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C^*-Ternary Biderivations and C^*-Ternary Bihomomorphisms

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Abstract: In this paper, we investigate C^*-ternary biderivations and C^*-ternary bihomomorphism in C^*-ternary algebras, associated with bi-additive s-functional inequalities.

Keywords: C^*-ternary biderivation; C^*-ternary algebra; C^*-ternary bihomomorphism; Hyers-Ulam stability; bi-additive s-functional inequality

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [6] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \le \|f(x+y)\| \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [7]. Fechner [8] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1).

Park [10,11] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [12–20]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z) \mapsto [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$, and satisfies $\|[x,y,z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,x,x]\| = \|x\|^3$ (see [21]).

If a C^*-ternary algebra $(A, [\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Let A and B be C^*-ternary algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary homomorphism* if

$$H([x,y,z]) = [H(x), H(y), H(z)]$$

for all $x,y,z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x,y,z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x,y,z \in A$ (see [22,23]).

Bae and Park [24] defined C^*-ternary bihomomorphisms and C^*-ternary biderivations in C^*-ternary algebras.

Definition 1. *[24] Let A and B be C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a C^*-ternary bihomomorphism if*

$$\begin{aligned} H([x,y,z],w) &= [H(x,w), H(y,w), H(z,w)], \\ H(x,[y,z,w]) &= [H(x,y), H(x,z), H(x,w)] \end{aligned} \tag{2}$$

for all $x,y,z,w \in A$. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a C^*-ternary biderivation if

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w), y, z] + [x, \delta(y,w), z] + [x, y, \delta(z,w)], \\ \delta(x,[y,z,w]) &= [\delta(x,y), z, w] + [y, \delta(x,z), w] + [y, z, \delta(x,w)] \end{aligned} \tag{3}$$

for all $x,y,z,w \in A$.

Replacing w by $2w$ in (2), we get

$$\begin{aligned} 2H([x,y,z],w) &= H([x,y,z],2w) = [H(x,2w), H(y,2w), H(z,2w)] \\ &= 8[H(x,w), H(y,w), H(z,w)] = 8H([x,y,z],w) \end{aligned}$$

and so $H([x,y,z],w) = 0$ for all $x,y,z,w \in A$.

Replacing w by iw in (3), we get

$$\begin{aligned} i\delta([x,y,z],w) &= \delta([x,y,z],iw) = [\delta(x,iw), y, z] + [x, \delta(y,iw), z] + [x, y, \delta(z,iw)] \\ &= i[\delta(x,w), y, z] - i[x, \delta(y,w), z] + i[x, y, \delta(z,w)] \neq i\delta([x,y,z],w) \end{aligned}$$

for all $x,y,z,w \in A$.

Now we correct the above definition as follows.

Definition 2. *Let A and B be C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a C^*-ternary bihomomorphism if*

$$\begin{aligned} H([x,y,z],[w,w,w]) &= [H(x,w), H(y,w), H(z,w)], \\ H([x,x,x],[y,z,w]) &= [H(x,y), H(x,z), H(x,w)] \end{aligned}$$

for all $x,y,z,w \in A$. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a C^*-ternary biderivation if

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w), y, z] + [x, \delta(y,w^*), z] + [x, y, \delta(z,w)], \\ \delta(x,[y,z,w]) &= [\delta(x,y), z, w] + [y, \delta(x^*,z), w] + [y, z, \delta(x,w)] \end{aligned}$$

for all $x,y,z,w \in A$.

In this paper, we prove the Hyers-Ulam stability of C^*-ternary bihomomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras.

This paper is organized as follows: In Sections 2 and 3, we correct and prove the results on C^*-ternary bihomomorphisms and C^*-ternary derivations in C^*-ternary algebras, given in [24]. In Sections 4 and 5, we investigate C^*-ternary biderivations and C^*-ternary bihomomorphisms in C^*-ternary algebras associated with the following bi-additive s-functional inequalities

$$\begin{aligned} &\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \\ &\leq \left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|, \end{aligned} \tag{4}$$

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right\| \\ &\leq \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\|, \end{aligned} \tag{5}$$

where s is a fixed nonzero complex number with $|s|<1$.

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Assume that s is a fixed nonzero complex number with $|s|<1$.

2. C^*-Ternary Bihomomorphisms in C^*-Ternary Algebras

In this section, we correct and prove the results on C^*-ternary bihomomorphisms in C^*-ternary algebras, given in [24].

Throughout this paper, assume that A and B are C^*-ternary algebras.

Lemma 1. *([24], Lemmas 2.1 and 2.2) Let $f: X\times X\to Y$ be a mapping such that*

$$f(\lambda(x+y),\mu(z-w))+f(\lambda(x-y),\mu(z+w))=2\lambda\mu f(x,z)-2\lambda\mu f(y,w)$$

for all $\lambda,\mu\in\mathbb{T}^1:=\{\xi\in\mathbb{C}:|\xi|=1\}$ and all $x,y,z,w\in V$. Then $f: X\times X\to Y$ is $\mathbb{C}$-bilinear.

For a given mapping $f: A\times A\to B$, we define

$$\begin{aligned} &D_{\lambda,\mu}f(x,y,z,w) \\ &:= f(\lambda(x+y),\mu(z-w))+f(\lambda(x-y),\mu(z+w))-2\lambda\mu f(x,z)+2\lambda\mu f(y,w) \end{aligned}$$

for all $\lambda,\mu\in\mathbb{T}^1$ and all $x,y,z,w\in A$.

We prove the Hyers-Ulam stability of C^*-ternary bihomomorphisms in C^*-ternary algebras.

Theorem 1. *Let $r<2$ and θ be nonnegative real numbers, and let $f: A\times A\to B$ be a mapping satisfying $f(0,0)=0$ and*

$$\|D_{\lambda,\mu}f(x,y,z,w)\|\leq\theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r), \tag{6}$$

$$\begin{aligned} &\|f([x,y,z],[w,w,w])-[f(x,w),f(y,w),f(z,w)]\| \\ &+\|f([x,x,x],[y,z,w])-[f(x,y),f(x,z),f(x,w)]\| \\ &\leq\theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \end{aligned} \tag{7}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$. *Then there exists a unique* C^**-ternary bi-homomorphism* $H : A \times A \to B$ *such that*

$$\|f(x,z) - H(x,z)\| \leq \frac{6\theta}{4 - 2^r}(\|x\|^r + \|z\|^r) \tag{8}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.3), there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (8). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is defined by

$$H(x,z) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$$

for all $x, z \in A$.

It follows from (7) that

$$\begin{aligned}
&\|H([x,y,z],[w,w,w]) - [H(x,w), H(y,w), H(z,w)]\| \\
&+\|H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)]\| \\
&= \lim_{n\to\infty} \frac{1}{64^n} \|f(8^n[x,y,z], 8^n[w,w,w]) - [f(2^n x, 2^n w), f(2^n y, 2^n w), f(2^n z, 2^n w)]\| \\
&+ \lim_{n\to\infty} \frac{1}{64^n} \|f(8^n[x,x,x], 8^n[y,z,w]) - [f(2^n x, 2^n y), f(2^n x, 2^n z), f(2^n x, 2^n w)]\| \\
&\leq \lim_{n\to\infty} \frac{2^{rn}}{64^n} \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0
\end{aligned}$$

for all $x, y, z, w \in A$. So

$$\begin{aligned}
H([x,y,z],[w,w,w]) &= [H(x,w), H(y,w), H(z,w)], \\
H([x,x,x],[y,z,w]) &= [H(x,y), H(x,z), H(x,w)]
\end{aligned}$$

for all $x, y, z, w \in A$, as desired. □

Similarly, we can obtain the following.

Theorem 2. *Let* $r > 6$ *and* θ *be nonnegative real numbers, and let* $f : A \times A \to B$ *be a mapping satisfying* $f(0,0) = 0$, (6) *and* (7). *Then there exists a unique* C^**-ternary bihomomorphism* $H : A \times A \to B$ *such that*

$$\|f(x,z) - H(x,z)\| \leq \frac{6\theta}{2^r - 4}(\|x\|^r + \|z\|^r) \tag{9}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.5), there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (9). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is defined by

$$H(x,z) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$$

for all $x, z \in A$.

It follows from (7) that

$$\begin{aligned}
&\|H([x,y,z],[w,w,w]) - [H(x,w),H(y,w),H(z,w)]\| \\
&+\|H([x,x,x],[y,z,w]) - [H(x,y),H(x,z),H(x,w)]\| \\
&= \lim_{n\to\infty} 64^n \left\| f\left(\frac{[x,y,z]}{8^n},\frac{[w,w,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right), f\left(\frac{y}{2^n},\frac{w}{2^n}\right), f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&+ \lim_{n\to\infty} 64^n \left\| f\left(\frac{[x,x,x]}{8^n},\frac{[y,z,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n},\frac{y}{2^n}\right), f\left(\frac{x}{2^n},\frac{z}{2^n}\right), f\left(\frac{x}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&\le \lim_{n\to\infty} \frac{64^n}{2^{rn}}\theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0
\end{aligned}$$

for all $x,y,z,w \in A$. So

$$\begin{aligned}
H([x,y,z],[w,w,w]) &= [H(x,w),H(y,w),H(z,w)], \\
H([x,x,x],[y,z,w]) &= [H(x,y),H(x,z),H(x,w)]
\end{aligned}$$

for all $x,y,z,w \in A$, as desired. □

Theorem 3. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers, and let $f : A \times A \to B$ be a mapping satisfying $f(0,0) = 0$ and*

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r, \tag{10}$$

$$\begin{aligned}
&\|f([x,y,z],[w,w,w]) - [f(x,w),f(y,w),f(z,w)]\| \\
&+\|f([x,x,x],[y,z,w]) - [f(x,y),f(x,z),f(x,w)]\| \\
&\le \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r
\end{aligned} \tag{11}$$

for all $\lambda,\mu \in \mathbb{T}^1$ and all $x,y,z,w \in A$. Then there exists a unique C^*-ternary bihomomorphism $H : A \times A \to B$ such that

$$\|f(x,z) - H(x,z)\| \le \frac{2\theta}{4-16^r}(\|x\|^r + \|z\|^r) \tag{12}$$

for all $x,z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.6), there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (12). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is defined by

$$H(x,z) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$$

for all $x,z \in A$.

The rest of the proof is similar to the proof of Theorem 1. □

Theorem 4. *Let $r > \frac{3}{2}$ and θ be nonnegative real numbers, and let $f : A \times A \to B$ be a mapping satisfying $f(0,0) = 0$, (10) and (11). Then there exists a unique C^*-ternary bihomomorphism $H : A \times A \to B$ such that*

$$\|f(x,z) - H(x,z)\| \le \frac{2\theta}{16^r-4}(\|x\|^r + \|z\|^r) \tag{13}$$

for all $x,z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.7), there exists a unique $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ satisfying (13). The $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is defined by

$$H(x,z) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 1. □

3. C^*-Ternary Biderivations on C^*-Ternary Algebras

In this section, we correct and prove the results on C^*-ternary biderivations on C^*-ternary algebras, given in [24].

Throughout this paper, assume that A is a C^*-ternary algebra.

Theorem 5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : A \times A \to A$ be a mapping satisfying $f(0,0) = 0$ and*

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r), \tag{14}$$

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\ &+\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\| \\ &\qquad \le \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \end{aligned} \tag{15}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \le \frac{6\theta}{4-2^r}(\|x\|^r + \|z\|^r) \tag{16}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorems 2.3 and 3.1), there exists a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ satisfying (16). The $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is defined by

$$\delta(x,z) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$$

for all $x, z \in A$.

It follows from (15) that

$$\begin{aligned} &\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\ &+\|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\ &= \lim_{n\to\infty} \frac{1}{16^n}(\|f(8^n[x,y,z],2^n w) - [f(2^n x,2^n w),2^n y,2^n w] \\ &\qquad -[2^n x, f(2^n y,2^n w^*),2^n z] - [2^n x,2^n y,f(2^n z,2^n w)]\|) \\ &+ \lim_{n\to\infty} \frac{1}{16^n}(\|f(2^n x,8^n[y,z,w]) - [f(2^n x,2^n y),2^n z,2^n w] \\ &\qquad -[2^n y, f(2^n x^*,2^n z),2^n w] - [2^n y,2^n z,f(2^n x,2^n w)]\|) \\ &\le \lim_{n\to\infty} \frac{2^{rn}}{16^n}\theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0 \end{aligned}$$

for all $x, y, z, w \in A$. So

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)], \\ \delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$, as desired. □

Similarly, we can obtain the following.

Theorem 6. *Let $r > 4$ and θ be nonnegative real numbers, and let $f : A \times A \to A$ be a mapping satisfying $f(0,0) = 0$, (14) and (15). Then there exists a unique C^*-ternary biderivation $\delta : A \times A \to A$ such that*

$$\|f(x,z) - \delta(x,z)\| \le \frac{6\theta}{2^r - 4}(\|x\|^r + \|z\|^r) \tag{17}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.5), there exists a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ satisfying (17). The $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is defined by

$$\delta(x,z) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$$

for all $x, z \in A$.

It follows from (15) that

$$\begin{aligned} &\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\ &+\|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\ &= \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{[x,y,z]}{8^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), \frac{y}{2^n}, \frac{w}{2^n}\right] \right.\right. \\ &\qquad \left.\left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right] \right\| \right) \\ &+ \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{x}{2^n}, \frac{[y,z,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{z}{2^n}, \frac{w}{2^n}\right] \right.\right. \\ &\qquad \left.\left. - \left[\frac{y}{2^n}, f\left(\frac{x^*}{2^n}, \frac{z}{2^n}\right), \frac{w}{2^n}\right] - \left[\frac{y}{2^n}, \frac{z}{2^n}, f\left(\frac{x}{2^n}, \frac{w}{2^n}\right)\right] \right\| \right) \\ &\le \lim_{n\to\infty} \frac{16^n}{2^{rn}} \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) = 0 \end{aligned}$$

for all $x, y, z, w \in A$. So

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)], \\ \delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$, as desired. □

Theorem 7. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers, and let $f : A \times A \to A$ be a mapping satisfying $f(0,0) = 0$ and*

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r, \tag{18}$$

$$\begin{aligned}&\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\ &+\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\| \\ &\qquad\le \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r\end{aligned} \tag{19}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x,y,z,w \in A$. Then there exists a unique C^*-ternary biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \le \frac{2\theta}{4-16^r}(\|x\|^r + \|z\|^r) \tag{20}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.6), there exists a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ satisfying (20). The $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is defined by

$$\delta(x,z) = \lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x, 2^n z\right)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 5. □

Theorem 8. *Let $r > \frac{3}{2}$ and θ be nonnegative real numbers, and let $f : A \times A \to A$ be a mapping satisfying $f(0,0) = 0$, (18) and (19). Then there exists a unique C^*-ternary biderivation $\delta : A \times A \to A$ such that*

$$\|f(x,z) - \delta(x,z)\| \le \frac{2\theta}{16^r - 4}(\|x\|^r + \|z\|^r) \tag{21}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of ([24] Theorem 2.7), there exists a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ satisfying (21). The $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is defined by

$$\delta(x,z) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$$

for all $x, z \in A$.

The rest of the proof is similar to the proof of Theorem 5. □

4. C^*-Ternary Biderivations on C^*-Ternary Algebras Associated with the Bi-Additive Functional Inequalities (4) and (5)

In [25], Park introduced and investigated the bi-additive s-functional inequalities (4) and (5) in complex Banach spaces.

Theorem 9. *([25] Theorem 2.2) Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\|f(x+y,z-w) + f(x-y,z+w) - 2f(x,z) + 2f(y,w)\| \\ &\le \left\|s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| \\ &+\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{22}$$

for all $x,y,z,w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \le \frac{2\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{23}$$

for all $x, z \in X$.

Theorem 10. *([25] Theorem 2.3) Let* $r < 1$ *and* θ *be nonnegative real numbers and let* $f : X^2 \to Y$ *be a mapping satisfying* (22) *and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in X$. *Then there exists a unique bi-additive mapping* $A : X^2 \to Y$ *such that*

$$\|f(x,z) - A(x,z)\| \leq \frac{2\theta}{2-2^r}\|x\|^r\|z\|^r \tag{24}$$

for all $x, z \in X$.

Theorem 11. *([25] Theorem 3.2) Let* $r > 1$ *and* θ *be nonnegative real numbers and let* $f : X^2 \to Y$ *be a mapping satisfying* $f(x,0) = f(0,z) = 0$ *and*

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right\| \\ &\quad \leq \|s\left(f(x+y,z-w) + f(x-y,z+w) - 2f(x,z) + 2f(y,w)\right)\| \\ &\quad +\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{25}$$

for all $x, y, z, w \in X$. *Then there exists a unique bi-additive mapping* $A : X^2 \to Y$ *such that*

$$\|f(x,z) - A(x,z)\| \leq \frac{2^{r-1}\theta}{2^r-2}\|x\|^r\|z\|^r \tag{26}$$

for all $x, z \in X$.

Theorem 12. *([25] Theorem 3.3) Let* $r < 1$ *and* θ *be nonnegative real numbers and let* $f : X^2 \to Y$ *be a mapping satisfying* (25) *and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in X$. *Then there exists a unique bi-additive mapping* $A : X^2 \to Y$ *such that*

$$\|f(x,z) - A(x,z)\| \leq \frac{\theta}{2(2-2^r)}\|x\|^r\|z\|^r \tag{27}$$

for all $x, z \in X$.

Now, we investigate C^*-ternary biderivations on C^*-ternary algebras associated with the bi-additive s-functional inequalities (4) and (5).

From now on, assume that A is a C^*-ternary algebra.

Theorem 13. *Let* $r > 2$ *and* θ *be nonnegative real numbers, and let* $f : A^2 \to A$ *be a mapping satisfying* $f(x,0) = f(0,z) = 0$ *and*

$$\begin{aligned} &\|f(\lambda(x+y), \mu(z-w)) + f(\lambda(x-y), \mu(z+w)) - 2\lambda\mu f(x,z) + 2\lambda\mu f(y,w)\| \\ &\quad \leq \left\|s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| \\ &\quad +\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{28}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$. *Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $D : A^2 \to A$ *such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{2\theta}{2^r-2}\|x\|^r\|z\|^r \tag{29}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ *satisfies* $f(2x,z) = 2f(x,z)$ *and*

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\ &\quad \le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|z\|^r), \end{aligned} \tag{30}$$

$$\begin{aligned} &\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z,w] - [y,z,f(x,w)]\| \\ &\quad \le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{31}$$

for all $x,y,z,w \in A$*, then the mapping* $f : A^2 \to A$ *is a* C^**-ternary biderivation.*

Proof. Let $\lambda = \mu = 1$ in (28). By Theorem 9, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (29) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x,z \in A$.

Letting $y = w = 0$ in (28), we get $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x,z \in A$ and all $\lambda,\mu \in \mathbb{T}^1$. By Lemma 1, the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-bilinear.

If $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then we can easily show that $D(x,z) = f(x,z)$ for all $x,z \in A$.

It follows from (30) that

$$\begin{aligned} &\|D([x,y,z],w) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)]\| \\ &= \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{[x,y,z]}{8^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right]\right.\right. \\ &\quad \left.\left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n},\frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\|\right) \\ &\le \lim_{n\to\infty} \frac{16^n\theta}{4^{rn}}(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) = 0 \end{aligned}$$

for all $x,y,z,w \in A$. Thus

$$D([x,y,z],w) = [D(x,w),y,z] + [x,D(y,w^*),z] + [x,y,D(z,w)]$$

for all $x,y,z,w \in A$.

Similarly, one can show that

$$D(x,[y,z,w]) = [D(x,y),z,w] - [y,D(x^*,z,w] - [y,z,D(x,w)]$$

for all $x,y,z,w \in A$. Hence the mapping $f : A^2 \to A$ is a C^*-ternary biderivation. □

Theorem 14. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : A^2 \to A$ *be a mapping satisfying* (28) *and* $f(x,0) = f(0,z) = 0$ *for all* $x,z \in A$*. Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $D : A^2 \to A$ *such that*

$$\|f(x,z) - D(x,z)\| \le \frac{2\theta}{2-2^r}\|x\|^r\|z\|^r \tag{32}$$

for all $x,z \in A$*.*

If, in addition, the mapping $f : A^2 \to A$ *satisfies* (30), (31) *and* $f(2x,z) = 2f(x,z)$ *for all* $x,z \in A$*, then the mapping* $f : A^2 \to A$ *is a* C^**-ternary biderivation.*

Proof. The proof is similar to the proof of Theorem 13. □

Similarly, we can obtain the following results.

Theorem 15. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\left\|2f\left(\lambda\frac{x+y}{2}, \mu(z-w)\right) + 2f\left(\lambda\frac{x-y}{2}, \mu(z+w)\right) - 2\lambda\mu f(x,z) + 2\lambda\mu f(y,w)\right\| \\ &\quad \le \|s\left(f(x+y,z-w) + f(x-y,z+w) - 2f(x,z) + 2f(y,w)\right)\| \\ &\qquad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{33}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x,z) - D(x,z)\| \le \frac{2^{r-1}\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{34}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (30), (31) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a C^*-ternary biderivation.

Theorem 16. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (33) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z) - D(x,z)\| \le \frac{\theta}{2(2 - 2^r)}\|x\|^r\|z\|^r \tag{35}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (30), (31) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a C^*-ternary biderivation.

5. C^*-Ternary Bihomomorphisms in C^*-Ternary Algebras Associated with the Bi-Additive Functional Inequalities (4) and (5)

In this section, we investigate C^*-ternary bihomomorphisms in C^*-ternary algebras associated with the bi-additive s-functional inequalities (4) and (5).

Theorem 17. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and (28). Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (29), where D is replaced by H in (29).*

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x,z) = 2f(x,z)$ and

$$\|f([x,y,z],[w,w,w]) - [f(x,w), f(y,w), f(z,w)]\| \le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r), \tag{36}$$

$$\|f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)]\| \le \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \tag{37}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to B$ is a C^*-ternary bihomomorphism.

Proof. By the same reasoning as in the proof of Theorem 13, there is a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$, which is defined by

$$H(x,z) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then we can easily show that $H(x,z) = f(x,z)$ for all $x,z \in A$.

It follows from (36) that

$$\begin{aligned}
&\|H([x,y,z],[w,w,w]) - [H(x,w),H(y,w),H(z,w)]\| \\
&= \lim_{n\to\infty} 4^{3n} \left\| f\left(\frac{[x,y,z]}{8^n}, \frac{[w,w,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), f\left(\frac{y}{2^n}, \frac{w}{2^n}\right), f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right] \right\| \\
&\le \lim_{n\to\infty} \frac{4^{3n}\theta}{4^{rn}} (\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) = 0
\end{aligned}$$

for all $x,y,z,w \in A$. Thus

$$H([x,y,z],[w,w,w]) = [H(x,w),H(y,w),H(z,w)]$$

for all $x,y,z,w \in A$.

Similarly, one can show that

$$H([x,x,x],[y,z,w]) = [H(x,y),H(x,z),H(x,w)]$$

for all $x,y,z,w \in A$. Hence the mapping $f : A^2 \to B$ is a C^*-ternary bihomomorphism. □

Theorem 18. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying (28) and $f(x,0) = f(0,z) = 0$ for all $x,z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (32), where D is replaced by H in (32).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (36), (37) and $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then the mapping $f : A^2 \to B$ is a C^*-ternary bihomomorphism.

Proof. The proof is similar to the proof of Theorem 17. □

Similarly, we can obtain the following results.

Theorem 19. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and (33). Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (34), where D is replaced by H in (34).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (36), (37) and $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then the mapping $f : A^2 \to B$ is a C^*-ternary bihomomorphism.

Theorem 20. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying (33) and $f(x,0) = f(0,z) = 0$ for all $x,z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (35), where D is replaced by H in (35).*

If, in addition, the mapping $f : A^2 \to B$ satisfies (36), (37) and $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then the mapping $f : A^2 \to B$ is a C^*-ternary bihomomorphism.

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References

1. Ulam, S.M. *A Collection of the Mathematical Problems*; Interscience Publishers: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* **1941**, *27*, 222–224.
3. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66.

4. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300.
5. Găvruta, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436.
6. Gilányi, A. Eine zur Parallelogrammgleichung äquivalente Ungleichung. *Aequ. Math.* **2001**, *62*, 303–309.
7. Rätz, J. On inequalities associated with the Jordan-von Neumann functional equation. *Aequ. Math.* **2003**, *66*, 191–200.
8. Fechner, W. Stability of a functional inequalities associated with the Jordan-von Neumann functional equation. *Aequ. Math.* **2006**, *71*, 149–161.
9. Gilányi, A. On a problem by K. Nikodem. *Math. Inequal. Appl.* **2002**, *5*, 707–710.
10. Park, C. Additive ρ-functional inequalities and equations. *J. Math. Inequal.* **2015**, *9*, 17–26.
11. Park, C. Additive ρ-functional inequalities in non-Archimedean normed spaces. *J. Math. Inequal.* **2015**, *9*, 397–407.
12. Cho, Y.; Jang, S.; Kwon, S.; Park, C.; Park, W. Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras: revisited. *Korean J. Math.* **2013**, *21*, 161–170.
13. Gordji, M.E.; Fazeli, A.; Park, C. 3-Lie multipliers on Banach 3-Lie algebras. *Int. J. Geom. Meth. Mod. Phys.* **2012**, *9*, 1250052.
14. Gordji, M.E.; Ghaemi, M.B.; Alizadeh, B. A fixed point method for perturbation of higher ring derivationsin non-Archimedean Banach algebras. *Int. J. Geom. Meth. Mod. Phys.* **2011**, *8*, 1611–1625.
15. Gordji, M.E.; Ghobadipour, N. Stability of (α, β, γ)-derivations on Lie C^*-algebras. *Int. J. Geom. Meth. Mod. Phys.* **2010**, *7*, 1097–1102.
16. Jung, S.; Lee, Y. A fixed point approach to the stability of a mean value type functional equation. *Mathematics* **2017**, *5*, 78.
17. Lee, Y. Stability of a monomial functional equation on a restricted domain. *Mathematics* **2017**, *5*, 53.
18. Shin, D.; Park, C.; Farhadabadi, S. On the superstability of ternary Jordan C^*-homomorphisms. *J. Comput. Anal. Appl.* **2014**, *16*, 964–973.
19. Shin, D.; Park, C.; Farhadabadi, S. Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation. *J. Comput. Anal. Appl.* **2014**, *17*, 125–134.
20. Amyari, M.; Baak, C.; Moslehian, M. Nearly ternary derivations. *Taiwan. J. Math.* **2007**, *11*, 1417–1424.
21. Zettl, H. A characterization of ternary rings of operators. *Adv. Math.* **1983**, *48*, 117–143.
22. Moslehian, M.S. Almost derivations on C^*-ternary rings. *Bull. Belg. Math. Soc. Simon Stevin* **2006**, *14*, 135–142.
23. Amyari, M.; Moslehian, M.S. Approximate homomorphisms of ternary semigroups. *Lett. Math. Phys.* **2006**, *77*, 1–9.
24. Bae, J.; Park, W. Approximate bi-homomorphisms and bi-derivations in C^*-ternary algebras. *Bull. Korean Math. Soc.* **2010**, *47*, 195–209.
25. Park, C. Bi-additive s-functional inequalities and quasi-multipliers on Banach algebras. Unpublished. (preprint)