nk}|a_n| \\ +\sum_{n=1}^{\infty}(|n-k|+\alpha\gamma)C_{nk}|b_n|\end{array}}{1-\gamma\sum_{n=2}^{\infty} C_{nk}|a_n|z^{n-1}+\gamma\sum_{n=1}^{\infty} C_{nk}|b_n|\overline{z}^{n-1}} \geq 0 \tag{12}$$

If the condition (6) does not hold, then the numerator in Equation (12) is negative for r sufficiently closed to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in Equation (12) is negative, therefore there is a contradicts the required condition for $f_k \in A_{\overline{H}}(k,\alpha,\gamma)$. □

2.2. Extreme Points

Here, we determine the extreme points of the closed convex hull of $A_{\overline{H}}(k,\alpha,\gamma)$, denoted by $clcoA_{\overline{H}}(k,\alpha,\gamma)$.

Theorem 3. *Let f_k given by* (1.2). *Then $f_k \in A_{\overline{H}}(k,\ \alpha,\gamma)$ if and only if*

$$f_k(z) = \sum_{n=1}^{\infty}(\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})$$

where

$$h_1(z) = z,\ h_n(z) = \ z - \frac{1}{\varnothing(n,k,\alpha,\gamma)}z^n,\ n = 2,3,\dots\ ,$$

$$g_{kn}(z) = z +\ (-1)^k\ \frac{1}{\psi(n,k,\alpha,\gamma)}\overline{z}^n,\ n = 1,2,\dots,$$

and

$$\mathcal{X}_n \geq 0,\ \mathcal{Y}_n \geq 0,\ \mathcal{X}_1 = 1 - \sum_{n=2}^{\infty}(\mathcal{X}_n + \mathcal{Y}_n) \geq 0$$

In particular the extreme points of $A_{\overline{H}}(k,\alpha,\gamma)$ are $\{h_n\}$ *and* $\{g_{kn}\}$.

Proof. Suppose

$$\begin{aligned} f_k(z) &= \sum_{n=1}^{\infty}(\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn}) \\ &= \sum_{n=1}^{\infty}(\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})z - \sum_{n=2}^{\infty}\tfrac{1}{\varnothing(n,k,\alpha,\gamma)}\mathcal{X}_n z^n + (-1)^k\sum_{n=1}^{\infty}\tfrac{1}{\psi(n,k,\alpha,\gamma)}\mathcal{Y}_n\overline{z}^n \\ &= z - \sum_{n=2}^{\infty}\tfrac{1}{\varnothing(n,k,\alpha,\gamma)}\mathcal{X}_n z^n + (-1)^{k-1}\sum_{n=1}^{\infty}\tfrac{1}{\psi(n,k,\alpha,\gamma)}\mathcal{Y}_n\overline{z}^n \end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=2}^{\infty} \varnothing(n,k,\alpha,\gamma)|a_n| + \sum_{n=1}^{\infty} \psi(n,k,\alpha,\gamma)|b_n| \\
&= \sum_{k=2}^{\infty} \varnothing(n,k,\alpha,\gamma)\left(\tfrac{1}{\varnothing(n,k,\alpha,\gamma)}\mathcal{X}_n\right) + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma)\left(\tfrac{1}{\psi(n,k,\alpha,\gamma)}\mathcal{Y}_n\right) \\
&= \sum_{n=2}^{\infty} \mathcal{X}_n + \sum_{n=1}^{\infty} \mathcal{Y}_n = 1 - \mathcal{X}_1 \le 1\,.
\end{aligned}$$

Therefore $f_k(z) \in \ clcoA_{\overline{H}}(k,\alpha,\gamma)$.
Conversely, if $f_k(z) \in \ clcoA_{\overline{H}}(k,\alpha,\gamma)$. Then

$$\text{Set } \mathcal{X}_n = \ \varnothing(n,k,\alpha,\gamma)|a_n|\ ,\ (n = 2,3,\ldots) \text{ and } \mathcal{Y}_n = \psi(n,k,\alpha,\gamma)|b_n|,$$
$$(n = 1,2,\ldots\,) \text{ and } \mathcal{X}_1 = 1 - \sum_{n=2}^{\infty} \mathcal{X}_n + \sum_{n=1}^{\infty} \mathcal{Y}_n$$

The required representation is obtained as

$$\begin{aligned}
f_k(z) \quad &= z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n \\
&= z - \sum_{n=2}^{\infty} \tfrac{1}{\varnothing(n,k,\alpha,\gamma)} \mathcal{X}_n z^n + (-1)^k \sum_{n=1}^{\infty} \tfrac{1}{\psi(n,k,\alpha,\gamma)} \mathcal{Y}_n \overline{z}^n \\
&= z - \sum_{n=2}^{\infty} [z - h_n(z)]\mathcal{X}_n + \sum_{n=1}^{\infty} [z - g_{kn}(z)]\mathcal{Y}_n \\
&= \left[1 - \sum_{n=2}^{\infty} \mathcal{X}_n - \sum_{n=1}^{\infty} \mathcal{Y}_n\right] z + \sum_{n=2}^{\infty} h_n(z)\mathcal{X}_n + \sum_{n=1}^{\infty} g_{kn}(z)\,\mathcal{Y}_n \ = \sum_{n=1}^{\infty} (\mathcal{X}_n h_n + \mathcal{Y}_n g_{kn})
\end{aligned}$$

□

2.3. Convex Combination

Here, we show that the class $A_{\overline{H}}(k,\alpha,\gamma)$ is closed under convex combination of its members.
Let the function $f_{k,i}(z)$ be defined, for $i = 1,2,\ldots,m$ by

$$f_{k,i}(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + (-1)^k \sum_{n=1}^{\infty} |b_{n,i}| \overline{z}^n \tag{13}$$

Theorem 4. *Let the functions $f_{k,i}(z)$, defined by* Equation (13) *be in the class $A_{\overline{H}}(k,\alpha,\gamma)$, for every $i = 1,2,\ldots,m$. Then the functions $c_i(z)$* defined by

$$c_i(z) = \sum_{i=1}^{\infty} t_i\, f_{k,i}(z),\ 0 \le t_i \le 1$$

are also in the class $A_{\overline{H}}(k,\alpha,\gamma)$, where $\sum_{i=1}^{\infty} t_i = 1$.

Proof. According to the definition of $c_i(z)$, we can write

$$c_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) z^n + (-1)^k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i\ |b_{n,i}| \right) \overline{z}^n$$

Further, since $f_{k,i}(z)$ are in $A_{\overline{H}}(k, \alpha, \gamma)$ for every $i = 1, 2, \ldots, m$, then by Theorem 2, we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \varnothing(n,k,\alpha,\gamma)\left(\sum_{i=1}^{\infty} t_i|a_{n,i}|\right) + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma)\left(\sum_{i=1}^{\infty} t_i\,|b_{n,i}|\right) \\ &= \sum_{i=1}^{\infty} t_i\left(\sum_{k=2}^{\infty} \varnothing(n,k,\alpha,\gamma)|a_{n,i}| + \sum_{k=1}^{\infty} \psi(n,k,\alpha,\gamma)|b_{n,i}|\right) \le \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

which is required coefficient condition. □

2.4. Convolution (Hadamard Product) Property

Here, we show that the class $A_{\overline{H}}(k, \alpha, \gamma)$ is closed under convolution.

The convolution of two harmonic functions

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n, \tag{14}$$

and

$$Q_n(z) = z - \sum_{n=2}^{\infty} |L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |M_n| \overline{z}^n \tag{15}$$

is defined as

$$\begin{aligned} (f_n * Q_n)(z) &= f_n(z) * Q_n(z) \\ &= z - \sum_{n=2}^{\infty} |a_n L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n M_n| \overline{z}^n \end{aligned} \tag{16}$$

Using Equations (12)–(14), we prove the following theorem.

Theorem 5. *For $0 \le \mu \le \alpha < 1$, $k \in \mathbb{N}_0$, let $f_n \in A_{\overline{H}}(k, \alpha, \gamma)$ and $Q_n \in A_{\overline{H}}(k, \mu, \gamma)$. Then*

$$f_n * Q_n \in A_{\overline{H}}(k, \alpha, \gamma) \subset A_{\overline{H}}(k, \mu, \gamma).$$

Proof. Let

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \overline{z}^n$$

be in the class $A_{\overline{H}}(k, \alpha, \gamma)$ and

$$Q_n(z) = z - \sum_{n=2}^{\infty} |L_n| z^n + (-1)^k \sum_{n=1}^{\infty} |M_n| \overline{z}^n,$$

be in $A_{\overline{H}}(k, \mu, \gamma)$.

Then the convolution $f_n * Q_n$ is given by Equation (16), we want to show that the coefficients of $f_n * Q_n$ satisfy the required condition given in Theorem 1.

For $Q_n \in A_{\overline{H}}(k, \mu, \gamma)$, we note that $|L_n| < 1$ and $|M_n| < 1$. Now consider convolution functions $f_n * Q_n$ as follows:

$$\begin{aligned} &\sum_{k=2}^{\infty} \varnothing(n,k,\mu,\gamma)|a_n||L_n| + \sum_{k=1}^{\infty} \psi(n,k,\mu,\gamma)|b_n||M_n| \\ &\le \sum_{k=2}^{\infty} \varnothing(n,k,\mu,\gamma)|a_n| + \sum_{k=1}^{\infty} \psi(n,k,\mu,\gamma)|b_n| \le 1. \end{aligned}$$

Since $0 \le \mu \le \alpha < 1$ and $f_n \in A_{\overline{H}}(k, \alpha, \gamma)$. Therefore $f_n * Q_n \in A_{\overline{H}}(k, \alpha, \gamma) \subset A_{\overline{H}}(k, \mu, \gamma)$. □

2.5. Integral Operator

Here, we examine the closure property of the class $A_{\overline{H}}(k,\alpha,\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator (see References [10,11]) $\mathcal{L}_u(f)$ which is defined by,

$$\mathcal{L}_u(f) = \frac{u+1}{z^u}\int_0^z t^{u-1}f(t)dt,\ u > -1. \tag{17}$$

Theorem 6. *Let $f_k(z) \in A_{\overline{H}}(k,\alpha,\gamma)$. Then*

$$\mathcal{L}_u(f_k(z)) \in A_{\overline{H}}(k,\alpha,\gamma)$$

Proof. From definition of $\mathcal{L}_u(f_k(z))$ given by Equation (17), it follows that

$$\begin{aligned}\mathcal{L}_u(f_k(z)) &= \tfrac{u+1}{z^u}\int_0^z t^{u-1}\left(t-\sum_{n=2}^{\infty}|a_n|t^n+(-1)^k\sum_{n=1}^{\infty}|b_n|\bar{t}^n\right)dt\\ &= z-\sum_{n=2}^{\infty}\tfrac{u+1}{u+n}|a_n|z^n+(-1)^k\sum_{n=1}^{\infty}\tfrac{u+1}{u+n}|b_n|z^n\\ &= z-\sum_{n=2}^{\infty}G_nz^n+(-1)^{n-1}\sum_{n=1}^{\infty}L_nz^n\end{aligned}$$

where

$$G_n = \tfrac{u+1}{u+n}|a_n|, \text{ and}$$
$$L_n = \tfrac{u+1}{u+n}|b_n|$$

Hence

$$\begin{aligned}&\sum_{k=2}^{\infty}\varnothing(n,k,\alpha,\gamma)\tfrac{u+1}{u+n}|a_n|+\sum_{k=1}^{\infty}\psi(n,k,\alpha,\gamma)\tfrac{u+1}{u+n}|b_n|\\ &\leq \sum_{n=2}^{\infty}\varnothing(n,k,\alpha,\gamma)|a_n|+\sum_{n=1}^{\infty}\psi(n,k,\alpha,\gamma)|b_n|\leq 1.\end{aligned}$$

by Theorem 2.

Therefore, we have $\mathcal{L}_u(f_k(z)) \in A_{\overline{H}}(k,\alpha,\gamma)$. □

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